Chapter 1

Magnetostatics

In electrostatics, we considered the electric field and flux density arising from charges. In magnetostatics, we are concerned with magnetic fields and flux density arising from currents. The basic equations of interest are Ampere's Law and Gauss' Law for magnetostatics:

Point or Differential Form:

$$\nabla \cdot \overline{B} = 0 \tag{1.1}$$

$$\nabla \times \overline{H} = \overline{J} \tag{1.2}$$

Integral Form:

$$\oint_S \overline{B} \cdot d\overline{s} = 0 \tag{1.3}$$

$$\oint_C \overline{H} \cdot d\overline{\ell} = \int_S \overline{J} \cdot d\overline{s} \tag{1.4}$$

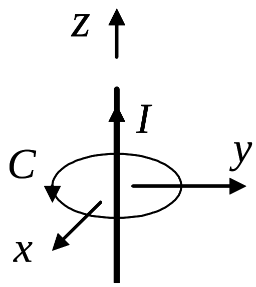
We will begin by using this integral form of Ampere's Law to determine magnetic fields much like we used Gauss' Law to determine electric fields in the prior section.

1.1 Ampere's Law

Suppose we are given a static current distribution, which means we have charges that move, but that motion is constant in time. We can use Ampere's Law to determine the electric field that emanates from the current. Let's try a few examples:

1.1.1 Line Current

Suppose we have a current of I amperes moving through a thin wire. The current flows in the $\hat{z}$ direction. Recall that we use the right hand rule which means that we put the thumb of our right hand in the direction of the current and our fingers will go in the direction of the magnetic field. The magnetic field will loop around the wire.



$$\oint_C \overline{H} \cdot d\overline{\ell} = \int_S \overline{J} \cdot d\overline{s} = I \tag{1.5}$$

$$\int_0^{2\pi} H_o \hat{\phi} \cdot \hat{\phi} r d\phi = I \tag{1.6}$$

$$2\pi H_o r = I \tag{1.7}$$

$$H_o = \frac{I}{2\pi r} \tag{1.8}$$

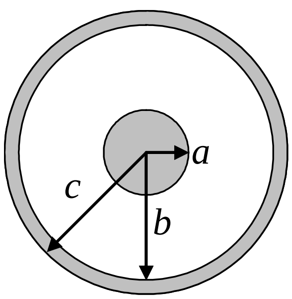
$$\overline{H} = \frac{I}{2\pi r}\hat{\phi} \tag{1.9}$$

The magnetic flux density is then

$$\overline{B} = \mu \overline{H} = \frac{\mu I}{2\pi r}\hat{\phi} \tag{1.10}$$

1.1.2 Coaxial Cable

On the conductors, the current density can be written as



$$\overline{J} = \begin{cases} \frac{I}{\pi a^2}\hat{z} & \text{inner conductor} \\ \frac{-I}{\pi(c^2-b^2)}\hat{z} & \text{outer conductor} \end{cases} \tag{1.11}$$

We can compute the magnetic field in 4 regions:

$r < a$:

$$\int_0^{2\pi} H_o \hat{\phi} \cdot \hat{\phi} r d\phi = \int_0^{2\pi}\int_0^r \frac{I}{\pi a^2}\hat{z} \cdot \hat{z} r_o dr_o d\phi \tag{1.12}$$

$$2\pi r H_o = \frac{I}{\pi a^2} 2\pi \frac{r^2}{2} = I\frac{r^2}{a^2} \tag{1.13}$$

$$H_o = \frac{Ir}{2\pi a^2} \tag{1.14}$$

$$\overline{H} = \frac{Ir}{2\pi a^2}\hat{\phi} \tag{1.15}$$

$a < r < b$: Note that the integral on the left hand side is the same for all regions.

