Chapter 1

Plane Waves

We are now ready to look at the simplest form of electromagnetic waves.

1.1 Wave Equation

We start with Maxwell's equations in the sinusoidal steady state.

$$\nabla \times \overline{E} = -j\omega\overline{B} = -j\omega\mu\overline{H} \qquad \nabla \cdot \overline{D} = \nabla \cdot \epsilon\overline{E} = \rho_v \tag{1.1}$$

$$\nabla \times \overline{H} = j\omega\overline{D} + \overline{J} = j\omega\epsilon\overline{E} + \overline{J} \qquad \nabla \cdot \overline{B} = \nabla \cdot \mu\overline{H} = 0 \tag{1.2}$$

First, we rewrite Ampere's Law: if we have a medium which has free charge which facilites the flow of current, then $\overline{J} = \sigma\overline{E}$. So:

$$\nabla \times \overline{H} = j\omega\epsilon\overline{E} + \sigma\overline{E} = j\omega[\epsilon + \sigma/j\omega]\overline{E} \tag{1.3}$$

$$= j\omega\underbrace{[\epsilon - j\sigma/\omega]}_{\epsilon_c}\overline{E} \tag{1.4}$$

We call ϵ_c the complex permittivity:

$$\epsilon_c = \epsilon - j\sigma/\omega = \epsilon' - j\epsilon'' \tag{1.5}$$

$$= \epsilon_0\left[\epsilon_r - j\frac{\sigma}{\omega\epsilon_0}\right] = \epsilon_0\epsilon_{cr} \tag{1.6}$$

So:

$$\nabla \times \overline{H} = j\omega\epsilon_c\overline{E} \tag{1.7}$$

Let's simplify matters a little bit. Suppose we have no sources in our region (no free charge to create fields). We can still have charges in motion that allow currents to flow. But, to the outside observer, all moving charge is accompanied by equal but opposite charge so that no electric fields are created by these charges. In other words, $\rho_v = 0$.

If we take the curl of Faraday's law:

$$\nabla \times \nabla \times \overline{E} = -j\omega\mu\nabla \times \overline{H} \tag{1.8}$$

$$= -j\omega\mu(j\omega\epsilon_c\overline{E}) = \omega^2\mu\epsilon_c\overline{E} \tag{1.9}$$

There is a vector identity

$$\nabla \times \nabla \times \overline{E} = \nabla(\nabla \cdot \overline{E}) - \nabla^2 \overline{E} \tag{1.10}$$

so that

$$\nabla(\nabla \cdot \overline{E}) - \nabla^2 \overline{E} = \omega^2 \mu \epsilon_c \overline{E} \tag{1.11}$$

From Gauss' law: $\nabla \cdot \epsilon_c \overline{E} = \epsilon_c \nabla \cdot \overline{E} = 0$ since $\rho_v = 0$. The **Homogeneous Wave Equation** is

$$\nabla^2 \overline{E} + \omega^2 \mu \epsilon_c \overline{E} = 0 \tag{1.12}$$

If $\gamma^2 = -\omega^2 \mu \epsilon_c$, we call γ the propagation constant. Therefore,

$$\nabla^2 \overline{E} - \gamma^2 \overline{E} = 0 \tag{1.13}$$

Note that we can follow a similar path for the magnetic field:

$$\begin{aligned}
\nabla \times \nabla \times \overline{H} &= j\omega\epsilon_c \nabla \times \overline{E} = j\omega\epsilon_c(-j\omega\mu\overline{H}) && (1.14)\\
\nabla(\nabla \cdot \overline{H}) - \nabla^2 \overline{H} &= \omega^2 \mu \epsilon_c \overline{H} = -\gamma^2 \overline{H} && (1.15)\\
\nabla^2 \overline{H} - \gamma^2 \overline{H} &= 0 && (1.16)
\end{aligned}$$

