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JS-H 1: 16

16 But, exerting all my powers to call upon God to deliver me out of the power of this enemy which had seized upon me, and at the very moment when I was ready to sink into despair and abandon myself to destruction—not to an imaginary ruin, but to the power of some actual being from the unseen world, who had such marvelous power as I had never before felt in any being—just at this moment of great alarm, I saw a pillar of light exactly over my head, above the brightness of the sun, which descended gradually until it fell upon me.
Lecture 13 – Network Analysis with Capacitors and Inductors

Phasors
Euler’s Identity

**Appendix A** reviews complex numbers

Complex exponential ($e^{j\theta}$) is a point on the complex plane

\[ e^{j\theta} = \cos \theta + j \sin \theta \]

\[ |e^{j\theta}| = 1 \rightarrow |\cos \theta + j \sin \theta| = \sqrt{\cos^2 \theta + \sin^2 \theta} = 1 \]

\[ Ae^{j\theta} = A \cos \theta + jA \sin \theta \]

\[ = A \angle \theta \]
Rewrite the expression for a general sinusoid signal:

\[ A \cos(\omega t + \theta) = \text{Re}\{A e^{j(\omega t + \theta)}\} \]

Complex phasor notation for the simplification:

\[ A \cos(\omega t + \theta) \rightarrow A \angle \theta = A e^{j\theta} \]

NB: The \( e^{j\omega t} \) term is implicit (it is there but not written)
Graphing in the frequency domain: helpful in order to understand Phasors

\[ \cos(\omega_0 t) \]

Time domain

\[ \pi [\delta(\omega - \omega_0) + \delta(\omega - \omega_0)] \]

Frequency domain
Phasors

1. Any sinusoidal signal can be represented by either:
   - **Time-domain form**: \( v(t) = A\cos(\omega t + \theta) \)
   - **Frequency-domain form**: \( V(j\omega) = Ae^{j\theta} = A\angle\theta \)

2. **Phasor**: a complex number expressed in polar form consisting of:
   - **Magnitude** (A)
   - **Phase angle** (\( \theta \))

3. Phasors do not explicitly include the sinusoidal frequency (\( \omega \)) but this information is still important
**Example 1**: compute the phasor voltage for the equivalent voltage $v_s(t)$

- $v_1(t) = 15\cos(377t + \pi/4)$
- $v_2(t) = 15\cos(377t + \pi/12)$

**Diagram:**

- $v_1(t) + \sim$
- $v_2(t) + \sim$
- $v_s(t) + \sim$
**Phasors**

- **Example 1**: compute the phasor voltage for the equivalent voltage $v_s(t)$
  - $v_1(t) = 15\cos(377t + \pi/4)$
  - $v_2(t) = 15\cos(377t + \pi/12)$

1. Write voltages in phasor notation

\[ V_1(j\omega) = 15e^{j\pi/4} = 15 \angle \frac{\pi}{4} \, V \]
\[ V_2(j\omega) = 15e^{j\pi/12} = 15 \angle \frac{\pi}{12} \, V \]
Phasors

Example 1: compute the phasor voltage for the equivalent voltage $v_s(t)$

- $v_1(t) = 15\cos(377t + \pi/4)$
- $v_2(t) = 15\cos(377t + \pi/12)$

1. Write voltages in phasor notation
2. Convert phasor voltages from polar to rectangular form (see Appendix A)

\[
V_1(j\omega) = 15 \angle \frac{\pi}{4} \ V
\]

Convert to rectangular:

\[
V_1(j\omega) = 15 \cos\left(\frac{\pi}{4}\right) + j15 \sin\left(\frac{\pi}{4}\right)

= 10.61 + j10.61 \ V
\]

\[
V_2(j\omega) = 15 \angle \frac{\pi}{12} \ V
\]

Convert to rectangular:

\[
V_2(j\omega) = 15 \cos\left(\frac{\pi}{12}\right) + j15 \sin\left(\frac{\pi}{12}\right)

