**A.1. Use the first seven relations to prove relations (A.10), (A.13), and (A.16).**

Prove \( (F \cup G)^c = F^c \cap G^c \) (A.10).
\[ (F \cup G)^c = (F^c \cap G^c)^c \] by A.6.
\[ (F \cup G)^c = F^c \cap G^c \] by A.4

Prove \( F \cup (F \cap G) = F = F \cap (F \cup G) \) (A.13).
\[ F \cap (F \cup G) = (F \cap F) \cup (F \cap G) \] by A.3.
\[ F \cap (F \cup G) = F \cup (F \cap G) \] by A.20 (proved in book)

Now let's look at:
\[ F \subset F \cup X \therefore F \subset F \cup (F \cap G) \]
\[ F \cap X \subset F \therefore F \cap (F \cup G) \subset F \]

Because \( F \cap (F \cup G) = F \cup (F \cap G) \),
\[ F \subset F \cap (F \cup G) \text{ and } F \cap (F \cup G) \subset F \]
\[ \therefore F = F \cap (F \cup G) \]
\[ F \cup (F \cap G) = F = F \cap (F \cup G). \]

Prove \( F \cup G = F \cup (F^c \cap G) = F \cup (G - F) \) (A.16).
\[ F \cup G = (F \cup G) \cap \Omega \] by A.7.
\[ (F \cup G) \cap \Omega = (F \cup G) \cap (F \cup F^c) \] by A.10.
\[ (F \cup G) \cap (F \cup F^c) = F \cup (G \cap F^c) \] by A.17.
\[ \therefore F \cup (G \cap F^c) = F \cup (F^c \cap G) \] by A.8.
\[ G - F = G \cap F^c \]
\[ \therefore F \cup (G \cap F^c) = F \cup (G - F) \]
\[ \therefore F \cup G = F \cup (F^c \cap G) = F \cup (G - F) \]

**A.4 Show that \( F \subset G \) implies that \( F \cap G = F \), \( F \cup G = G \), and \( G^c \subset F^c \).**

\[ F \subset G \Rightarrow F \cap G = F \]
\[ F \subset G \text{ means that } \omega \in F \Rightarrow \omega \in G. \]
\[ \omega \in F \cap G \Leftrightarrow \omega \in F \text{ and } \omega \in G \]
\[ \Rightarrow F \cap G \subset F \text{ and } F \subset G \cap F \]
\[ \Rightarrow F \cap G = F \]

\[ F \subset G \Rightarrow F \cup G = G \]
\[ \omega \in F \cup G \Rightarrow \omega \in F \text{ or } \omega \in G \]
\[ \Rightarrow \omega \in G \text{ or } \omega \in G \text{ because } F \subset G. \]
\[ \Rightarrow \omega \in G \]
\[ \Rightarrow F \cup G \subset G. \]
\[ \omega \in G \Rightarrow \omega \in G \cap \Omega \]
\[ \Rightarrow \omega \in G \cap (F \cup F^c) \]
\[ \Rightarrow \omega \in (G \cap F) \cup (G \cap F^c) \]
\[ \Rightarrow \omega \in F \text{ or } \omega \in (G \cap F^c) \]
\[ \Rightarrow \omega \in F \text{ or } \omega \in F \]
\[ \Rightarrow \omega \in F \cup G \]
\[ \Rightarrow G \subset F \cup G \]
\[ \Rightarrow F \cup G = G \]

\[ F \subset G \Rightarrow G^c \subset F^c \]
\[ \omega \in G^c \Leftrightarrow \omega \notin G \]
\[ \Rightarrow \omega \notin F \]
\[ \Rightarrow \omega \in F^c \]
\[ \Rightarrow G^c \subset F^c \]
A.8 Prove the countably infinite version of deMorgan’s “laws.” For example, given a sequence of sets $F_i; i = 1, 2, \ldots$, then

$$\bigcap_{i=1}^{\infty} F_i = \left(\bigcup_{i=1}^{\infty} F^c_i\right)^c.$$  

To do this, we start by proving two subset relationships  

$\omega \in \bigcap_{i=1}^{\infty} F_i$  
$\Rightarrow \omega \in F_i$ for all $i,$  
$\Rightarrow \omega \notin F^c_i$ for any $i.$  
$\Rightarrow \omega \notin \bigcup_{i=1}^{\infty} F^c_i$  
$\Rightarrow \omega \in \left(\bigcup_{i=1}^{\infty} F^c_i\right)^c$  
$\therefore \bigcap_{i=1}^{\infty} F_i \subset \left(\bigcup_{i=1}^{\infty} F^c_i\right)^c.$  

