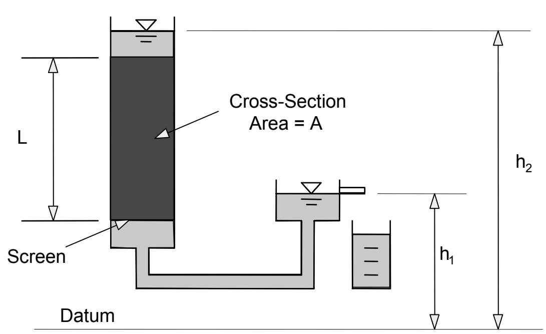
Darcy’s Law

Brigham Young University

CE En 547

Darcy’s Experiment

- Darcy did some lab tests on flow through sand in 1856



Darcy's Law

Darcy experimented with different soils and with different values of L, h_1, and h_2. The results showed that:

$$q = kA\frac{h_2 - h_1}{L}$$

where:

q = ΔVol/Δt = flow rate [L^3/T]
k = coefficient of permeability
 or hydraulic conductivity [L/T]
A = gross cross-sectional area of flow [L^2]
h = total head [L]
L = length of flow path [L]

Alternate Formulation

Darcy's law is often written as:

q = kiA

where

i = hydraulic gradient

$$i = \frac{h_2 - h_1}{L} = \frac{\Delta h}{L}$$

$$i = -\frac{dh}{ds}$$ where s is the flow path length

$$q = -k\frac{dh}{ds}A$$

Similar Governing Equations

- There are other governing equations that have this same form.
- Examples
  - Flow of electricity (Ohm's Law)
  - Heat flow

Ohm's Law

George Ohm (1789-1854) did a series of tests like Darcy did and came up with "Ohm's Law"

$$i = -K \frac{v_a - v_b}{L} A = -K \frac{dV}{dx} A$$

where

i = current
K = electrical conductivity
K = 1/ ρ where ρ = resistivity
V = voltage
L = distance
A = area

Sometimes written as:

$V = IR$

where

$R = \rho(L/A)$

Heat Flow

$$q = -K \frac{dT}{dx} A$$

where

q = rate of heat flow
K = thermal conductivity
T = temperature
x = distance
A = cross-sectional area

Applicability

- Darcy's law is only valid for laminar flow.
- Upper bound for laminar flow is based on Reynold's number
  - Re > ~1

Reynold's Number

$$R_e = \frac{vD\rho}{\mu}$$

where

- v = discharge velocity [ft/s]
- D = mean particle size [ft]
- ρ = mass density of water [1.94 $lb\text{-}sec^2/ft^4$]
- μ = dynamic viscosity of water [2.35×10^{-5} $lb\text{-}sec/ft^2$]

Velocity

Darcy's law can be rewritten as:

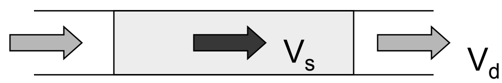
$$q = kiA$$

$$\frac{q}{A} = ki\frac{A}{A}$$

$$v_d = ki$$

Darcian Velocity

v_d = "discharge velocity" or "Darcian velocity"



v_s = "seepage velocity"

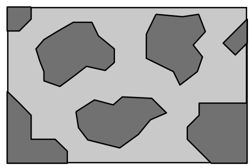
Seepage Velocity

Seepage velocity can be related to discharge velocity as follows:

$$\frac{\text{Area of flow}}{\text{Total area}} = \frac{A_f}{A} = \frac{V_v}{V} = n$$

V_v = Volume of soil voids
V = Total volume
n = porosity

Seepage Velocity, cont.



$$n = \frac{\square\ \text{area}}{\text{total area}}$$

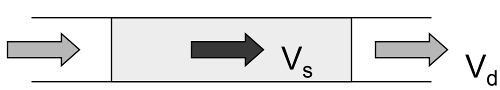
$$v_s = \frac{q}{A_f} = \frac{q}{A}\frac{A}{A_f} = v_d\frac{1}{n}$$

$$v_s = \frac{v_d}{n} = \frac{ki}{n}$$

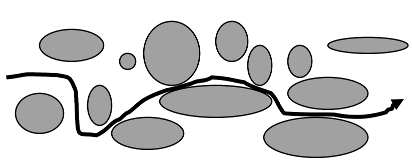
n	v_s/v_d
0.33	3
0.5	2
0.67	1.5
0.75	1.3

Tortuosity

Note: v_s is average linear velocity



Actual velocity is higher because the water follows a longer path through the soil particles.

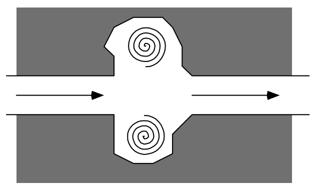


“tortuous path”

"Tortuosity" is a measure of the distance traveled vs. linear distance.