= 14.49 + j3.88 \ V
\]
**Phasors**

◆ **Example 1**: compute the phasor voltage for the equivalent voltage $v_s(t)$
  - $v_1(t) = 15\cos(377t + \pi/4)$
  - $v_2(t) = 15\cos(377t + \pi/12)$

1. Write voltages in phasor notation
2. Convert phasor voltages from polar to rectangular form (see Appendix A)
3. Combine voltages

$$V_s(j\omega) = V_1(j\omega) + V_2(j\omega) = 25.10 + j14.49$$
Phasors

**Example 1**: compute the phasor voltage for the equivalent voltage $v_s(t)$

- $v_1(t) = 15\cos(377t + \pi/4)$
- $v_2(t) = 15\cos(377t + \pi/12)$

1. Write voltages in phasor notation
2. Convert phasor voltages from polar to rectangular form (see Appendix A)
3. Combine voltages
4. Convert rectangular back to polar

\[ V_s(j\omega) = 25.10 + j14.49 \]

Convert to polar:

\[
\begin{align*}
  r &= \sqrt{(25.10)^2 + (14.49)^2} \\
  &= 28.98 \\
  \theta &= \tan^{-1}\left(\frac{14.49}{25.10}\right) \\
  &= \frac{\pi}{6} \\
  V_s(j\omega) &= 28.98 \angle \frac{\pi}{6}
\end{align*}
\]
**Example 1**: compute the phasor voltage for the equivalent voltage \( v_s(t) \)

\[ v_1(t) = 15\cos(377t + \pi/4) \]
\[ v_2(t) = 15\cos(377t + \pi/12) \]

1. Write voltages in phasor notation
2. Convert phasor voltages from polar to rectangular form (see Appendix A)
3. Combine voltages
4. Convert rectangular back to polar
5. Convert from phasor to time domain

Bring \( \omega t \) back

\[ V_S(j\omega) = 28.98 \angle \frac{\pi}{6} \]
\[ v_s(t) = 28.98 \cos \left(377t + \frac{\pi}{6}\right) \]

**NB**: the answer is NOT simply the addition of the amplitudes of \( v_1(t) \) and \( v_2(t) \) (i.e. 15 + 15), and the addition of their phases (i.e. \( \pi/4 + \pi/12 \))
Phasors

- **Example 1**: compute the phasor voltage for the equivalent voltage $v_s(t)$
  - $v_1(t) = 15\cos(377t + \pi/4)$
  - $v_2(t) = 15\cos(377t + \pi/12)$

\[
\begin{align*}
  v_1(t) &= 15\cos(377t + \pi/4) \\
  v_2(t) &= 15\cos(377t + \pi/12) \\
  v_s(t) &= v_1(t) + \mathbf{j}v_2(t)
\end{align*}
\]

\[
\begin{align*}
  V_s(j\omega) &= 28.98 \angle \frac{\pi}{6} \\
  v_s(t) &= 28.98 \cos \left( 377t + \frac{\pi}{6} \right)
\end{align*}
\]
**Phasors of Different Frequencies**

**Superposition of AC signals**: when signals do not have the same frequency ($\omega$) the $e^{j\omega t}$ term in the phasors can no longer be implicit.

\[
i(t) = i_1(t) + i_2(t)
\]

\[
I(j\omega) = I_1(j\omega_1) + I_2(j\omega_2)
\]

\[
= A_1e^{j0}e^{j\omega_1t} + A_2e^{j0}e^{j\omega_2t}
\]

\[
\ne^{j\omega t} \text{ can no longer be implicit}
\]
Phasors of Different Frequencies

**Superposition of AC signals:** when signals do **not have the same frequency** ($\omega$) solve the circuit separately for each different frequency ($\omega$) – then add the individual results
Phasors of Different Frequencies