$\nabla^2 \overline{E}$ is the Laplacian. In Cartesian coordinates:

$$\nabla^2 \overline{E} = \frac{\partial^2 \overline{E}}{\partial x^2} + \frac{\partial^2 \overline{E}}{\partial y^2} + \frac{\partial^2 \overline{E}}{\partial z^2} \tag{1.17}$$

1.2 Lossless Media

Note that $\overline{J} = \sigma \overline{E}$ is like $I = GV = V/R$ (G = conductance). So, $\sigma \neq 0$ means energy will be dissipated (loss). If $\sigma = 0$, we call the medium a *lossless medium*. In this case:

$$\begin{aligned}
\gamma^2 &= -\omega^2 \mu \epsilon_c = -\omega^2 \mu \epsilon && (1.18)\\
\gamma &= j\omega\sqrt{\mu\epsilon} = jk && (1.19)
\end{aligned}$$

We call $k = \omega\sqrt{\mu\epsilon}$ the wavenumber. Recall that for transmission lines, we had $\beta = \omega\sqrt{L'C'}$.

1.2.1 Plane Waves

So, $\nabla^2 \overline{E} - k^2 \overline{E} = 0$. Let's solve this in Cartesian coordinates. If $\overline{E} = \hat{x}E_x + \hat{y}E_y + \hat{z}E_z$,

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right)(\hat{x}E_x + \hat{y}E_y + \hat{z}E_z) + k^2(\hat{x}E_x + \hat{y}E_y + \hat{z}E_z) = 0 \tag{1.20}$$

Let's focus on the $\hat{x}$ component of this vector equation (there are similar equations for the other components):

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} + k^2\right) E_x = 0 \tag{1.21}$$

A **uniform plane wave** is a wave for which there is *no variation of the fields within a plane.* For example, let the x-y plane represent the plane of no variation. Then

$$\frac{\partial \overline{E}}{\partial x} = \frac{\partial \overline{E}}{\partial y} = 0 \tag{1.22}$$

$$\frac{\partial \overline{H}}{\partial x} = \frac{\partial \overline{H}}{\partial y} = 0 \tag{1.23}$$

We obtain

$$\frac{d^2 E_x}{dz^2} + k^2 E_x = 0 \tag{1.24}$$

There are similar equations for E_y, H_x, and H_y. What about E_z and H_z? Using Ampere's law with only the $\hat{z}$ component:

$$\nabla \times \overline{H} = j\omega\epsilon\overline{E} \tag{1.25}$$

$$\hat{z}\left(\frac{\partial H_y}{\partial x} - \frac{\partial H_x}{\partial y}\right) = \hat{z} j\omega\epsilon E_z \tag{1.26}$$

But since

$$\frac{\partial H_y}{\partial x} = \frac{\partial H_x}{\partial y} = 0 \tag{1.27}$$

we have that $E_z = 0$. Using Faraday's law in the same way shows that $H_z = 0$ as well. So, all we have to do is solve the differential equation for E_x. Assume

$$E_x = Ae^{mz} \tag{1.28}$$

Putting this into the differential equation leads to

$$Am^2 e^{mz} + k^2 A e^{mz} = 0 \tag{1.29}$$

$$m^2 + k^2 = 0 \tag{1.30}$$

$$m = \pm jk \tag{1.31}$$

So,

$$E_x(z) = E_{xo}^{+} e^{-jkz} + E_{xo}^{-} e^{jkz} \tag{1.32}$$

where the first term is a *forward traveling wave* and the second is a *reverse traveling wave.*

Let's assume:

1. $E_y(z) = 0$: Only an $\hat{x}$ component of the electric field exists
2. $E_x(z) = E_{xo}^{+} e^{-jkz}$: Only a forward traveling wave exists

$$\overline{E}(z) = \hat{x} E_{xo}^{+} e^{-jkz} \tag{1.33}$$

$$\nabla \times \overline{E} = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ E_x & 0 & 0 \end{vmatrix} = \hat{y}\frac{\partial E_x}{\partial z} - \hat{z}\frac{\partial E_x}{\partial y} = \hat{y}\frac{\partial E_x}{\partial z} = -j\omega\mu\overline{H} \tag{1.34}$$

$$\hat{y} E_{xo}^{+} e^{-jkz}(-jk) = -j\omega\mu\hat{y} H_y \tag{1.35}$$

$$H_y = E_{xo}^{+}\frac{k}{\omega\mu} e^{-jkz} = E_{xo}^{+}\frac{\omega\sqrt{\mu\epsilon}}{\omega\mu} e^{-jkz} = \frac{E_{xo}^{+}}{\sqrt{\mu/\epsilon}} e^{-jkz} = H_{yo}^{+} e^{-jkz} \tag{1.36}$$

So: $H_{yo}^+ = E_{xo}^+/\eta$ where

$$\eta = \sqrt{\mu/\epsilon} \tag{1.37}$$

η has units of ohms. We call it the **intrinsic impedance** of the propagation medium. It is analogous to the characteristic impedance of a transmission line, and simply represents the ratio of the electric to magnetic fields. However, in this case, we must worry about the vector components of the fields. For example, in our case

$+z$ traveling wave
$+x$ directed electric field
$+y$ directed magnetic field

Note that $\overline{E} \times \overline{H}$ gives the direction of propagation ($+z$). So,

$$\frac{E_{xo}^+}{H_{yo}^+} = \eta \tag{1.38}$$

Note that if we had used the $-z$ traveling wave:

$$\begin{aligned}
\hat{y}E_{xo}^- e^{jkz}(jk) &= -j\omega\mu\hat{y}H_y && (1.39)\\
H_y &= -E_{xo}^- \frac{k}{\omega\mu} e^{jkz} = -\frac{E_{xo}^-}{\eta} e^{jkz} = H_{yo}^- e^{jkz} && (1.40)\\
\frac{E_{xo}^-}{H_{yo}^-} &= -\eta && (1.41)
\end{aligned}$$

Let's again examine:

$-z$ traveling wave
$+x$ directed electric field
$-y$ directed magnetic field

$\overline{E} \times \overline{H}$ again points in the $-\hat{z}$ direction (the direction of propagation).

Let's go back to the forward traveling wave:

$$\begin{aligned}
\overline{E}(z) &= \hat{x}E_{xo}^+ e^{-jkz} && (1.42)\\
\overline{H}(z) &= \hat{y}\frac{E_{xo}^+}{\eta} e^{-jkz} && (1.43)
\end{aligned}$$

$\overline{E}$ and $\overline{H}$ are perpendicular to each other and are both perpendicular to the direction of travel.

If the plane wave is traveling in an arbitrary direction the equation for the electric field becomes

$$\overline{E} = \overline{E}_o e^{-j\,(k_x x + k_y y + k_z z)}, \tag{1.44}$$

where $\overline{E}_o$ is a complex vector constant. For a convenient shorthand we define a wavevector

$$\overline{k} = k_x\hat{x} + k_y\hat{y} + k_z\hat{z}. \tag{1.45}$$

$\overline{k}$ points in the direction of propagation and has a magnitude of

$$\left|\overline{k}\right| = \sqrt{k_x^2 + k_y^2 + k_z^2} = \omega\sqrt{\mu\epsilon} = \frac{2\pi}{\lambda} \tag{1.46}$$

Now we can write $\overline{E}$ as

$$\overline{E} = \overline{E}_o e^{-j\overline{k}\cdot\overline{r}}. \tag{1.47}$$

Since

$$\overline{k}\cdot\overline{r} = (k_x\hat{x} + k_y\hat{y} + k_z\hat{z})\cdot(x\hat{x} + y\hat{y} + z\hat{z}) = k_x x + k_y y + k_z z \tag{1.48}$$

If we use the notation $\hat{k}$ to designate the direction of propagation ($\hat{k} = \hat{z}$ for our example), then the conditions $\overline{E} \perp \overline{H}$, $\overline{E} \perp \hat{k}$, and $\overline{H} \perp \hat{k}$ indicate that the wave is a **Transverse Electromagnetic (TEM) Wave**.