Effective Porosity

Not all voids in the soil conduct flow.



n_e = "effective" porosity

$$= \frac{A_f}{A}$$

where A_f = actual area of flow

$$v_s = \frac{ki}{n_e}$$

Effective Porosity, cont.

$n_e = \lambda n$

λ = effective porosity factor
λ is determined experimentally
$\lambda \approx 1$ for sands and gravels
$\lambda = 0.01 - 0.5$ for clays

Permeability

In the equation for Darcy's law:

$$q = kiA$$

k is a constant which depends on both the fluid and the soil.
Darcy's law can be rewritten as:

$$q = -K \frac{\gamma_f}{\mu} \frac{dh}{dx} A$$

where:

K = "constant of proportionality" [L^2]
or "intrinsic permeability"
or "permeability"

K is based on soil properties only

γ_f = unit weight of fluid [F/L^3]

μ = viscosity of fluid [ML/T]

Permeability, cont.

Can also be written as:

$$q = \frac{K}{\mu}\left[\rho g \frac{dh}{dx}\right] A$$

This form is used a lot in the petroleum industry.

K is often expressed in units of Darcys

1 Darcy = the permeability for a flow of 1 $cm^3/sec/cm^2$ for a viscosity of 1 centipoise and [$\rho g(dh/dx)$] = 1 atm/cm.

1 poise = 1 dyne-sec/cm^2
1 Darcy = 0.987×10^{-8} cm^2

For water at 20° C:

$$K = \frac{k\mu_w}{\gamma_w}$$

μ_w = 0.01002 poise
$\gamma_w = \rho_w g$
ρ_w = 1 g/cm^3
g = 980.7 cm/sec^2

for k = 1 cm/s: K = 1.02×10^{-5} cm^2

See handout for tables of typical values of k & K
We will use k (hydraulic conductivity) in this class.

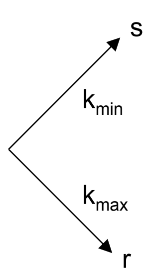
Darcy's Law in 2D

- Isotropic medium

$$v_x = -k \frac{\partial h}{\partial x}$$

$$v_y = -k \frac{\partial h}{\partial y}$$

Anisotropic Medium



$$\begin{bmatrix} v_r \\ v_s \end{bmatrix} = -\begin{bmatrix} k_{max} & 0 \\ 0 & k_{min} \end{bmatrix} \begin{bmatrix} \partial h / \partial r \\ \partial h / \partial s \end{bmatrix}$$

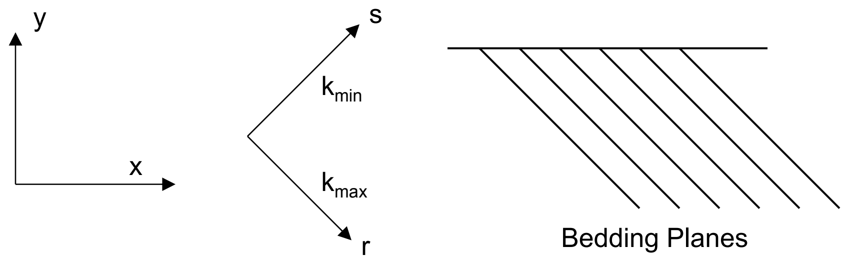
$$\begin{bmatrix} k_{max} & 0 \\ 0 & k_{min} \end{bmatrix}$$ = "conductivity matrix" or "conductivity tensor"

$$v_r = -k_{max} \frac{\partial h}{\partial r}$$

$$v_s = -k_{min} \frac{\partial h}{\partial s}$$

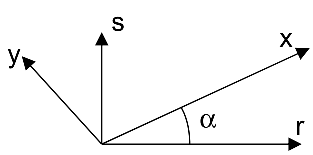
r & s may or may not coincide with x & y

Example



You often need k in terms of x & y (k_x, k_y).

Coordinate Transformation



$$v_x = -k_{xx}\frac{\partial h}{\partial x} - k_{xy}\frac{\partial h}{\partial y}$$

$$\begin{bmatrix} v_x \\ v_y \end{bmatrix} = -\begin{bmatrix} k_{xx} & k_{xy} \\ k_{yx} & k_{yy} \end{bmatrix}\begin{bmatrix} \partial h/\partial x \\ \partial h/\partial y \end{bmatrix}$$

$$v_y = -k_{yx}\frac{\partial h}{\partial x} - k_{yy}\frac{\partial h}{\partial y}$$

Coord. Transformation, cont.