Because these two are subsets of each other,  

$\bigcap_{i=1}^{\infty} F_i = \left(\bigcup_{i=1}^{\infty} F^c_i\right)^c.$

A.12 Show that inverse images preserve set theoretic operations, that is, given $f : \Omega \to A$ and sets $F$ and $G$ in $A,$ then

$$f^{-1}(F^c) = \left(f^{-1}(F)\right)^c,$$

$$f^{-1}(F \cup G) = f^{-1}(F) \cup f^{-1}(G),$$

and

$$f^{-1}(F \cap G) = f^{-1}(F) \cap f^{-1}(G).$$

If \{ $F_i, \ i \in I$ \} is an indexed family of subsets of $A$ that partitions $A,$ show that \{ $f^{-1}(F_i), \ i \in I$ \} is a partition of $\Omega.$ Do images preserve set theoretic operations in general? (Prove that they do or provide a counterexample).

For $f^{-1}(F^c) = \left(f^{-1}(F)\right)^c,$  
$\omega \in f^{-1}(F^c) \iff f(\omega) \in F^c \iff f(\omega) \notin F \iff \omega \notin f^{-1}(F) \iff \omega \in \left[f^{-1}(\omega)\right]^c$  
For $f^{-1}(F \cup G) = f^{-1}(F) \cup f^{-1}(G),$  
$\omega \in f^{-1}(F \cup G) \iff f(\omega) \in F \cup G \iff f(\omega) \in F \text{ or } f(\omega) \in G$  
$\iff \omega \in f^{-1}(F) \text{ or } \omega \in f^{-1}(G) \iff \omega \in f^{-1}(F) \cup f^{-1}(G)$  
For $f^{-1}(F \cap G) = f^{-1}(F) \cap f^{-1}(G),$  
$\omega \in f^{-1}(F \cap G) \iff f(\omega) \in F \cap G \iff f(\omega) \in F \text{ and } f(\omega) \in G$  
$\iff \omega \in f^{-1}(F) \text{ and } \omega \in f^{-1}(G) \iff \omega \in f^{-1}(F) \cap f^{-1}(G)$

Take \{ $F_i, \ i \in I$ \}. If it is an indexed family of subsets of $A$ that partitions $A,$ this means that  

$F_i \cap F_j = \emptyset; \text{ all } i, j \in I, \ i \neq j$  
and that  

$\bigcup_{i \in I} F_i = A.$  
We now need to show the same for the inverse image \{ $f^{-1}(F_i), \ i \in I$ \}.  
Our proofs above show that set theoretic operations are preserved for inverse images.  
$f^{-1}(F_i) \cap f^{-1}(F_j) = f^{-1}(\emptyset) = \emptyset; \text{ all } i, j \in I, \ i \neq j$  
We now need to show that $\bigcup_{i \in I} f^{-1}(F_i) = \Omega.$  
This is true because $\bigcup_{i \in I} f^{-1}(F_i) = f^{-1}(\bigcup_{i \in I} F_i) = f^{-1}(A) = \Omega.$
Images do not preserve set theoretic operations in general. This is particularly well-illustrated for the case of non one-to-one mappings.

Let $\Omega = \{a, b, c\}$, $A = \{d, e\}$ with $f (a) = f (b) = d$ and $f (c) = e$.

Let $F = \{a\}$, $F^c = \{b, c\}$.

$f (F) = \{d\}$

$f (F^c) = f (\{b, c\}) = \{d, e\} \neq [f (F)]^c = \{e\}$

2.3 Describe the sigma-field of subsets of $\mathbb{R}$ generated by the points or singleton sets. Does this sigma-field contain intervals of the form $(a, b)$ for $b > a$?

The sigma-field $\mathcal{S}$ generated by the points must have all countable unions of distinct points of the form $\cup_i \{a_i\}$ together with the complements of such sets of the form $(\cup_i \{a_i\})^c = \cap_i \{a_i\}^c$, which are intersections of the sample space minus an individual point. Since $\mathcal{S}$ is a field, it must contain simple unions of the form

$$F = \cup_i \{a_i\} \cup \cap_j \{b_j\}^c.$$

The sigma-field does not contain intervals since intervals do not have the form of $F$.