Let's look at the time domain forms. If $E_{xo}^+ = |E_{xo}^+|e^{j\phi^+}$,

$$\begin{aligned}
\overline{E}(z,t) = \text{Re}\{\tilde{\overline{E}}(z)e^{j\omega t}\} &= \hat{x}\ \text{Re}\{|E_{xo}^+|e^{j(\omega t - kz + \phi^+)}\} && (1.49)\\
&= \hat{x}|E_{xo}^+|\cos(\omega t - kz + \phi^+) && (1.50)\\
\overline{H}(z,t) = \text{Re}\{\tilde{\overline{H}}(z)e^{j\omega t}\} &= \hat{y}\frac{|E_{xo}^+|}{\eta}\cos(\omega t - kz + \phi^+) && (1.51)
\end{aligned}$$

Note that $\overline{E}(z,t)$ and $\overline{H}(z,t)$ are in phase for this case.

Let's explore the properties of these waves like we did with transmission lines.

1. **Phase Velocity:** Recall that this is how fast we need to travel if we want to stay at the same point on the wave.

$$\begin{aligned}
\xi &= \omega t - kz + \phi^+ = \text{constant} && (1.52)\\
z &= \frac{-\xi + \omega t + \phi^+}{k} && (1.53)\\
u_p &= \frac{dz}{dt} = \frac{\omega}{k} = \frac{\omega}{\omega\sqrt{\mu\epsilon}} = \frac{1}{\sqrt{\mu\epsilon}} && (1.54)\\
& && (1.55)
\end{aligned}$$

2. **Properties in Vacuum:** In a vacuum (free-space), $\mu = \mu_0 = 4\pi \times 10^{-7}$ H/m and $\epsilon = \epsilon_0 = 8.854 \times 10^{-12}$ F/m.

$$\begin{aligned}
u_p &= \frac{1}{\sqrt{\mu_0\epsilon_0}} = c = 3 \times 10^8 \text{ m/s (speed of light in vacuum)} && (1.56)\\
\eta &= \eta_0 = \sqrt{\frac{\mu_0}{\epsilon_0}} = 377\ \Omega \approx 120\pi\ \Omega \text{ (intrinsic impedance of vacuum)} && (1.57)
\end{aligned}$$

3. **Wavelength:** Wavelength is the distance in z necessary to go one complete cycle of the sinusoid.

$$\begin{aligned} kz|_{z=\lambda} &= k\lambda = 2\pi && (1.58)\\ \lambda &= \frac{2\pi}{k} = \frac{2\pi}{\omega/u_p} = \frac{u_p}{f} && (1.59)\\ u_p &= f\lambda && (1.60)\\ k &= \frac{2\pi}{\lambda} = \omega\sqrt{\mu\epsilon} = \frac{\omega}{u_p} && (1.61) \end{aligned}$$

1.2.2 Relation between $\overline{E}$ and $\overline{H}$

Consider the ∇ operator acting on an arbitrary plane wave. For example the gradient

$$\begin{aligned} \nabla e^{-j\overline{k}\cdot\overline{r}} &= \left(\hat{x}\frac{\partial}{\partial x} + \hat{y}\frac{\partial}{\partial y} + \hat{z}\frac{\partial}{\partial z}\right) e^{-jk_x x - jk_y y - jk_z z} && (1.62)\\ &= \left[\hat{x}(-jk_x) + \hat{y}(-jk_y) + \hat{z}(-jk_z)\right] e^{-j\overline{k}\cdot\overline{r}} && (1.63)\\ &= -j\left[k_x\hat{x} + k_y\hat{y} + k_z hatz\right] e^{-j\overline{k}\cdot\overline{r}} && (1.64)\\ &= -j\overline{k}\, e^{-j\overline{k}\cdot\overline{r}} && (1.65) \end{aligned}$$

