## Schedule...

<table>
<thead>
<tr>
<th>Date</th>
<th>Day</th>
<th>Class No.</th>
<th>Title</th>
<th>Chapters</th>
<th>HW Due date</th>
<th>Lab Due date</th>
<th>Exam</th>
</tr>
</thead>
<tbody>
<tr>
<td>27 Oct</td>
<td>Mon</td>
<td>16</td>
<td>Sinusoidal Frequency Response</td>
<td>6.1</td>
<td></td>
<td></td>
<td>LAB 5</td>
</tr>
<tr>
<td>28 Oct</td>
<td>Tue</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>29 Oct</td>
<td>Wed</td>
<td>17</td>
<td>Operational Amplifiers</td>
<td>8.1 – 8.2</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>30 Oct</td>
<td>Thu</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>31 Oct</td>
<td>Fri</td>
<td></td>
<td>Recitation</td>
<td></td>
<td></td>
<td></td>
<td>HW 7</td>
</tr>
<tr>
<td>1 Nov</td>
<td>Sat</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>2 Nov</td>
<td>Sun</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>3 Nov</td>
<td>Mon</td>
<td>18</td>
<td>Operational Amplifiers</td>
<td>8.3 – 8</td>
<td></td>
<td></td>
<td>LAB 6</td>
</tr>
<tr>
<td>4 Nov</td>
<td>Tue</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td>Exam 1</td>
</tr>
</tbody>
</table>
Easier to Maintain Than to Retake

**Alma 59:9**

9 And now as Moroni had supposed that there should be men sent to the city of Nephihah, to the assistance of the people to maintain that city, and knowing that it was easier to keep the city from falling into the hands of the Lamanites than to retake it from them, he supposed that they would easily maintain that city.
Lecture 16 – Frequency Response

Back to AC Circuits and Phasors

• Frequency Response
• Filters
Frequency Response

**Frequency Response $H(j\omega)$**: a measure of how the voltage/current/impedance of a load responds to the voltage/current of a source

\[
H_V(j\omega) = \frac{V_L(j\omega)}{V_S(j\omega)}
\]

\[
H_I(j\omega) = \frac{I_L(j\omega)}{I_S(j\omega)}
\]

\[
H_Z(j\omega) = \frac{V_L(j\omega)}{I_S(j\omega)}
\]
Frequency Response

\( V_L(j\omega) \) is a phase-shifted and amplitude-scaled version of \( V_S(j\omega) \)

\[
\frac{V_L(j\omega)}{V_S(j\omega)} = H_V(j\omega)
\]

\[
V_L(j\omega) = H_V(j\omega)V_S(j\omega)
\]
**Frequency Response**

**V_L(jω)** is a phase-shifted and amplitude-scaled version of **V_S(jω)**

\[ \frac{V_L(j\omega)}{V_S(j\omega)} = H_V(j\omega) \]

\[ V_L(j\omega) = H_V(j\omega)V_S(j\omega) \]

\[ V_L e^{j\phi_L} = |H_V| e^{j\angle H_V} |V_S| e^{j\angle V_S} \]

\[ V_L e^{j\phi_L} = |H_V| |V_S| e^{j(\angle H_V + \angle V_S)} \]

**Amplitude scaled**

**Phase shifted**

\[ V_L = |H_V| |V_S| \]

\[ \phi_L = \angle H_V + \angle V_S \]

**NB**: any complex number can be expressed as

\[ A = |A| e^{j\angle A} \]

\[ A = |A| \angle A \]
**Example 1**: compute the frequency response $H_V(j\omega)$

$R_1 = 1\,\text{k}\Omega$, $R_L = 10\,\text{k}\Omega$, $C = 10\,\text{\mu F}$

![Circuit Diagram]

$v_s(t)$ $\rightarrow C \rightarrow + \quad - \quad + \quad R_L \rightarrow$
**Example 1**: compute the frequency response \( H_V(j\omega) \)

\[
R_1 = 1k\Omega, \quad R_L = 10k\Omega, \quad C = 10\mu F
\]

