

## Chapter 2

# Electrostatics

### 2.1 Vectors

The laws of electromagnetics were originally formulated using a system of many partial differential equations. Today, we use a more compact notation that is much more convenient. But in order to be able to use the notation, one must first understand it.

The fundamental definition of electric field is in terms of force on a test charge. That force has a magnitude and direction. We represent the ratio of that force to the strength of the charge mathematically as a vector:

$$\vec{E} = E_x \hat{x} + E_y \hat{y} + E_z \hat{z} \quad (2.1)$$

where  $\hat{x}$  is a unit vector of length one in the  $+x$  direction and the other two unit vectors are defined similarly.  $E_x$ ,  $E_y$ , and  $E_z$  are real or complex numbers called the components of  $\vec{E}$ . We can also express the components of the vector using other sets of linearly independent unit vectors. The magnitude of the vector is given by the same character without an accent:

$$E = \|\vec{E}\| = \sqrt{|E_x|^2 + |E_y|^2 + |E_z|^2} \quad (2.2)$$

**Vector field.** A vector field assigns a vector to each point in space, so the components of the vector are functions of position:

$$\vec{E}(x, y, z) = E_x(x, y, z) \hat{x} + E_y(x, y, z) \hat{y} + E_z(x, y, z) \hat{z} \quad (2.3)$$

The components may also depend on other independent variables such as time or frequency.

#### Examples

Scalar fields: temperature  $T(x, y, z)$ , pressure  $p(x, y, z)$ , electric potential  $V(x, y, z)$ .

Vector fields: wind velocity  $\vec{v}(x, y, z)$ , electric field intensity  $\vec{E}(x, y, z)$ , magnetic field intensity  $\vec{H}(x, y, z)$ .

**Unit vectors.** Unit vectors have length one, so that  $\|\hat{x}\| = \|\hat{y}\| = \|\hat{z}\| = 1$ . We can also come up with a unit vector in the direction of an arbitrary vector  $\overline{A}$  using

$$\hat{a} = \frac{\overline{A}}{\|\overline{A}\|} \quad (2.4)$$

**Position vector.** The position vector is defined by

$$\overline{r} = x\hat{x} + y\hat{y} + z\hat{z} \quad (2.5)$$

This is not really a vector, but is merely a compact way to represent the point  $(x, y, z)$ .

**Dot product.** Two vectors can be combined to form a scalar:

$$\overline{A} \cdot \overline{B} = A_x B_x + A_y B_y + A_z B_z \quad (2.6)$$

$$= AB \cos \psi \quad (2.7)$$

where  $\psi$  is the angle between the two vectors. If  $\overline{A} \cdot \overline{B} = 0$ , the vectors are orthogonal. Also,  $\overline{A} \cdot \overline{A} = \|\overline{A}\|^2$ .

**Cross product.** Two vectors can also be combined to form another vector:

$$\overline{A} \times \overline{B} = \hat{n} AB \sin \psi \quad (2.8)$$

where  $\hat{n}$  is a unit vector in the direction given by the right hand rule applied to the vectors  $\overline{A}$  and  $\overline{B}$  and  $\psi$  is the angle between the vectors. If we switch the order of  $\overline{A}$  and  $\overline{B}$ , the cross product changes sign. For the rectangular unit vectors,

$$\begin{aligned} \hat{x} \times \hat{x} &= 0, & \hat{x} \times \hat{y} &= \hat{z}, & \hat{x} \times \hat{z} &= -\hat{y} \\ \hat{y} \times \hat{x} &= -\hat{z}, & \hat{y} \times \hat{y} &= 0, & \hat{y} \times \hat{z} &= \hat{x} \\ \hat{z} \times \hat{x} &= \hat{y}, & \hat{z} \times \hat{y} &= -\hat{x}, & \hat{z} \times \hat{z} &= 0 \end{aligned} \quad (2.9)$$

These relationships can be used to express the cross product using components as

$$\overline{A} \times \overline{B} = (A_y B_z - A_z B_y)\hat{x} + (A_z B_x - A_x B_z)\hat{y} + (A_x B_y - A_y B_x)\hat{z} \quad (2.10)$$

Another handy rule for computing the cross product is

$$\overline{A} \times \overline{B} = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ A_x & A_y & A_z \\ B_x & B_y & B_z \end{vmatrix} \quad (2.11)$$

where the vertical bars denote the matrix determinant operation. An identity that connects the dot and cross products is

$$\overline{A} \cdot (\overline{B} \times \overline{C}) = \overline{B} \cdot (\overline{C} \times \overline{A}) = \overline{C} \cdot (\overline{A} \times \overline{B}) \quad (2.12)$$

## Integrals

Vector fields can be integrated over paths and surfaces. Paths and surfaces can be represented using parameterizations. A path is defined by three functions such that the point

$$\bar{r}(t) = (f(t), g(t), h(t)), \quad a \leq t \leq b \quad (2.13)$$

traces out the path as the parameter  $t$  ranges from  $a$  to  $b$ . A surface is parameterized by functions of two parameters:

$$\bar{r}(s, t) = (f(s, t), g(s, t), h(s, t)), \quad a \leq s \leq b, c \leq t \leq d \quad (2.14)$$