Thus, $\nabla \to -j\overline{k}$ for plane waves. Maxwell's equations become:

$$\begin{array}{ccc} \textbf{Time Harmonic Equation} & & \textbf{Plane Wave} \\ \nabla \times \overline{E} = -j\omega\mu\overline{H} & \Rightarrow & -j\overline{k} \times \overline{E} = -j\omega\mu\overline{H} \\ \nabla \times \overline{H} = j\omega\epsilon\overline{E} & \Rightarrow & -j\overline{k} \times \overline{H} = j\omega\epsilon\overline{E} \\ \nabla \cdot \left(\epsilon\overline{E}\right) & \Rightarrow & -j\overline{k} \cdot \epsilon\overline{E} = 0 \\ \nabla \cdot \left(\mu\overline{H}\right) & \Rightarrow & -j\overline{k} \cdot \mu\overline{H} = 0 \end{array} \qquad (1.66)$$

Now let $\hat{k}$ be the unit vector in the direction of propagation ($\overline{k}$ direaction), so that

$$\hat{k} = \frac{\overline{k}}{|\overline{k}|} = \frac{\overline{k}}{\omega\sqrt{\mu\epsilon}}. \qquad (1.67)$$

Then Maxwell's equations become

$$\begin{array}{rcll} \omega\sqrt{\mu\epsilon}\,\hat{k} \times \overline{E} = \omega\mu\overline{H} & \Rightarrow & \hat{k} \times \overline{E} = \eta\overline{H} & \\ \omega\sqrt{\mu\epsilon}\,\hat{k} \times \overline{H} = -\epsilon\overline{E} & \Rightarrow & \hat{k} \times \overline{H} = -\frac{1}{\eta}\overline{E} & \\ -j\omega\sqrt{\mu\epsilon}\epsilon\,\hat{k} \cdot \overline{E} = 0 & \Rightarrow & \hat{k} \cdot \overline{E} = 0 & \hat{k} \perp \overline{E} \\ -j\omega\sqrt{\mu\epsilon}\mu\,\hat{k} \cdot \overline{H} = 0 & \Rightarrow & \hat{k} \cdot \overline{H} = 0 & \hat{k} \perp \overline{H} \end{array} \qquad (1.68)$$

1.3 Plane Wave Polarization

Polarization of a wave is the shape the tip of the electric field vector $\overline{E}$ traces as a function of time at a point in space.

In general, this shape is an ellipse, so we say the wave has *elliptical polarization*. Special cases: *circular polarization*, *linear polarization*

We will assume a forward traveling wave and suppress the '+' superscript.

$$\overline{E} = \hat{x}E_x(z) + \hat{y}E_y(z) = \hat{x}E_{xo}e^{-jkz} + \hat{y}E_{yo}e^{-jkz} = E_{xo}\left(\hat{x} + \hat{y}\frac{E_{yo}}{E_{xo}}\right)e^{-jkz} \tag{1.69}$$

1.3.1 Linear Polarization

We say that the wave is *linearly polarized*. If $\alpha = E_{yo}/E_{xo}$ is real, we will have linear polarization. So,

$$\begin{aligned} \overline{E}(z) &= E_{xo}\left(\hat{x} + \alpha\hat{y}\right)e^{-jkz} && (1.70)\\ \overline{\mathcal{E}}(z,t) &= |E_{xo}|\left\{\hat{x}\cos(\omega t - kz + \phi_x) + \hat{y}\alpha\cos(\omega t - kz + \phi_x)\right\} && (1.71)\\ &= |E_{xo}|(\hat{x} + \alpha\hat{y})\cos(\omega t - kz + \phi_x) && (1.72) \end{aligned}$$