$$2\pi r H_o = \int_0^{2\pi}\int_0^a \frac{I}{\pi a^2}\hat{z} \cdot \hat{z} r dr d\phi = \frac{I}{\pi a^2} 2\pi \frac{a^2}{2} = I \tag{1.16}$$

$$H_o = \frac{I}{2\pi r} \tag{1.17}$$

$$\overline{H} = \frac{I}{2\pi r}\hat{\phi} \tag{1.18}$$

$b < r < c$:

$$
\begin{aligned}
2\pi r H_o &= I - \int_0^{2\pi}\int_b^r \frac{I}{\pi(c^2-b^2)} r_o dr_o d\phi && (1.19)\\
&= I - \frac{I}{\pi(c^2-b^2)} 2\pi \left(\frac{r^2}{2} - \frac{b^2}{2}\right) && (1.20)\\
&= I - I\left(\frac{r^2-b^2}{c^2-b^2}\right) && (1.21)\\
&= I\left[\frac{c^2-b^2-r^2+b^2}{c^2-b^2}\right] && (1.22)\\
&= I\left(\frac{c^2-r^2}{c^2-b^2}\right) && (1.23)\\
\overline{H} &= \frac{I}{2\pi r}\left(\frac{c^2-r^2}{c^2-b^2}\right)\hat{\phi} && (1.24)
\end{aligned}
$$

$r > c$: Note that on the right hand side of (1.23), we simply replace r with the outer limit of integration c (the left hand side remains the same). Therefore, the right hand side will be zero, so

$$\overline{H} = 0 \quad (1.25)$$

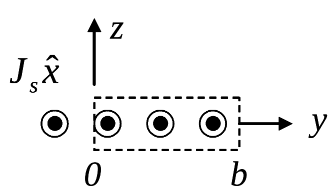
This makes sense, since there is no *net* current enclosed.

1.1.3 Infinite Current Sheet

Consider a surface current in the x-y plane flowing in the $\hat{x}$ direction. The magnetic field will be

$$\overline{H} = \begin{cases} -H_o\hat{y} & z > 0 \\ H_o\hat{y} & z < 0 \end{cases} \quad (1.26)$$

Let's consider doing the integration over a square path of side length b:

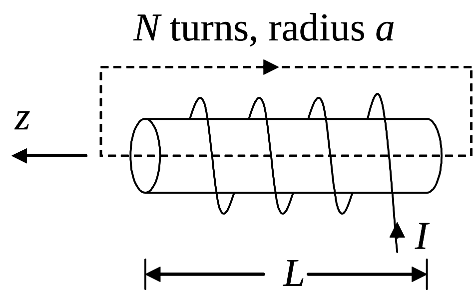


$$
\begin{aligned}
\int_0^b H_o dy - \int_b^0 H_o dy &= \int_0^b J_s dy && (1.27)\\
2H_o b &= J_s b && (1.28)\\
H_o &= \frac{J_s}{2} && (1.29)\\
\overline{H} &= \pm\frac{J_s}{2}\hat{y} && (1.30)
\end{aligned}
$$

Note that we only integrated the surface current density over the line from 0 to b in y. We did not integrate in z since the surface current density is in units of A/m (ie we only need to integrate in one dimension to get the total current enclosed).

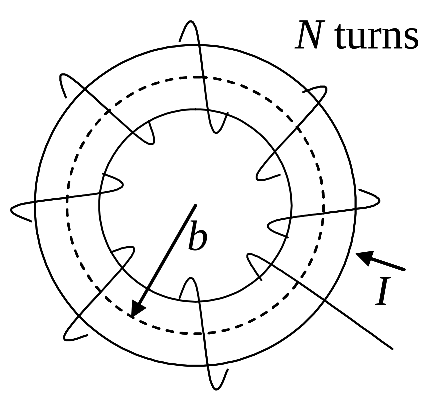
1.1.4 Solenoid (coil)

In this case, $\overline{H} = H_o\hat{z}$ and the total current enclosed by the surface is NI, where N is the number of turns of wire and I is the current flowing in the wire. Also, we let the contour of the surface outside of the solenoid go off to infinity where the fields are zero. The main contribution to the line integral then comes from the integration within the solenoid of length L:



$$\begin{aligned}
\int_0^L H_o\hat{z}\cdot\hat{z}dz &= NI && (1.31)\\
H_o &= \frac{NI}{L} && (1.32)\\
\overline{H} &= \frac{NI}{L}\hat{z} && (1.33)\\
\overline{B} &= \mu\frac{NI}{L}\hat{z} && (1.34)
\end{aligned}$$