1. Note frequencies of AC sources

Only one AC source so frequency response \( H_V(j\omega) \) will be the function of a single frequency
**Example 1**: compute the frequency response $H_v(j\omega)$

$R_1 = 1k\Omega$, $R_L = 10k\Omega$, $C = 10uF$

1. Note frequencies of AC sources
2. Convert to phasor domain
**Example 1**: compute the frequency response $H_V(j\omega)$

$R_1 = 1k\Omega$, $R_L = 10k\Omega$, $C = 10\mu F$

1. Note frequencies of AC sources
2. Convert to phasor domain
3. Solve using network analysis
   - Thévenin equivalent

$$Z_T = Z_1 \parallel Z_C$$
**Example 1**: compute the frequency response $H_V(j\omega)$

- $R_1 = 1\,\text{k}\Omega$, $R_L = 10\,\text{k}\Omega$, $C = 10\,\mu\text{F}$

1. Note frequencies of AC sources
2. Convert to phasor domain
3. Solve using network analysis
   - Thévenin equivalent

![Circuit diagram](image)

$V_T(j\omega) = V_S(j\omega) \frac{Z_C}{Z_1 + Z_C}$

$Z_T = Z_1 \parallel Z_C$
**Example 1**: compute the frequency response $H_V(j\omega)$

$R_1 = 1k\Omega$, $R_L = 10k\Omega$, $C = 10uF$

1. Note frequencies of AC sources
2. Convert to phasor domain
3. Solve using network analysis
   - Thévenin equivalent
4. Find an expression for the load voltage

$$V_L(j\omega) = V_T(j\omega) \frac{Z_{LD}}{Z_T + Z_{LD}}$$

$$Z_T = Z_1 \parallel Z_C$$

$$V_T(j\omega) = V_S(j\omega) \frac{Z_C}{Z_1 + Z_C}$$
**Example 1**: compute the frequency response $H_V(j\omega)$

$R_1 = 1\,k\Omega$, $R_L = 10\,k\Omega$, $C = 10\,\mu F$

5. Find an expression for the frequency response

$$H_V(j\omega) = \frac{V_L(j\omega)}{V_S(j\omega)} = \frac{Z_C}{Z_1 + Z_C} \cdot \frac{Z_{LD}}{(Z_1 \parallel Z_C) + Z_{LD}}$$
Example 1: compute the frequency response $H_V(j\omega)$

$R_1 = 1k\Omega$, $R_L = 10k\Omega$, $C = 10\mu F$

5. Find an expression for the frequency response

$$H_V(j\omega) = \frac{Z_C}{Z_1 + Z_C} \cdot \frac{Z_{LD}}{(Z_1 \parallel Z_C) + Z_{LD}}$$

$$= \frac{Z_{LD}Z_C}{Z_{LD}(Z_1 + Z_C) + Z_1Z_C}$$

$$= \frac{10^4 / (j\omega \times 10^{-5})}{10^4[10^3 + 1/(j\omega \times 10^{-5})] + 10^3 / (j\omega \times 10^{-5})}$$

$$= \frac{100}{110 + j\omega}$$

$$= \frac{100}{\sqrt{110^2 + \omega^2}} \angle - \arctan \left( \frac{\omega}{110} \right)$$
Example 1: compute the frequency response $H_V(j\omega)$

Let $R_1 = 1\, \text{k}\Omega$, $R_L = 10\, \text{k}\Omega$, $C = 10\, \text{uF}$

5. Find an expression for the frequency response
   • Look at response for low frequencies ($\omega = 10$) and high frequencies ($\omega = 10000$)

$$H_V(j\omega) = \frac{100}{\sqrt{110^2 + \omega^2}} \angle -\arctan\left(\frac{\omega}{110}\right)$$

For $\omega = 10$:

$$H_V(j10) = 0.905 \angle -0.0907$$

For $\omega = 10000$:

$$H_V(j10000) = 0.010 \angle -\frac{\pi}{2}$$
Example2: compute the frequency response $H_Z(j\omega)$