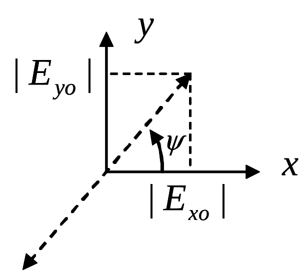
At a point in space (let's choose $z = 0$)

$$\overline{\mathcal{E}}(0,t) = |E_{xo}|(\hat{x} + \alpha\hat{y})\cos(\omega t + \phi_x) \tag{1.73}$$

So, the vector oscillates in time but always points in the same direction. The angle ψ from the x-axis is:

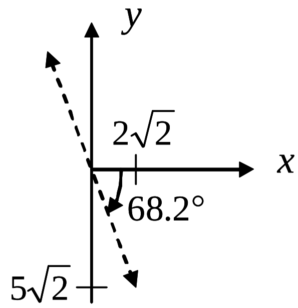
$$\psi = \tan^{-1}\alpha = \tan^{-1}\left(\frac{E_{yo}}{E_{xo}}\right) \tag{1.74}$$

Note that if $\alpha < 0$, $\psi < 0$. Note also that E_{yo} and E_{xo} can be complex, but their ratio must be real.



Example: $E_{xo} = 2 + j2$, $E_{yo} = -5 - j5$

$$\begin{aligned} \frac{E_{yo}}{E_{xo}} &= \frac{-5 - j5}{2 + j2} = -\frac{5}{2}\frac{1 + j1}{1 + j1} = -\frac{5}{2} = -2.5 && (1.75)\\ |E_{xo}| &= 2\sqrt{2} && (1.76)\\ |E_{yo}| &= 5\sqrt{2} && (1.77)\\ \psi &= \tan^{-1}\left(\frac{E_{yo}}{E_{xo}}\right) = -68.2^\circ && (1.78) \end{aligned}$$



1.3.2 Circular Polarization

If

$$\alpha = \frac{E_{yo}}{E_{xo}} = \pm j \tag{1.79}$$

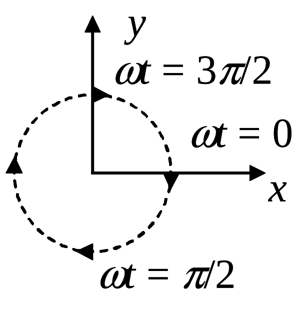
we have circular polarization. Let's consider $\alpha = +j$.

$$\begin{aligned}
\overline{E}(z) &= E_{xo}(\hat{x} + j\hat{y})e^{-jkz} = E_{xo}(\hat{x} + \hat{y}e^{j\pi/2})e^{-jkz} && (1.80)\\
\overline{\mathcal{E}}(z,t) &= |E_{xo}|\left\{\hat{x}\cos(\omega t - kz + \phi_x) + \hat{y}\cos(\omega t - kz + \phi_x + \pi/2)\right\} && (1.81)\\
&= |E_{xo}|\left\{\hat{x}\cos(\omega t - kz + \phi_x) - \hat{y}\sin(\omega t - kz + \phi_x)\right\} && (1.82)
\end{aligned}$$

At $z = 0$ (we will assume $\phi_x = 0$ for simplicity)

$$\overline{\mathcal{E}}(0,t) = |E_{xo}|\left\{\hat{x}\cos(\omega t) - \hat{y}\sin(\omega t)\right\} \tag{1.83}$$

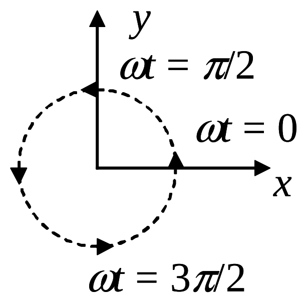
So, $\overline{\mathcal{E}}(0,t)$ traces out a circle. We assign a sense to the circle. Put your thumb in the direction of propagation, and your fingers in the direction of rotation of $\overline{\mathcal{E}}(0,t)$. In this case, this works out for the left-hand. We therefore call it *Left-Hand Circular Polarization (LHCP)*.