**Path integrals.** A path integral of a vector field is written as

$$\int_P \bar{A} \cdot d\bar{\ell} \quad (2.15)$$

where  $P$  represents a path and  $d\bar{\ell}$  is a vector tangent to the path with a differential length. Using a parameterization to change the integration variable from a point in the  $x, y, z$  plane to the parameter of the path, a path integral can be transformed into a standard scalar integral:

$$\begin{aligned} \int_P \bar{A} \cdot d\bar{\ell} &= \int_a^b [A_x \hat{x} + A_y \hat{y} + A_z \hat{z}]_{(x,y,z)=(f(t),g(t),h(t))} \cdot d[f(t)\hat{x} + g(t)\hat{y} + h(t)\hat{z}] \\ &= \int_a^b [A_x(t)f'(t) + A_y(t)g'(t) + A_z(t)h'(t)] dt \end{aligned} \quad (2.16)$$

Example: To integrate  $\bar{A} = x\hat{x} + y\hat{y}$  over a straight line from  $(0, 0)$  to  $(1, 0)$ , we parameterize the path using  $(x, y) = (t, 0)$ ,  $0 \leq t \leq 1$ . The integral is

$$\begin{aligned} \int_P \bar{A} \cdot d\bar{\ell} &= \int_0^1 (t\hat{x} + 0\hat{y}) \cdot \hat{x} dt \\ &= \int_0^1 t dt \\ &= \frac{1}{2} \end{aligned}$$

**Surface integrals.** A surface integral is written as

$$\int_S \bar{A} \cdot d\bar{S} \quad (2.17)$$

where  $S$  represents a surface and  $d\bar{S}$  is a normal differential area element vector. A surface integral can be evaluated using a parameterization as with a path integral. In many cases, however, the integration surface is simple enough that we can write down  $d\bar{S}$  by inspection using

$$d\bar{S} = \hat{n} dS \quad (2.18)$$

where  $\hat{n}$  is a unit vector normal to the surface and  $dS$  is a differential area element. If a vector field represents flow, then the surface integral represents the total amount of flow through the surface.

Example: We want to integrate  $\bar{A} = 3(z + 1)\hat{z}$  over a square with corners  $(0, 0, 0)$ ,  $(1, 0, 0)$ ,  $(1, 1, 0)$ , and  $(0, 1, 0)$ . Because the surface is confined to the  $z = 0$  plane, the differential area element is  $dx dy$  and the surface normal vector is  $\hat{z}$ . The integral is

$$\begin{aligned}\int_S \bar{A} \cdot d\bar{S} &= \int_0^1 \int_0^1 3\hat{z} \cdot \hat{z} dx dy \\ &= \int_0^1 \int_0^1 3 dx dy \\ &= 3\end{aligned}$$

## Derivatives

Because vector fields can change with position, we can measure the amount of change using derivatives of vector fields, much like a scalar derivative gives the rate of change of a scalar function. Vector derivatives are derived in terms of the operator

$$\nabla = \hat{x} \frac{\partial}{\partial x} + \hat{y} \frac{\partial}{\partial y} + \hat{z} \frac{\partial}{\partial z} \quad (2.19)$$

With this operator, four main derivatives can be defined:

**Gradient.** The gradient operation transforms a scalar to a vector:

$$\nabla V(x, y, z) = \hat{x} \frac{\partial V(x, y, z)}{\partial x} + \hat{y} \frac{\partial V(x, y, z)}{\partial y} + \hat{z} \frac{\partial V(x, y, z)}{\partial z} \quad (2.20)$$

This vector points in the direction of most rapid increase of the function  $V(x, y, z)$ .

Example: If  $f(x, y) = x^2 + y^2$ , then  $\nabla f = 2x\hat{x} + 2y\hat{y}$ .

**Curl.** The curl operation transforms a vector to a vector:

$$\nabla \times \bar{A} = \hat{x} \left( \frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z} \right) + \hat{y} \left( \frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x} \right) + \hat{z} \left( \frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y} \right) \quad (2.21)$$

This vector gives the amount of rotation of the vector field  $\bar{A}$ .

Example:  $\nabla \times (y\hat{x} - x\hat{y}) = \hat{z}(-1 - 1) = -2\hat{z}$ .

**Gradient.** The gradient operation transforms a vector to a scalar:

$$\nabla \cdot \bar{A} = \frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z} \quad (2.22)$$

The value of this scalar is positive at sources of the vector field and negative at sinks.

Example: If  $\nabla \cdot (x\hat{x} + y\hat{y}) = 1 + 1 = 2$ .

**Laplacian.** The Laplacian transforms a scalar to a scalar or a vector to a vector:

$$\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} \quad (2.23)$$

When this operator acts on a vector field, it does not mix components:  $\nabla^2 \bar{A} = (\nabla^2 A_x)\hat{x} + (\nabla^2 A_y)\hat{y} + (\nabla^2 A_z)\hat{z}$ .

Some important identities are

$$\nabla \times (\nabla V) = 0 \quad (2.24)$$

$$\nabla \cdot (\nabla \times \bar{A}) = 0 \quad (2.25)$$

$$\nabla \cdot (\nabla V) = \nabla^2 V \quad (2.26)$$

$$-\nabla \times (\nabla \times \bar{A}) + \nabla(\nabla \cdot \bar{A}) = \nabla^2 \bar{A} \quad (2.27)$$