**Example 2**: compute the resistor voltages

\[ i_s(t) = 0.5\cos[2\pi(100t)] \text{ A} \]

\[ v_s(t) = 20\cos[2\pi(1000t)] \text{ V} \]

\[ R_1 = 150\Omega, \ R_2 = 50\ \Omega \]
Phasors of Different Frequencies

**Example 2**: compute the resistor voltages

- \( i_s(t) = 0.5 \cos[2\pi(100t)] \) A
- \( v_s(t) = 20 \cos[2\pi(1000t)] \) V
- \( R_1 = 150 \Omega, \ R_2 = 50 \Omega \)

1. Since the sources have different frequencies \((\omega_1 = 2\pi*100)\) and \((\omega_2 = 2\pi*1000)\) use superposition

   - first consider the \((\omega_1 = 2\pi*100)\) part of the circuit
   - When \( v_s(t) = 0 \) – short circuit
Phasors of Different Frequencies

**Example 2**: compute the resistor voltages
- \( i_s(t) = 0.5\cos[2\pi(100t)] \) A
- \( v_s(t) = 20\cos[2\pi(1000t)] \) V
- \( R_1 = 150\Omega, R_2 = 50\Omega \)

1. Since the sources have different frequencies \((\omega_1 = 2\pi*100)\) and \((\omega_2 = 2\pi*1000)\) use superposition
   - first consider the \((\omega_1 = 2\pi*100)\) part of the circuit

\[
I_s(j\omega) = 0.5\angle 0
\]
\[
V_{i1}(j\omega) = V_{i2}(j\omega) = I_s \cdot R_1 \parallel R_2
\]
\[
= I_s \cdot \frac{R_1 R_2}{R_1 + R_2}
\]
\[
= 0.5\angle 0 \cdot \frac{(50)(150)}{(50) + (150)}
\]
\[
= 18.75\angle 0
\]
**Example 2:** compute the resistor voltages

- $i_s(t) = 0.5\cos[2\pi(100t)] \ A$
- $v_s(t) = 20\cos[2\pi(1000t)] \ V$
- $R_1 = 150\Omega$, $R_2 = 50 \Omega$

1. Since the sources have different frequencies $(\omega_1 = 2\pi*100)$ and $(\omega_2 = 2\pi*1000)$ use superposition
   - first consider the $(\omega_1 = 2\pi*100)$ part of the circuit
   - Next consider the $(\omega_2 = 2\pi*1000)$ part of the circuit

\[
\begin{align*}
V_s(j\omega) &= 20 \angle 0 \\
V_{v1}(j\omega) &= V_s \cdot \frac{R_1}{R_1 + R_2} \\
&= 20 \angle 0 \cdot \frac{150}{50 + 150} \\
&= 15 \angle 0
\end{align*}
\]

\[
\begin{align*}
V_s(j\omega) &= 20 \angle 0 \\
V_{v2}(j\omega) &= -V_s \cdot \frac{R_2}{R_1 + R_2} \\
&= -20 \angle 0 \cdot \frac{50}{50 + 150} \\
&= -5 \angle 0 \\
&= 5 \angle \pi
\end{align*}
\]
**Example 2**: compute the resistor voltages

- \( i_s(t) = 0.5\cos[2\pi(100t)] \) A
- \( v_s(t) = 20\cos[2\pi(1000t)] \) V
- \( R_1 = 150\Omega, R_2 = 50\Omega \)

1. Since the sources have different frequencies \((\omega_1 = 2\pi*100)\) and \((\omega_2 = 2\pi*1000)\) use superposition
   - first consider the \((\omega_1 = 2\pi*100)\) part of the circuit
   - Next consider the \((\omega_2 = 2\pi*1000)\) part of the circuit
   - Add the two together

\[
V_1(j\omega) = V_{i1}(j\omega) + V_{v1}(j\omega) = 18.75\angle 0 + 15\angle 0
\]

\[
v_1(t) = 18.75\cos[2\pi(100t)] + 15\cos[2\pi(1000t)]
\]

\[
V_2(j\omega) = V_{i2}(j\omega) + V_{v2}(j\omega) = 18.75\angle 0 - 5\angle 0
\]

\[
v_1(t) = 18.75\cos[2\pi(100t)] - 5\cos[2\pi(1000t)]
\]
Impedance: complex resistance (has no physical significance)
- will allow us to use network analysis methods such as node voltage, mesh current, etc.
- Capacitors and inductors act as frequency-dependent resistors
Impedance – Resistors

