Simple Interest

Interest is defined as a rate per unit time charged for the use of capital (the cost of borrowed money); it may also refer to the money so paid. In general it is common to consider interest as a predetermined amount or rate, much as occurs in the case of a bank loan.

The amount of interest earned or charged to a principle is called simple interest when the interest is found by the following formula:

\[ I = Pin \]

where, 
- \( I \) = total interest
- \( P \) = amount of principle
- \( i \) = interest rate per interest period, fraction
- \( n \) = number of interest periods

Consider the following example. Individual A agrees to loan B $1000 for a time period of 3 years. B agrees to pay A the $1000 at the end of the 3 years plus an amount of interest determined by applying a simple interest rate of 10% per year. The total interest charge will be:

\[ I = 1000 (0.10) (3) = 300 \]

Therefore, at the end of 3 years, B will pay a total of $1300 to A, the $1000 initially borrowed plus $300 interest for the use of A’s money.

Compound Interest

Simple interest concepts are used infrequently in today’s business dealings. Instead, a compound interest rate is used to compute the total interest charge. Compound interest is computed by applying the interest rate to the remaining unpaid principal plus any accumulated interest. In the example discussed above, suppose B agrees to pay an amount of interest found by applying a compound interest rate of 10% per year along with the $1000 at the end of 3 years. Now the total interest is found by the following:

\[ I_1 = 1000(0.10) = 100 \]
\[ I_2 = (1000 + 100) (0.10) = 110 \]
\[ I_3 = (1000 + 100 + 110) (0.10) = 121 \]
\[ I = I_1 + I_2 + I_3 = 331 \]

By applying compound interest principles, B will pay to A the $1000 initially borrowed plus $331 of interest for the use of the money. The total interest charge is $31 more when using the compound interest rate than when using the simple interest rate. The difference in the total amount of interest determined by compounding and that determined with a simple interest rate will become larger as the principal, interest rate, and number of interest periods increase.

**Interest Formulas for Discrete Compounding**

The following section contains the derivation and sample calculations of six interest formulas used in most interest calculations. The formulas are based on discrete compounding, i.e. the interest is compounded at the end of each finite interest period. Before deriving the formulas, it is necessary to define the following terms:

- \( P \) = present sum of money, the present is defined as any point from which the analyst wishes to measure time
- \( F \) = future sum of money
- \( A \) = end of interest period cash flow in a uniform series of equal payments
- \( i \) = compound interest rate per period
- \( n \) = number of compounding periods

The first formula to be derived allows the calculation of the equivalent amount of a present sum, \( P \), at some later time. Suppose \( P \) is placed in a bank account which draws \( i\% \) interest per period. It will grow to a future amount, \( F \) at the end of \( n \) interest periods, equal to:

\[ F = P(1 + i)^n \]  \hspace{1cm} (1)

The derivation of Equation 1 is given by the table on the next page.
<table>
<thead>
<tr>
<th>period</th>
<th>Amount of principal at beginning of period</th>
<th>Amount of interest earned during period</th>
<th>Total amount of principal and interest at end of period</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>P</td>
<td>Pi</td>
<td>P(1+i)\textsuperscript{1}</td>
</tr>
<tr>
<td>2</td>
<td>P(1+i)\textsuperscript{1}</td>
<td>P(1+i)\textsuperscript{1}i</td>
<td>P(1+i)\textsuperscript{2}</td>
</tr>
<tr>
<td>3</td>
<td>P(1+i)\textsuperscript{2}</td>
<td>P(1+i)\textsuperscript{2}i</td>
<td>P(1+i)\textsuperscript{3}</td>
</tr>
<tr>
<td>.</td>
<td>.</td>
<td>.</td>
<td>.</td>
</tr>
<tr>
<td>n</td>
<td>P(1+i)\textsuperscript{n-1}</td>
<td>P(1+i)\textsuperscript{n-1}i</td>
<td>P(1+i)\textsuperscript{n}</td>
</tr>
</tbody>
</table>

The factor \((1+i)^n\) is symbolized by \(F/P_{i,n}\).

If a future amount, \(F\), is known and it is desired to calculate the equivalent present sum, \(P\), then Equation 1 is simply rearranged and solved for \(P\).

\[
P = F(1+i)^{-n}
\] (2)

The term \((1+i)^{-n}\) is symbolized by \(P/F_{i,n}\).

**UNIFORM SERIES (ANNUITIES).** It is often necessary to know the amount of a uniform series payment, \(A\), which would be equivalent to a present sum, \(P\), or a future sum, \(F\). An example might be the calculation of a loan payment for a car or a house. The payment would be the \(A\) and the amount of the loan would be the present sum or \(P\). In the following formulas relating \(P\), \(F\), and \(A\), note that: 1) \(P\) occurs one interest period before the first \(A\); 2) \(A\) occurs at the end of each interest period; and 3) \(F\) occurs at the same time as the last \(A\).