$R_1 = 1k\Omega$, $R_L = 4k\Omega$, $L = 2mH$
**Frequency Response**

**Example 2:** compute the frequency response $H_Z(j\omega)$

$R_1 = 1k\Omega$, $R_L = 4k\Omega$, $L = 2mH$

1. Note frequencies of AC sources

Only one AC source so frequency response $H_Z(j\omega)$ will be the function of a single frequency
**Frequency Response**

**Example 2**: compute the frequency response $H_Z(j\omega)$

$R_1 = 1k\Omega$, $R_L = 4k\Omega$, $L = 2mH$

1. Note frequencies of AC sources
2. Convert to phasor domain

\[ i_s(t) \quad \text{R}_1 \quad \text{R}_L \quad + \quad - \quad I_s(j\omega) \quad I_s(j\omega) \]

\[ Z_L = j\omega L \quad Z_1 = R_1 \quad V_L \quad Z_{LD} = R_L \]
**Example 2**: compute the frequency response $H_Z(j\omega)$

$R_1 = 1\,\text{k}\Omega$, $R_L = 4\,\text{k}\Omega$, $L = 2\,\text{mH}$

1. Note frequencies of AC sources
2. Convert to phasor domain
3. Solve using network analysis
   - Current divider & Ohm’s Law

$$I_L(j\omega) = I_s(j\omega) \frac{Z_1 \| (Z_L + Z_{LD})}{(Z_L + Z_{LD})}$$

$$V_L(j\omega) = I_L(j\omega)Z_{LD}$$

$$= \left( I_s(j\omega) \frac{Z_1 \| (Z_L + Z_{LD})}{(Z_L + Z_{LD})} \right) Z_{LD}$$
**Example 2:** compute the frequency response $H_Z(j\omega)$

$R_1 = 1k\Omega$, $R_L = 4k\Omega$, $L = 2mH$

4. Find an expression for the frequency response

$$H_Z(j\omega) = \frac{V_L(j\omega)}{I_S(j\omega)} = \frac{Z_1 \parallel (Z_L + Z_{LD})}{(Z_L + Z_{LD})} Z_{LD}$$

$$= \left. \frac{R_L}{1 + R_L / R_1 + j\omega L / R_1} \right|_{800} = \frac{800}{1 + j\omega 0.4 \times 10^{-6}}$$
**Example 2**: compute the frequency response \( H_Z(j\omega) \)

\[ R_1 = 1k\Omega, \quad R_L = 4k\Omega, \quad L = 2mH \]

4. Find an expression for the frequency response
   - Look at response for low frequencies (\( \omega = 10 \)) and high frequencies (\( \omega = 10000 \))

\[
H_Z(j\omega) = \frac{800}{1 + j\omega 0.4 \times 10^{-6}}
\]

\[
H_Z(j\omega) = 800 \angle -4.0
\]  \( \omega = 10 \)

\[
H_Z(j\omega) = 800 \angle -0.0040
\]  \( \omega = 10000 \)
1\textsuperscript{st} and 2\textsuperscript{nd} Order RLC Filters

Graphing in Frequency Domain
Filter Orders
Resonant Frequencies
Basic Filters
Graphing in the frequency domain: helpful in order to understand filters

\[ \cos(\omega_0 t) \]

Time domain

\[ \pi[\delta(\omega - \omega_0) + \delta(\omega - \omega_0)] \]

Frequency domain
Graphing in the frequency domain: helpful in order to understand filters

\[ X(j\omega) = \begin{cases} 
1, & |\omega| < W \\
0, & |\omega| > W 
\end{cases} \]

\[ x(t) = \text{sinc}(\omega_0 t) \]

Time domain

Frequency domain
Electromagnetic (Frequency) Spectrum
Basic Filters

**Electric circuit filter**: attenuates (reduces) or eliminates signals at unwanted frequencies

- **Low-pass**
- **High-pass**
- **Band-pass**
- **Band-stop**
Filter Orders

Higher filter orders provide a higher quality filter

![Diagram showing frequency response with gain in dB vs frequency for 1st, 2nd, 3rd, 4th, and 32nd order filters.](image)
Impedance