If $\alpha = -j$,

$$\begin{aligned}
\overline{E}(z) &= E_{xo}(\hat{x} - \hat{y}e^{j\pi/2})e^{-jkz} && (1.84)\\
\overline{\mathcal{E}}(z,t) &= |E_{xo}|\left\{\hat{x}\cos(\omega t - kz + \phi_x) + \hat{y}\sin(\omega t - kz + \phi_x)\right\} && (1.85)\\
\overline{\mathcal{E}}(0,t) &= |E_{xo}|\left\{\hat{x}\cos(\omega t) + \hat{y}\sin(\omega t)\right\} \text{ for } \phi_x = 0 && (1.86)
\end{aligned}$$

The rule works out for the right hand. Therefore, we call this *Right-Hand Circular Polarization.*



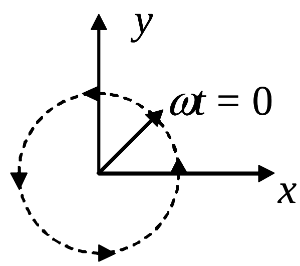
Example: $E_{xo} = 2 + j2$, $E_{yo} = 2 - j2$

$$\frac{E_{yo}}{E_{xo}} = \frac{2-j2}{2+j2} = \frac{(2-j2)(2-j2)}{8} = -\frac{j8}{8} = -j \tag{1.87}$$

$$|E_{xo}| = |E_{yo}| = 2\sqrt{2} \tag{1.88}$$

$$E_{xo} = 2\sqrt{2}e^{j\pi/4} \tag{1.89}$$

So, this will be RHCP with a radius of $2\sqrt{2}$. Because $\phi_x = \pi/4$, at $z = 0$ and $t = 0$, the vector will point at 45°.



Example: $E_{xo} = 2 - j2$, $E_{yo} = 2 + j2$

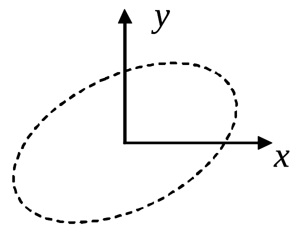
$$\frac{E_{yo}}{E_{xo}} = +j \tag{1.90}$$

$$E_{xo} = 2\sqrt{2}e^{-j\pi/4} \tag{1.91}$$

So, this will be LHCP with a radius of $2\sqrt{2}$. Because $\phi_x = -\pi/4$, at $z = 0$ and $t = 0$, the vector will point at $-45°$.

1.3.3 Elliptical Polarization

This is the most general case. The tip of the electric field will trace out an ellipse. We can in general have an ellipse rotated from the x-y axes as shown. However, we won't analyze this case in detail.



However, consider the case where

$$\alpha = \frac{E_{yo}}{E_{xo}} = -j\frac{|E_{yo}|}{|E_{xo}|} \tag{1.92}$$

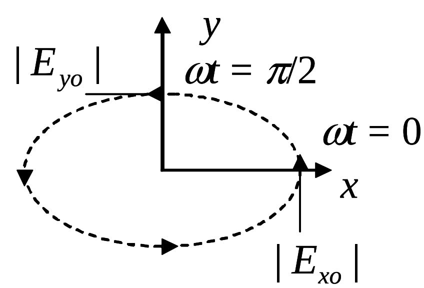
$$\overline{E}(z) = E_{xo}\left(\hat{x} + \hat{y}\frac{|E_{yo}|}{|E_{xo}|}e^{-j\pi/2}\right)e^{-jkz} \tag{1.93}$$

$$\overline{\mathcal{E}}(z,t) = |E_{xo}|\left\{\hat{x}\cos(\omega t - kz + \phi_x) + \hat{y}\frac{|E_{yo}|}{|E_{xo}|}\sin(\omega t - kz + \phi_x)\right\} \tag{1.94}$$

$$= \hat{x}|E_{xo}|\cos(\omega t - kz + \phi_x) + \hat{y}|E_{yo}|\sin(\omega t - kz + \phi_x) \tag{1.95}$$

In this case for $z = 0$ and $\phi_x = 0$, we get the picture below. Notice that the ellipse has a major (longer) axis and a minor (shorter) axis. The ratio of the length of the major axis to the length of the minor axis is call the *Axial Ratio* ($1 \leq$ axial ratio $< \infty$).