2.7 Let $\Omega = [0, \infty)$ be a sample space and let $\mathcal{F}$ be the sigma-field of subsets of $\Omega$ generated by all sets of the form $(n, n+1)$ for $n=0, 1, 2, ...$

(a) Are the following subsets of $\Omega$ in $\mathcal{F}$? (i) $[0, \infty)$, (ii) $\mathbb{Z}_+ = \{0, 1, 2, ...\}$, (iii) $[0, k] \cup [k+1, \infty)$ for any positive integer $k$, (iv) $\{k\}$ for any positive integer $k$, (v) $[0, k]$ for any positive integer $k$, (vi) $(1/3, 2)$.

(b) Define the following set function on subsets of $\Omega$:

$$P (F) = c \sum_{i \in \mathbb{Z}_+: i+1/2 \in F} 3^{-i}.$$ (If there is no $i$ for which $i+1/2 \in F$, then the sum is taken as zero.) Is $P$ a probability measure on $(\Omega, \mathcal{F})$ for an appropriate choice of $c$? If so, what is $c$?

(c) Repeat part (b) with $\mathcal{B}$, the Borel field, replacing $\mathcal{F}$ as the event space.

(d) Repeat part (b) with the power set of $[0, \infty)$ replacing $\mathcal{F}$ as the event space.

(e) Find $P (F)$ for the sets $F$ considered in part (a).

(a) i) Yes, because $\Omega$ is always in $\mathcal{F}$.

ii) Yes, because this is the set that is formed by the complement of all of the subsets $(n, n+1)$ for all $n=0, 1, 2, ...$. This can be written

$$\mathbb{Z}_+ = \left( \bigcup_{n=0}^{\infty} (n, n+1) \right)^c \in \mathcal{F}$$

iii) $[0, k] \cup [k+1, \infty) = (k, k+1)^c \in \mathcal{F}$

iv) $\{k\} \notin \mathcal{F}$ since the set cannot be constructed as a countable combination of set theoretic operations on generating sets.

v) $[0, k] \notin \mathcal{F}$ since the set cannot be constructed as a countable combination of set theoretic operations on generating sets.

vi) $(1/3, 2)$ since the set cannot be constructed as a countable combination of set theoretic operations on generating sets.

b) This is a suitable probability measure if $P(\Omega) = 1$. It also satisfies the properties of nonnegativity and countable additivity.

$$P(\Omega) = 1 = c \sum_{k=0}^{\infty} 3^{-i} = \frac{c}{1 - 1/3} = \frac{3c}{2}$$

This means that $c = 2/3$.

c) This is going to be the same as in part (b), so $P$ is a valid probability measure with $c = 2/3$.

d) This is going to be the same as in part (b) since $P$ was defined for all sets and is thus a probability measure on the power set.

e) i) $\Omega = [0, \infty)$ and thus $P(F) = 1$. 

ii) \( P(Z_+) = 0 \) as there are no \( i \) for which \( i + 1/2 \in Z_+ \).

iii) \( P(\{0, k\} \cup [k + 1, \infty)) = P((k, k + 1)^c) = 1 - P((k, k + 1)) = 1 - \frac{2}{3}(3^{-k}) \).

iv) \( P(\{k\}) = 0 \) as there are no \( k \) for which \( k + 1/2 \in Z_+ \).

v) \( P(\{0, k\}) = \sum_{i=0}^k 3^{-i} = 1 - 3^{-(k+1)} \)

vi) \( P((1/3, 2)) = c(3^{-0} + 3^{-1}) = \frac{2}{3}(1 + \frac{1}{3}) = \frac{8}{9} \).

2.9 Consider the measurable space \((0, 1), \mathcal{B}([0, 1])\). Define a set function \( P \) on this space as follows:

\[
P(F) = \begin{cases} 
1/2 & \text{if } 0 \in F \text{ or } 1 \in F \text{ but not both} \\
1 & \text{if } 0 \in F \text{ and } 1 \in F \\
0 & \text{otherwise}
\end{cases}
\]

Is \( P \) a probability measure?