Impedance of resistors, inductors, and capacitors

\[ Z_L(j\omega) = \frac{1}{j\omega C} \]

\[ Z_C(j\omega) = j\omega L \]

\[ Z_R(j\omega) = R \]
Impedance of capacitors

\[ Z_C(j\omega) = \frac{1}{j\omega C} \]

\[ Z_C(j\omega) \to \frac{1}{0} = \infty \quad \text{as} \ \omega \to 0 \]

\[ Z_C(j\omega) \to \frac{1}{\infty} = 0 \quad \text{as} \ \omega \to \infty \]
Impedance of inductors

\[ Z_L(j\omega) = j\omega L \]

As \( \omega \to 0 \),
\[ Z_C(j\omega) \to 0 \]

As \( \omega \to \infty \),
\[ Z_C(j\omega) \to \infty \]
**Resonant Frequency**

**Resonant Frequency** \((\omega_n)\): the frequency at which capacitive impedance and inductive impedance are equal and opposite (in 2nd order filters)

\[
Z_L(j\omega) = -Z_C(j\omega)
\]

\[
j\omega L = -\frac{1}{j\omega C}
\]

\[
\omega_n = \frac{1}{\sqrt{LC}}
\]
**Resonant Frequency**

**Resonant Frequency** ($\omega_n$): the frequency at which capacitive impedance and inductive impedance are equal and opposite (in 2nd order filters)

**Impedances in series**

\[
Z_L = j\omega L \\
= j\left(\frac{1}{\sqrt{LC}}\right)L \\
= \frac{jL\sqrt{LC}}{LC} \\
= \frac{j\sqrt{LC}}{C}
\]

\[
Z_C = \frac{1}{j\omega C} \\
= \frac{1}{j\left(\frac{1}{\sqrt{LC}}\right)C} \\
= \frac{\sqrt{LC}}{jC} \\
= -\frac{j\sqrt{LC}}{C}
\]
**Resonant Frequency** ($\omega_n$): the frequency at which capacitive impedance and inductive impedance are equal and opposite (in 2nd order filters)

**Impedances in series**

\[ Z_L = \frac{j\sqrt{LC}}{C} \quad Z_C = -\frac{j\sqrt{LC}}{C} \]

\[ Z_{EQ} = Z_L + Z_C = 0 \]

\[ V_o(j\omega) = 0 \]
Resonant Frequency ($\omega_n$): the frequency at which capacitive impedance and inductive impedance are equal and opposite (in 2nd order filters)

Impedances in parallel

\[ Z_L = j\omega L \]
\[ = j \left( \frac{1}{\sqrt{LC}} \right) L \]
\[ = j \left( \frac{1}{\sqrt{LC}} \right) L \cdot \left( \frac{\sqrt{LC}}{\sqrt{LC}} \right) \]
\[ = jL \frac{\sqrt{LC}}{LC} \]
\[ = \frac{j\sqrt{LC}}{C} \]

\[ Z_C = \frac{1}{j\omega C} \]
\[ = \frac{1}{j} \left( \frac{1}{\sqrt{LC}} \right) C \]
\[ = \frac{\sqrt{LC}}{jC} \]
\[ = - \frac{j\sqrt{LC}}{C} \]
Resonant Frequency \( (\omega_n) \): the frequency at which capacitive impedance and inductive impedance are equal and opposite (in 2\textsuperscript{nd} order filters)

Impedances in parallel

\[ Z_L = \frac{j\sqrt{LC}}{C} \quad Z_C = -\frac{j\sqrt{LC}}{C} \]

\[ Z_{EQ} = \frac{Z_L Z_C}{Z_L + Z_C} = \frac{L/C}{\infty} = \frac{L/C}{0} = \infty \]
Low-Pass Filters

**Low-pass Filters**: only allow signals under the cutoff frequency \( \omega_0 \) to pass

\[
v_i(t) = \cos(\omega_1 t) + \cos(\omega_2 t) + \cos(\omega_3 t)
\]

\[
H_L(j\omega) \rightarrow v_o(t) = \cos(\omega_1 t)
\]

1st Order

\[
\begin{align*}
R & \quad + \\
C & \quad - \\
v_i(t) & \quad + \\
& \quad v_o(t) \\
\end{align*}
\]