The value of a future sum, \(F\), of a series of uniform payments, each worth \(A\), can be found by summing the future worth of each of the payments.

\[
F = A (1+i)^{n-1} + A (1+i)^{n-2} + ... + A (1+i)^{1} + A
\] (3)

Multiplying both sides of Equation 3 by \(1+i\) gives:

\[
F(1+i) = A (1+i)^{n} + A (1+i)^{n-1} + ... + A (1+i)^{2} + A (1+i)
\] (4)

 Subtracting Equation 3 from Equation 4 yields:
Solving for F yields the future sum of a uniform series of n payments, each having a value of A, with compound interest of i%, and is given by Equation 6:

\[ F = A \left[ \frac{(1+i)^n - 1}{i} \right] \]  

(6)

The term in the brackets in Equation 6 is symbolized by \( F/A_{i,n} \).

Rearranging Equation 6 and solving for A yields:

\[ A = F \left[ \frac{i}{(1+i)^n - 1} \right] \]  

(7)

The bracketed term is symbolized \( A/F_{i,n} \).

Substitution of Equation 6 into Equation 2 yields Equation 8 which contains the factor for calculating the present value of a series of annuity payments and is symbolized by \( P/A_{i,n} \).

\[ P = A \left[ \frac{(1+i)^n - 1}{i(1+i)^n} \right] \]  

(8)

Rearranging Equation 8 and solving for A gives:

\[ A = P \left[ \frac{i(1+i)^n}{(1+i)^n - 1} \right] \]  

(9)

The term in the brackets in Equation 9 is designated \( A/P_{i,n} \).

The six interest formulas derived above are summarized in Table 1 along with their factor name and symbol.
### TABLE 1. SUMMARY OF DISCRETE COMPOUNDING INTEREST FORMULAS

<table>
<thead>
<tr>
<th>Formula</th>
<th>Factor Name</th>
<th>Factor Symbol</th>
<th>Use</th>
</tr>
</thead>
<tbody>
<tr>
<td>$F = P(1+i)^n$</td>
<td>Single Payment</td>
<td>$F/P_{i,n}$</td>
<td>Find $F$</td>
</tr>
<tr>
<td></td>
<td>Compound Amount</td>
<td></td>
<td>given $P$</td>
</tr>
<tr>
<td>$P = F(1+i)^{-n}$</td>
<td>Single Payment</td>
<td>$P/F_{i,n}$</td>
<td>Find $P$</td>
</tr>
<tr>
<td></td>
<td>Present Worth</td>
<td></td>
<td>given $F$</td>
</tr>
<tr>
<td>$F = A[((1+i)^n - 1)/i]$</td>
<td>Uniform Series</td>
<td>$F/A_{i,n}$</td>
<td>Find $F$</td>
</tr>
<tr>
<td></td>
<td>Compound Amount</td>
<td></td>
<td>given $A$</td>
</tr>
<tr>
<td>$A = F[i/((1+i)^n -1)]$</td>
<td>Sinking Fund</td>
<td>$A/F_{i,n}$</td>
<td>Find $A$</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>given $F$</td>
</tr>
<tr>
<td>$P = A[((1+i)^n - 1)/(i(1+i)^n)]$</td>
<td>Uniform Series</td>
<td>$P/A_{i,n}$</td>
<td>Find $P$</td>
</tr>
<tr>
<td></td>
<td>Present Worth</td>
<td></td>
<td>given $A$</td>
</tr>
<tr>
<td>$A = P[(i(1+i)^n)/(1+i)^n - 1)]$</td>
<td>Capital Recovery</td>
<td>$A/P_{1,n}$</td>
<td>Find $A$</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>given $P$</td>
</tr>
</tbody>
</table>

### EXAMPLE PROBLEMS FOR DISCRETE COMPOUNDING

1. If $10,000 is invested in a fund earning 15\%$ compounded annually, what will it grow to in 10 years?

   $F = P(F/P_{i,n}) = 10,000 (F/P_{15,10}) = 10,000(1+0.15)^{10} = $40,456

2. It is desired to accumulate $5000 at the end of a 15 year period. What amount will need to be invested if the interest rate is 10\% compounded annually?

   $P = F(P/F_{i,n}) = 5000 (P/F_{10,15}) = 5000(1+0.10)^{-15} = $1197
3. An individual wishes to have $6000 available after 8 years. If the interest rate is 7% compounded annually, what uniform amount must be deposited at the end of each year?

\[ A = F(A/F_{i,n}) = 6000 \cdot (A/F_{0.07,8}) = 6000 \cdot \left[ \frac{0.07}{(1+0.07)^8 - 1} \right] = $585 \]

4. An individual wishes to place an amount of money in a savings account and then at the end of 1 month and every month there after for 30 months draw out $1000. What amount must be placed in the account if the interest rate is 1% compounded monthly?

\[ P = A(P/A_{i,n}) = 1000 \cdot (P/A_{0.01,30}) = 1000 \cdot \left[ \frac{(1+0.01)^{30} - 1}{0.01(1+0.01)^{30}} \right] = $25,808 \]

5. A principal of $50,000 is to be borrowed at an interest rate of 15% compounded monthly for 30 years. What will be the monthly payment to pay back the loan?

\[ i = \frac{0.15}{12} = 0.0125 \]

\[ A = P(A/P_{i,n}) = P(A/P_{0.0125,360}) = 50,000 \cdot \left[ \frac{0.0125(1+0.0125)^{360}}{(1+0.0125)^{360} - 1} \right] = $632 \]