Yes. \( P \) is a probability measure if it satisfies the three axioms for probability measures. It satisfies the property of nonnegativity and the property \( P(\Omega) = 1 \). We need to demonstrate countable additivity:

(a) \( 0 \notin F_i \) and \( 0 \notin F_j \) for all \( i \). Then \( P(\cup_i F_i) = 0 = \sum_i P(F_i) \).

(b) \( 0 \in F_i \) for some \( i \) and \( 0 \notin F_j \) for all \( i \), or \( 1 \in F_i \) for some \( i \) and \( 1 \notin F_j \) for all \( i \). Then \( P(\cup_i F_i) = 1/2 = \sum_i P(F_i) \).

(c) \( 0 \in F_i \) for some \( i \) and \( 0 \in F_j \) for some \( j \neq k \). Then \( P(\cup_i F_i) = 1 = \sum_i P(F_i) \).

(d) \( 0 \in F_i \) and \( 0 \notin F_k \) for some \( k \). Then \( P(\cup_i F_i) = 1 = \sum_i P(F_i) \).

Thus \( P \) is a probability measure.

2.10 Let \( S \) be a sphere in \( \mathbb{R}^3 \): \( S = \{(x, y, z) : x^2 + y^2 + z^2 \leq r^2\} \), where \( r \) is a fixed radius. In the sphere are fixed \( N \) molecules of gas, each molecule being considered as an infinitesimal volume (that is, it occupies only a point in space). Define for any subset \( F \) of \( S \) the function

\[
\#(F) = \text{the number of molecules in } F
\]

Show that \( P(F) = \#(F)/N \) is a probability measure on the measurable space consisting of \( S \) and its power set.

We need to demonstrate that this measure satisfies the three axioms for probability measures.

\( \#(S) = N \Rightarrow P(\Omega) = 1 \).

Normalization

Now we need to prove countable additivity.

For disjoint sets described by \( \{F_i : i = 0, 1, \ldots, k-1\} \), we can say that any particle in \( F_i \) is not in \( F_j \) for \( i \neq j \). Then \( \#(\bigcup_{i=0}^{k-1} F_i) = \sum_{i=0}^{k-1} \#(F_i) \) and this implies \( P(\bigcup_{i=0}^{k-1} F_i) = \sum_{i=0}^{k-1} P(F_i) \)

Suppose now that the disjoint sets are a countable collection \( \{F_i : i = 0, 1, \ldots\} \), let \( M \) be the largest integer \( i \) such that \( \#(F) > 0 \) (there must be such a finite integer since there are only \( N \) particles).

Then \( \#(\bigcup_{i=M+1}^{\infty} F_i) = 0 \) and

\[
\#(\bigcup_{i=M-1}^{\infty} F_i) = \#(\bigcup_{i=0}^M F_i) + \#(\bigcup_{i=M+1}^{\infty} F_i) \\
= \#(\bigcup_{i=0}^M F_i) \\
= \sum_{i=0}^M \#(F_i) = \sum_{i=0}^{\infty} \#(F_i)
\]

This implies that \( P(\bigcup_{i=0}^{\infty} F_i) = \sum_{i=0}^{\infty} P(F_i) \) and hence \( P \) is a probability measure.

2.16 Prove that \( P(F \cup G) \leq P(F) + P(G) \). Prove more generally that for any sequence (i.e., countable collection) of events \( F_i \),

\[
P\left(\bigcup_{i=1}^{\infty} F_i\right) \leq \sum_{i=1}^{\infty} P(F_i).
\]
This inequality is called the union bound or the Bonferroni inequality. (Hint: use Problem A.2 or Problem 2.1).

We know from 2.1 that in general, \( P(F \cup G) = P(F) + P(G) - P(F \cap G) \).

From non-negativity, we know that \( P(F \cap G) \geq 0 \) and thus \( P(F \cup G) \leq P(F) + P(G) \).

Let \( G_i = F_i - \bigcup_{j<i} F_j \) which makes these sets of \( G \) disjoint.

\[
P(\bigcup_i F_i) = P(\bigcup_i G_i) = \sum_i P(G_i) = \sum_i P(F_i - \bigcup_{j<i} F_j)\]

We know that \( P(F_i - \bigcup_{j<i} F_j) \leq P(F_i) \)

\[
\leq \sum_i P(F_i) \leq P(F) + P(G) \]

2.23 Answer true or false for each of the following statements. Answers must be justified.

(a) The following is a valid probability measure on the sample space \( \Omega = \{1, 2, 3, 4, 5, 6\} \) with event space \( F = \) all subsets of \( \Omega \):

\[
P(F) = \frac{1}{21} \sum_{i \in F} i; \text{ all } F \in F
\]

True.