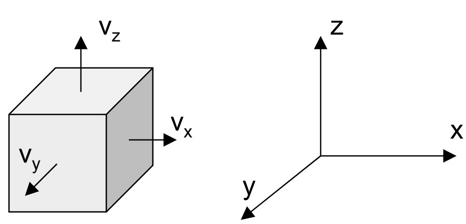
Solving in terms of k_r and k_s:

$$k_{xx} = k_r \cos^2\alpha + k_s \sin^2\alpha$$

$$k_{yy} = k_r \sin^2\alpha + k_s \cos^2\alpha$$

$$k_{xy} = k_{yx} = -\frac{1}{2}(k_r - k_s)\sin^2\alpha$$

Darcy's Law in 3D



If x, y, and z axes coincide with principal axes of permeability:

$$\begin{bmatrix} v_x \\ v_y \\ v_z \end{bmatrix} = \begin{bmatrix} k_{xx} & 0 & 0 \\ 0 & k_{yy} & 0 \\ 0 & 0 & k_{zz} \end{bmatrix} \begin{bmatrix} \partial h / \partial x \\ \partial h / \partial y \\ \partial h / \partial z \end{bmatrix}$$

In most cases
$k_{xx} = k_{yy}$
$k_{xx} > k_{zz}$

Darcy's Law in 3D, cont.

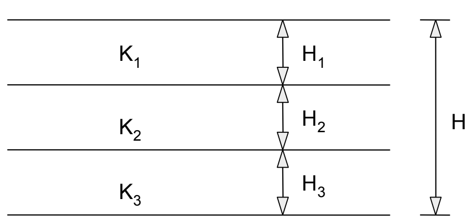
General case:

$$\begin{bmatrix} v_x \\ v_y \\ v_z \end{bmatrix} = \begin{bmatrix} k_{xx} & k_{xy} & k_{xz} \\ k_{yx} & k_{yy} & k_{yz} \\ k_{zx} & k_{zy} & k_{zz} \end{bmatrix} \begin{bmatrix} \partial h / \partial x \\ \partial h / \partial y \\ \partial h / \partial z \end{bmatrix}$$

The tensor is symmetric, i.e.:

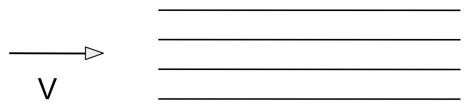
$k_{yx} = k_{xy}$
$k_{zx} = k_{xz}$
$k_{zy} = k_{yz}$

Layered Systems



It is useful to compute an equivalent k for entire system

Flow Parallel to Layering



For this case, the following must hold:

1. Head loss is the same through each layer

   $i_1 = i_2 = i_3$

2. $q_{total} = q_1 + q_2 + q_3$

Parallel Flow, cont.

Thus,

$q = q_1 + q_2 + q_3$

$vH = v_1H_1 + v_2H_2 + v_3H_3$ (width = 1)

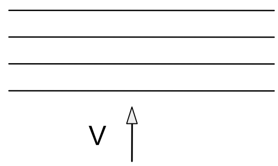
$k_{eq}i_{eq}H = k_1i_1H_1 + k_2i_2H_2 + k_3i_3H_3$

since i's are the same, divide by i:

$k_{eq}H = k_1H_1 + k_2H_2 + k_3H_3$

$$k_{eq} = \frac{k_1H_1 + k_2H_2 + k_3H_3}{H} \qquad k_{eq} = \frac{\sum_{i=1}^{n} k_iH_i}{\sum_{i=1}^{n} H_i}$$

Flow Perpendicular to Layering



For this case, the following must hold:

1. q will be the same through each layer

2. $\Delta h_{total} = \Delta h_1 + \Delta h_2 + \Delta h_3$

Perpendicular Flow, cont.

$q = q_1 = q_2 = q_3$

$v = v_1 = v_2 = v_3$

$k_{eq} i_{eq} = k_1 i_1 = k_2 i_2 = k_3 i_3$

$k_{eq}(\Delta h/H) = k_1 i_1 = k_2 i_2 = k_3 i_3$

$\Delta h = \Delta h_1 + \Delta h_2 + \Delta h_3$

$\Delta h = H_1 i_1 + H_2 i_2 + H_3 i_3$

Perpendicular Flow, cont.

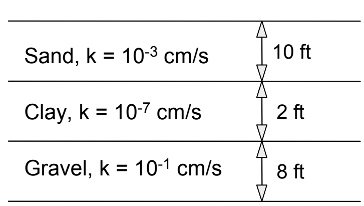
$$i_1 = \frac{k_{eq}\Delta h}{Hk_1} \qquad i_2 = \frac{k_{eq}\Delta h}{Hk_2} \qquad i_3 = \frac{k_{eq}\Delta h}{Hk_3}$$

$$\Delta h = \frac{H_1 k_{eq}\Delta h}{Hk_1} + \frac{H_2 k_{eq}\Delta h}{Hk_2} + \frac{H_3 k_{eq}\Delta h}{Hk_3}$$

Divide by Δh and solve for k_{eq}:

$$k_{eq} = \frac{H}{\frac{H_1}{k_1} + \frac{H_2}{k_2} + \frac{H_3}{k_3}} \qquad k_{eq} = \frac{\sum_{i=1}^{n} H_i}{\sum_{i=1}^{n} \frac{H_i}{k_i}}$$

Example



(not to scale)

k_{eq} (horizontal) = ?

k_{eq} (vertical) = ?