1.1.5 Toroid



$$\begin{aligned}
\int_0^{2\pi} H_o\hat{\phi}\cdot\hat{\phi}rd\phi &= NI && (1.35)\\
2\pi rH_o &= NI && (1.36)\\
H_o &= \frac{NI}{2\pi r} && (1.37)\\
\overline{H} &= \frac{NI}{2\pi r}\hat{\phi} && (1.38)\\
\overline{B} &= \mu\frac{NI}{2\pi r}\hat{\phi} && (1.39)
\end{aligned}$$

1.2 Magnetic Vector Potential

In electrostatics, we had the notion of a potential. This concept is useful, since sometimes it is more convenient to compute the potential and then compute the electric field using $\overline{E} = -\nabla V$. It would be convenient to also define a magnetic potential to assist in the computation of magnetic fields. It turns out that we have not identified a measurable quantity that we could call a magnetic potential. However, we can define one mathematically to assist us in computation. We call this quantity the *magnetic vector potential* and denote it as $\overline{A}$.

In constructing an equation for $\overline{A}$, we use Gauss's Law of magnetics that $\nabla\cdot\overline{B} = 0$. We also take advantage of a vector identity that for any vector $\overline{A}$,

$$\nabla\cdot(\nabla\times\overline{A}) = 0 \tag{1.40}$$

Therefore, we will use

$$\overline{B} = \nabla\times\overline{A} \tag{1.41}$$

so that we are guaranteed to satisfy Gauss's Law. Since $\overline{B}$ has units Wb/m^2, $\overline{A}$ has units of Wb/m.

Since $\overline{B} = \mu \overline{H}$, Ampere's Law gives

$$\nabla \times \overline{B} = \mu \overline{J} \tag{1.42}$$

$$\nabla \times (\nabla \times \overline{A}) = \mu \overline{J} \tag{1.43}$$

Now, we invoke a second vector identity that states

$$\nabla^2 \overline{A} = \nabla(\nabla \cdot \overline{A}) - \nabla \times (\nabla \times \overline{A}) \tag{1.44}$$

where $\nabla^2 \overline{A}$ is the vector Laplacian operation. We can therefore write

$$\nabla(\nabla \cdot \overline{A}) - \nabla^2 \overline{A} = \mu \overline{J} \tag{1.45}$$

Now, when we define $\overline{B} = \nabla \times \overline{A}$, we have not uniquely specified the vector $\overline{A}$. In order to uniquely specify a vector, we must specify its curl *and* it divergence. Therefore, we are still free to choose the divergence of $\overline{A}$ any way we like. It is clearly convenient to specify

$$\nabla \cdot \overline{A} = 0 \tag{1.46}$$

Our equation therefore becomes

$$\nabla^2 \overline{A} = -\mu \overline{J} \tag{1.47}$$

So, we now have a differential equation for $\overline{A}$ much like we had an equation for V. A solution to this differential equation (which we will not prove) is

$$\overline{A} = \frac{\mu}{4\pi} \int_V \frac{\overline{J}}{R'} dV' \tag{1.48}$$

where R is the distance from the integration point to the point where the field is observed, or $R' = |\overline{R} - \overline{R}_i|$. Here, $\overline{R}$ is the point where $\overline{A}$ is evaluated. $\overline{R}_i$ is the point of integration. So, if we have a current distribution, we can compute $\overline{A}$ using this integral and then compute $\overline{B} = \nabla \times \overline{A}$.

1.3 Magnetic Properties of Materials

We need to briefly discuss the magnetic permeability μ much like we discussed the permittivity for electrostatics. Using a classical description of matter, all atoms have electrons which orbit the nucleus. This orbiting charge represents a current loop which creates a magnetic moment. In most materials, called diamagnetic materials, these atoms are randomly aligned so that there is no net magnetic effect.