To prove this, we have to show that the probability measure satisfies the different axioms.

\( P(F) \geq 0 \) so nonnegativity is satisfied.

\( P(\Omega) = 1 \) so normalization is satisfied.

Now countable additivity needs to be proved.

If \( F \) and \( G \) are disjoint, then \( P(F \cup G) = P(F) + P(G) \)

\[
P(F) = \frac{1}{21} \sum_{i \in F} i
\]

\[
P(G) = \frac{1}{21} \sum_{i \notin F} i
\]

\[
P(F \cup G) = \frac{1}{21} \sum_{i \in (F \cup G)} i = \frac{1}{21} \left( \sum_{i \in F} i + \sum_{i \notin F} i \right) = P(F) + P(G)
\]

(b) The following is a valid probability measure on the sample space \( \Omega = \{1, 2, 3, 4, 5, 6\} \) with event space \( F = \) all subsets of \( \Omega \):

\[
P(F) = \begin{cases} 
1 & \text{if } 2 \in F \text{ or } 6 \in F \\
0 & \text{otherwise}
\end{cases}
\]

False.

This is not a valid probability measure because countable additivity is not satisfied.

\( P(\{2\}) = 1 \).

\( P(\{6\}) = 1 \).

\( P(\{2, 6\}) = 1 \).

\( P(\{2, 6\}) \neq P(\{2\}) + P(\{6\}) \)

(c) If \( P(G \cup F) = P(F) + P(G) \), then \( F \) and \( G \) are independent.

False.

If \( F \) and \( G \) are independent, then \( P(F \cap G) = P(F)P(G) \)

By definition, \( P(G \cup F) = P(F) + P(G) - P(F \cap G) \).

Because \( P(F) > 0 \) and \( P(G) > 0 \), then if \( F \) and \( G \) are independent, \( P(F \cap G) > 0 \) and

\( P(G \cup F) = P(F) + P(G) \)
(d) \( P(F|G) \geq P(G) \) for all events \( F \) and \( G \).

False.

Just pick two disjoint events \( F \) and \( G \) with nonzero probability for \( P(G) \).

Then, \( P(F|G) = 0 \) and \( P(G) > 0 \).

(e) Mutually exclusive (disjoint) events with nonzero probability cannot be independent.

True.

Suppose that \( F \) and \( G \) have nonzero probability so that \( P(F)P(G) > 0 \). Since the events are disjoint, \( P(F \cap G) = 0 \) and thus \( P(F|G) = P(F \cap G)/P(G) = 0 \neq P(F) \). Thus, the events cannot be independent.

(f) For any finite collection of events \( F_i; i = 1, 2, \ldots, N \)

\[
P(\bigcup_{i=1}^{N} F_i) \leq \sum_{i=1}^{N} P(F_i)
\]

True.

Define \( G_n = F_n - \cup_{j<i} F_j \). Then \( G_n \subset F_n \) and the \( G_n \) are disjoint so that

\[
P(\bigcup_{i=1}^{N} F_i) = P(\bigcup_{i=1}^{N} G_i) = \sum_{i=1}^{N} P(G_i) \leq \sum_{i=1}^{N} P(F_i)
\]

2.26 Given a sample space \( \Omega = \{0, 1, 2, \ldots\} \) define

\[
p(k) = \frac{\gamma}{2^k}; \quad k = 0, 1, 2, \ldots
\]

(a) What must \( \gamma \) be in order for \( p(k) \) to be a pmf?

To be a valid pmf, it must be positive for all values of \( k \) and satisfy \( \sum_{k=0}^{\infty} p(k) = 1 \).

The infinite sum of a geometric progression with ratio \( a \), \( |a| < 1 \) is

\[
\sum_{k=0}^{\infty} a^k = \frac{1}{1-a}
\]

Thus, we can write:

\[
\sum_{k=0}^{\infty} \left(\frac{1}{2}\right)^k = \frac{1}{1-1/2} = 2 = \frac{1}{2}
\]

\( \gamma = \frac{1}{2} \) and our pmf is \( p(k) = \frac{1}{2^{k+1}} \).