Impedance of a Resistor:

Consider Ohm’s Law in phasor form:

\[ v_s(t) = A \cos(\omega t) \]

\[ i(t) = \frac{v_s(t)}{R} \]

\[ = \frac{A}{R} \cos(\omega t) \]

\[ Z_R(j\omega) = \frac{V_Z(j\omega)}{I_Z(j\omega)} = R \]

\[ V_Z(j\omega) = A \angle 0 \]

\[ I_Z(j\omega) = \frac{A}{R} \angle 0 \]

NB: Ohm’s Law works the same in DC and AC
Impedance – Inductors

**Impedance of an Inductor:**

First consider voltage and current in the **time-domain**

\[ v_L(t) = L \frac{di_L(t)}{dt} = v_s(t) \]

\[ i_L(t) = \frac{1}{L} \int v_L(\tau)d\tau = \frac{1}{L} \int A\cos(\omega \tau)d\tau = \frac{A}{\omega L} \sin(\omega t) \]

\[ v_s(t) = v_L(t) = A\cos(\omega t) \]

\[ i_L(t) = \frac{A}{\omega L} \sin(\omega t) \]

\[ = \frac{A}{\omega L} \cos\left(\omega t - \frac{\pi}{2}\right) \]

**NB:** current is shifted 90° from voltage
Impedance – Inductors

Impedance of an Inductor:

Now consider voltage and current in the phasor-domain.

\[ v_s(t) = v_L(t) = A \cos(\omega t) \]
\[ i_L(t) = \frac{A}{\omega L} \sin(\omega t) \]
\[ Z_L(j\omega) = \frac{V_z(j\omega)}{I_z(j\omega)} = j\omega L \]

Phasor domain
Impedance – Capacitors

Impedance of a capacitor:

First consider voltage and current in the time-domain.

\[ v_c(t) = \frac{1}{C} \int i_c(\tau) d\tau = v_s(t) \]

\[ i_c(t) = C \frac{dv_c(t)}{dt} \]

= \[ C \frac{d}{dt} [A \cos(\omega t)] \]

= \[ -C[A \omega \sin(\omega t)] \]

= \[ \omega CA \cos\left(\omega t + \frac{\pi}{2}\right) \]

\[ v_c(t) + \sim C v_c(t) - \]

\[ i(t) + \sim \]

\[ V_s(j\omega) + \sim V_z(j\omega) - \]

\[ V_z(j\omega) = A \angle 0 \]

\[ I_z(j\omega) = \omega CA \angle \frac{\pi}{2} \]
Impedance – Capacitors

Impedance of a capacitor:

Next consider voltage and current in the phasor-domain

\[ V_s(j\omega) \sim \frac{+}{-} Z \frac{+}{-} V_Z(j\omega) \]

\[ V_Z(j\omega) = A \angle 0 \]
\[ I_Z(j\omega) = \omega CA \angle -\frac{\pi}{2} \]

\[ Z_L(j\omega) = \frac{V_Z(j\omega)}{I_Z(j\omega)} = \frac{1}{\omega C} \angle -\frac{\pi}{2} \]

\[-j = e^{-j\frac{\pi}{2}} = \frac{1}{j}\]
Impedance of resistors, inductors, and capacitors

- The impedance of a resistor is $Z_R(j\omega) = R$.
- The impedance of an inductor is $Z_L(j\omega) = j\omega L$.
- The impedance of a capacitor is $Z_C(j\omega) = \frac{1}{j\omega C}$.
Impedance