The electrons also spin which creates another contribution to the magnetic moment. In atoms with even numbers of electrons, there are always two spins that are equal but opposite, resulting in zero net magnetic moment from spin. For an odd number of electrons, there is a net magnetic effect from the single unpaired electron.

When a material is exposed to a magnetic field $\overline{H}$, we can express the magnetic flux density as

$$\overline{B} = \mu_0 \overline{H} + \mu_0 \overline{M} = \mu_0(\overline{H} + \overline{M}) \tag{1.49}$$

where $\overline{M}$ is called the magnetization vector of a material. This vector represents the vector sum of the magnetic dipole moments of the atoms. Physically, the magnetic field is aligning the atomic magnetic dipoles. The degree to which these dipoles can be aligned is represented by the magnitude of $\overline{M}$. Much like we did in electrostatics, we define a magnetic susceptibility χ_m and write

$$\begin{aligned} \overline{M} &= \chi_m \overline{H} && (1.50) \\ \overline{B} &= \mu_0(\overline{H} + \chi_m \overline{H}) = \mu_0 \underbrace{(1+\chi_m)}_{\mu_r} \overline{H} && (1.51) \\ \overline{B} &= \mu \overline{H} && (1.52) \end{aligned}$$

The units of μ are H/m. For most materials, χ_m is so small that we can write $\mu_r = 1$. Ferromagnetic materials, however, which are suceptible to magnetic alignment, can have high values of μ_r. For example, pure iron has $\mu_r = 2 \times 10^5$.

1. Diamagnetic $\chi_m < 0$ ($\chi_m \sim 10^{-5}$, $\mu_r \sim 1$)
2. Paramagnetic $\chi_m < 0$ ($\chi_m \sim 10^{-5}$, $\mu_r \sim 1$)
3. Ferromagnetic $|\chi_m| \gg 1$

1.4 Inductance

The physical behavior of currents and magnetic fields is interesting. For example, we know that currents create magnetic fields. However, a static magnetic field (created by a magnet) which cuts through a loop of wire will not create a current. The key concept is that sources (charges, currents) create fields.

We will see that time-varying magnetics fields can induce currents. For example, if we have two separate loops, and drive a current through one (that changes with time), we will observe a current in the other. This magnetic linking is mutual inductance. Similarly, in a solenoid, the behavior of the current in one loop is influenced by time-varying current in another loop. This is self inductance, since the actual current flowing through both loops is the same.

Despite the fact that we need time-varying currents/fields to have this coupling, we can compute the strength of the coupling using static analysis. The strength of the coupling is the inductance of the system:

$$\text{Inductance: } L = \frac{\Lambda}{I} \tag{1.53}$$

where

Λ = magnetic flux linkage = flux linking current
I = current producing flux linkage

1.4.1 Steps for Computing Inductance

1. Assume a current and a form for $\overline{H}$
2. Calculate $\overline{H}$ using Ampere's Law

3. Calculate $\overline{B} = \mu \overline{H}$
4. Calculate $\Lambda = \int_s \overline{B} \cdot d\overline{s}$
5. Calculate $L = \Lambda / I$

1.4.2 Solenoid

We will change the notation so that the length of the solenoid is ℓ, since we will be using the symbol L to mean inductance.

1. We know the current and $\overline{H}$ was computed previously in class
2. $$\overline{H} = \frac{NI}{\ell}\hat{z} \tag{1.54}$$
3. $$\overline{B} = \mu\frac{NI}{\ell}\hat{z} \tag{1.55}$$
4. For a single loop, the flux is:
$$\begin{aligned}
\Lambda_1 &= \int_0^{2\pi}\int_0^a \overline{B} \cdot d\overline{s} && (1.56)\\
&= \int_0^{2\pi}\int_0^a \mu\frac{NI}{\ell}\hat{z} \cdot \hat{z} r dr d\phi && (1.57)\\
&= \mu 2\pi \frac{a^2}{2}\frac{NI}{\ell} && (1.58)\\
&= \mu\pi a^2 \frac{NI}{\ell} && (1.59)
\end{aligned}$$
For N loops, the flux linkage is
$$\Lambda = N\Lambda_1 = \mu\pi a^2 \frac{N^2 I}{\ell} \tag{1.60}$$
5. $$L = \frac{\Lambda}{I} = \frac{\mu\pi a^2 N^2}{\ell} \tag{1.61}$$