(b) Find the probabilities \( P(\{0, 2, 4, 6, \ldots\}) \), \( P(\{1, 3, 5, 7, \ldots\}) \), and \( P(\{1, 2, 3, 4, \ldots, 20\}) \).

Even outcomes:

\[
P(\{0, 2, 4, 6, \ldots\}) = p(0) + p(2) + p(4) + \cdots = \sum_{i=0}^{\infty} p(2i) = \sum_{i=0}^{\infty} \frac{1}{2^{i+1}}
\]

\[
= \frac{1}{2} \sum_{i=0}^{\infty} \frac{1}{2^i} = \frac{1}{2} \sum_{i=0}^{\infty} \left(\frac{1}{2}\right)^i = \frac{1}{2} \cdot \frac{1}{1-1/2} = \frac{2}{3}
\]

Odd outcomes:

\[
P(\{1, 3, 5, 6, \ldots\}) = 1 - P(\{0, 2, 4, 5, \ldots\}) = 1 - \frac{2}{3} = \frac{1}{3}
\]

Finite outcomes:

Formula for finite sum of \( N+1 \) successive terms of geometric progression with ratio \( a \):

\[
\sum_{k=n}^{N+n} a^k = a^n \cdot \frac{1-a^{N+1}}{1-a}
\]

\[
P(\{1, 2, 3, 4, \ldots, 20\}) = \sum_{k=0}^{20} p(k) = \sum_{k=0}^{20} \frac{1}{2^{k+1}} = \frac{1}{2} \sum_{k=0}^{20} \left(\frac{1}{2}\right)^k = \frac{1}{2} \cdot \frac{1-(1/2)^{21}}{1-1/2} = 1 - \left(\frac{1}{2}\right)^{21} \approx
\]

\[
1 - 4.8 \times 10^{-7}
\]
(c) Suppose that $K$ is a fixed integer. Find $P(\{0, K, 2K, 3K, \ldots \})$.
This is very similar to computing the even outcomes case above:

$$P(\{0, K, 2K, 3K, \ldots \}) = p(0) + p(K) + p(2K) + \cdots = \sum_{i=0}^{\infty} p(Ki) = \sum_{i=0}^{\infty} \frac{1}{2^i} = \frac{1}{2^{i-1}}.$$ 

(d) Find the mean, second moment, and variance of this pmf.
We know from a geometric pmf that $p(k) = (1 - p)^{k-1} p; k=1, 2, \ldots$, where $p \in (0, 1)$ is a parameter that the mean is $1/p$ and the variance is $(1-p)/p^2$. Suppose that $p = 1/2$. This means that $p(1/2) = (1/2)(1/2)^{k-1} = (1/2)^k$. This is useful because this define the following sums:

$$m = \sum_{k=0}^{\infty} k \left(\frac{1}{2}\right)^k = 1/p = 2$$
$$\sigma^2 = \sum_{k=0}^{\infty} (k - m)^2 \left(\frac{1}{2}\right)^k = \frac{1-p}{p^2} = \frac{1}{1/4} = 2$$
$$m^{(2)} = \sigma^2 + m^2 = 2 + 4 = 6 = \sum_{k=0}^{\infty} k^2 \left(\frac{1}{2}\right)^k$$

The pmf of this problem can then be considered in relation to the geometric pmf

$$m = \sum_{k=0}^{\infty} kp(k) = \sum_{k=0}^{\infty} \frac{k}{2^k} = \sum_{k=0}^{\infty} \frac{1}{2} \cdot k \left(\frac{1}{2}\right)^k = \frac{1}{2} \cdot \sum_{k=0}^{\infty} k \left(\frac{1}{2}\right)^k = \frac{1}{2} \cdot 2 = 1$$
$$m^{(2)} = \sum_{k=0}^{\infty} k^2 p(k) = \sum_{k=0}^{\infty} k^2 \cdot \frac{1}{2^k} = \sum_{k=0}^{\infty} \frac{1}{2} \cdot k^2 \left(\frac{1}{2}\right)^k = \frac{1}{2} \cdot \sum_{k=0}^{\infty} k^2 \left(\frac{1}{2}\right)^k = \frac{1}{2} \cdot 6 = 3$$
$$\sigma^2 = m^{(2)} - m^2 = 3 - 1 = 2$$