Impedance of resistors, inductors, and capacitors

\[ Z(j\omega) = R(j\omega) + jX(j\omega) \]

Impedance in general:

- Not a phasor but a complex number
- AC resistance
- Reactance

Phasor domain
Impedance

**Practical capacitors**: in practice capacitors contain a real component (represented by a resistive impedance $Z_R$)

- **At high frequencies or high capacitances**
  - Ideal capacitor acts like a short circuit

- **At low frequencies or low capacitances**
  - Ideal capacitor acts like an open circuit

**NB**: the ratio of $Z_C$ to $Z_R$ is highly frequency dependent
Impedance

**Practical inductors**: in practice inductors contain a real component (represented by a resistive impedance $Z_R$)

- At **low frequencies** or **low inductances** $Z_R$ has a strong influence
  - Ideal inductor acts like a **short circuit**
- At **high frequencies** or **high inductances** $Z_L$ dominates $Z_R$
  - Ideal inductor acts like an **open circuit**
  - At high frequencies a capacitor is also needed to correctly model a practical inductor

**NB**: the ratio of $Z_L$ to $Z_R$ is highly frequency dependent
**Example 3**: impedance of a practical capacitor

- Find the impedance
- \( \omega = 377 \text{ rad/s}, \ C = 1\text{nF}, \ R = 1\text{M}\Omega \)

![Circuit Diagram]

Impedance of a practical capacitor with a practical capacitor in parallel.
Example 3: impedance of a practical capacitor

- Find the impedance
- \( \omega = 377 \text{ rads/s}, \ C = 1\text{nF}, \ R = 1\text{M} \Omega \)

\[
Z = Z_R \parallel Z_C = \frac{R \cdot (1/ j\omega C)}{R + (1/ j\omega C)} = \frac{R}{1 + j\omega CR} = \frac{10^6}{1 + j(377)(10^{-9})(10^6)} = \frac{10^6}{1 + j0.377} = 9.36 \times 10^5 \angle (-0.36) \Omega
\]
Example 4: find the equivalent impedance ($Z_{EQ}$)

$\omega = 10^4$ rads/s, $C = 10\mu F$, $R_1 = 100\Omega$, $R_2 = 50\Omega$, $L = 10\text{mH}$
Example 4: find the equivalent impedance \( (Z_{EQ}) \)

\[\begin{align*}
\omega &= 10^4 \text{ rad/s}, \quad C = 10 \text{uF}, \quad R_1 = 100\Omega, \quad R_2 = 50\Omega, \quad L = 10\text{mH}
\end{align*}\]

\[
Z_{EQ} = Z_{R2} \parallel Z_C \\
= \frac{R_2 (1/j\omega C)}{R_2 + (1/j\omega C)} \\
= \frac{R_2}{1 + j\omega CR_2} \\
= \frac{50}{1 + j(10^4)(10 \times 10^{-6})(50)} \\
= \frac{50}{1 + j5} \\
= 1.92 - j9.62 \\
= 9.81 \angle (-1.37) \Omega
\]
**Example 4:** find the equivalent impedance \( (Z_{EQ}) \)

\[ \omega = 10^4 \text{ rads/s}, \ C = 10\mu F, \ R_1 = 100\Omega, \ R_2 = 50\Omega, \ L = 10\text{mH} \]

\[
Z_{EQ} = Z_{R1} + Z_L + Z_{EQ1} \\
= R_1 + j\omega L + 9.81\angle(-1.37) \\
= 100 + j(10^4)(10^{-2}) + 1.92 - j9.62 \\
= 101.92 + j90.38 \\
= 136.2\angle0.723\Omega
\]

**NB:** at this frequency (\( \omega \)) the circuit has an **inductive impedance** (reactance or phase is positive)

\[
Z_{EQ1} = 9.81\angle(-1.37)\Omega
\]