You may wonder why we use $\Lambda = N\Lambda_1$. The concept is that if flux passes through one loop, it will influence current in that loop (time-varying fields). If it passes through multiple loops, it will have an effect on each one. But since the current through each loop is the same, the overall effect on the solenoid current is additive.

1.4.3 Toroid

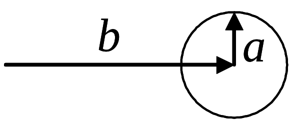
Based on our prior work, steps 1-3 yield:

$$\overline{B} = \mu \frac{NI}{2\pi r}\hat{\phi} \tag{1.62}$$

The integral in step 4 is relatively difficult to perform, so let's make an approximation. If the radius a of the core of the toroid is small compared to the radius of the toroid (ie $b \gg a$), then

$$\overline{B} \approx \mu \frac{NI}{2\pi b}\hat{\phi} \tag{1.63}$$

Now, the integration is over the area of the toroid cross-section:



$$\begin{aligned}
\Lambda_1 &= \int\int \mu \frac{NI}{2\pi b}\hat{\phi}\cdot\hat{\phi}drdz && (1.64)\\
&= \mu\frac{NI}{2\pi b}\pi a^2 = \mu\frac{NIa^2}{2b} && (1.65)\\
\Lambda &= N\Lambda_1 = \mu\frac{N^2Ia^2}{2b} && (1.66)
\end{aligned}$$

The inductance is therefore:

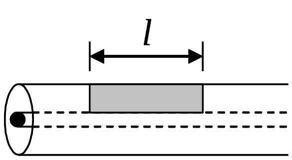
$$L = \frac{\Lambda}{I} = \mu\frac{N^2a^2}{2b} \tag{1.67}$$

1.4.4 Coax

We have already computed

$$\overline{B} = \frac{\mu I}{2\pi r}\hat{\phi} \tag{1.68}$$

The flux linking the two conductors is now the flux passing through the area between the conductors:



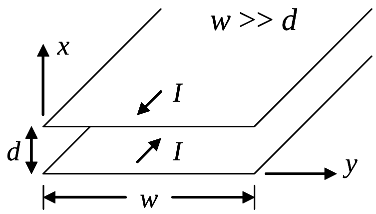
$$\begin{aligned}
\Lambda &= \int_a^b\int_0^\ell \frac{\mu I}{2\pi r}\hat{\phi}\cdot\hat{\phi}dzdr && (1.69)\\
&= \frac{\mu I\ell}{2\pi}\ln r|_a^b && (1.70)\\
&= \frac{\mu I\ell}{2\pi}\ln\left(\frac{b}{a}\right) && (1.71)
\end{aligned}$$

The inductance is:

$$\begin{aligned}
L &= \frac{\mu\ell}{2\pi}\ln\left(\frac{b}{a}\right) && (1.72)\\
\frac{L}{\ell} &= \frac{\mu}{2\pi}\ln\left(\frac{b}{a}\right) && (1.73)
\end{aligned}$$

1.4.5 Parallel Flat Conductors

$\overline{H} \approx H_o\hat{y}$



$$\int_0^w H_o\hat{y}\cdot\hat{y}dy = H_o w = I \tag{1.74}$$

$$\overline{H} = \frac{I}{w}\hat{y} \tag{1.75}$$

$$\overline{B} = \frac{\mu I}{w}\hat{y} \tag{1.76}$$

$$\Lambda = \int_0^d\int_0^\ell \frac{\mu I}{w}dzdx = \frac{\mu I}{w}\ell d \tag{1.77}$$

$$\frac{L}{\ell} = \mu\frac{d}{w} \tag{1.78}$$