2nd Order

\[
\begin{align*}
R & \quad + \\
L & \quad + \\
v_i(t) & \quad + \\
C & \quad - \\
& \quad v_o(t) \\
\end{align*}
\]
Low-Pass Filters

1st Order Low-pass Filters:

\[ Z_R(v_{o}(t) = v_{i}(t)) \]

\[ v_{o}(t) = v_{i}(t) \]

\[ Z_R \]

\[ Z_C = \frac{1}{j\omega C} \]

\[ \omega_1: \omega \to 0 \]

\[ \omega_3: \omega \to \infty \]

ECEN 301 Discussion #16 – Frequency Response
Low-Pass Filters

2nd Order Low-pass Filters:

\[ Z_R + V_i(j\omega) \]
\[ -V_o(j\omega) \]

\[ Z_L = j\omega L \]
\[ Z_C = 1/j\omega C \]

\[ V_i(j\omega) \]
\[ + \]
\[ - \]
\[ V_o(j\omega) \]

\[ \omega_1: \omega \to 0 \]
\[ \omega_2: \omega = \omega_n \]

Resonant frequency

\[ \omega_3: \omega \to \infty \]

\[ v_o(t) = v_i(t) \]
\[ v_o(t) = 0 \]

\[ v_o(t) = 0 \]
High-Pass Filters: only allow signals above the cutoff frequency ($\omega_0$) to pass

$$v_i(t) = \cos(\omega_1 t) + \cos(\omega_2 t) + \cos(\omega_3 t)$$

$$H_H(j\omega)$$

$$v_o(t) = \cos(\omega_3 t)$$
High-Pass Filters

1st Order High-pass Filters:

$$V_i(j\omega) \quad Z_R \quad V_o(j\omega)$$

$$\omega_1: \omega \to 0$$

$$\omega_3: \omega \to \infty$$

$$V_i(j\omega) \quad Z_C = 1/j\omega C \quad V_o(j\omega)$$

$$Z_R = R$$

$$v_o(t) = 0$$

$$v_o(t) = v_i(t)$$
**High-Pass Filters**

### 2nd Order High-pass Filters:

- **ω₁:** ω → 0
- **ω₂:** ω = ωₙ (Resonant frequency)
- **ω₃:** ω → ∞

\[
Z_C = \frac{1}{j\omega C} \quad Z_R
\]

\[
V_i(j\omega) \quad \Rightarrow \quad Z_R \quad \Rightarrow \quad V_o(j\omega)
\]

\[
v_o(t) = 0
\]
Band-Pass Filters

**Band-pass Filters**: only allow signals between the passband \((\omega_a \text{ to } \omega_b)\) to pass

\[ v_i(t) = \cos(\omega_1 t) + \cos(\omega_2 t) + \cos(\omega_3 t) \]

\[ H_B(j\omega) \]

\[ v_o(t) = \cos(\omega_2 t) \]
**Band-Pass Filters**

### 2nd Order Band-pass Filters:

**Band-pass**

**ω₁:** \( \omega \rightarrow 0 \)

**ω₂:** \( \omega = \omega_n \)  
**Resonant frequency**

**ω₃:** \( \omega \rightarrow \infty \)

\[
Z_R V_i(j\omega) \quad \rightarrow \quad \frac{1}{j\omega C} Z_L = j\omega L \quad \rightarrow \quad V_o(j\omega)
\]

\[v_o(t) = 0\]

\[v_o(t) = v_i(t)\]

\[v_o(t) = 0\]
**Band-Stop Filters**: allow signals except those between the **passband** \((\omega_a \text{ to } \omega_b)\) to pass

\[
v_i(t) = \cos(\omega_1 t) + \cos(\omega_2 t) + \cos(\omega_3 t)
\]

\[
H_N(j\omega)
\]

\[
v_o(t) = \cos(\omega_1 t) + \cos(\omega_3 t)
\]
2nd Order Band-stop Filters:

ω₁: ω → 0

ω₂: ω = ωₚ

Resonant frequency

ω₃: ω → ∞