An axial ratio of ∞ represents linear polarization
1 represents circular polarization



1.4 Lossy Media

If we go back to the wave equation without assuming $\sigma = 0$:

$$\nabla^2\overline{E} - \gamma^2\overline{E} = 0 \tag{1.96}$$

$$\gamma^2 = -\omega^2\mu\epsilon_c = -\omega^2\mu(\epsilon - j\sigma/\omega) = -\omega^2\mu(\epsilon' - j\epsilon'') \tag{1.97}$$

If we express: $\gamma = \alpha + j\beta$, we obtain

$$\gamma^2 = (\alpha + j\beta)^2 = -\omega^2\mu\epsilon' + j\omega^2\mu\epsilon'' \tag{1.98}$$

$$\alpha^2 - \beta^2 + j2\alpha\beta = -\omega^2\mu\epsilon' + j\omega^2\mu\epsilon'' \tag{1.99}$$

Equating real and imaginary parts and solving, we get:

$$\alpha = \omega \left\{ \frac{\mu\epsilon'}{2} \left[\sqrt{1 + \left(\frac{\epsilon''}{\epsilon'}\right)^2} - 1 \right] \right\}^{1/2} \quad \text{Np/m} \tag{1.100}$$

$$\beta = \omega \left\{ \frac{\mu\epsilon'}{2} \left[\sqrt{1 + \left(\frac{\epsilon''}{\epsilon'}\right)^2} + 1 \right] \right\}^{1/2} \quad \text{rad/m} \tag{1.101}$$

1.4.1 Plane Waves

Now, if we simplify the wave equation for a uniform plane wave just like we did in the lossless case:

$$\frac{d^2 E_x}{dz^2} - \gamma^2 E_x = 0 \tag{1.102}$$

$$E_x(z) = E_{xo}^+ e^{-\gamma z} + E_{xo}^- e^{\gamma z} \tag{1.103}$$

$$= E_{xo}^+ e^{-\alpha z} e^{-j\beta z} + E_{xo}^- e^{\alpha z} e^{j\beta z} \tag{1.104}$$

So, the wave decays as it propagates. Note that this also means we take $\alpha > 0$, $\beta > 0$ when we take the square root of γ^2.

To determine $\overline{H}$, we used Faraday's law: $\nabla \times \overline{E} = -j\omega\mu\overline{H}$

We find that:

$$\overline{H} = \frac{1}{\eta_c} \hat{k} \times \overline{E} \tag{1.105}$$

$$\eta_c = \sqrt{\frac{\mu}{\epsilon_c}} \tag{1.106}$$

η_c is the intrinsic impedance of the lossy medium. Since η_c is a complex number, $\overline{E}$ and $\overline{H}$ are no longer in phase.

1.4.2 Skin Depth

For a $+z$-traveling wave:

$$|E_x(z)| = |E_{xo}^+ E^{-\alpha z} e^{-j\beta z}| = |E_{xo}^+| e^{-\alpha z} \tag{1.107}$$

The propagation distance required to attenuate the wave by a factor of e^{-1} is called the *skin depth* δ_s:

$$|E_x(z = \delta_s)| = |E_{xo}^+| e^{-1} \rightarrow \delta_s = \frac{1}{\alpha} \tag{1.108}$$

Extremes:	Perfect Conductor	$\sigma = \infty$	$\alpha = \infty$	$\delta_s = 0$
	Dielectric	$\sigma = 0$	$\alpha = 0$	$\delta_s = \infty$

Since $\overline{J} = \sigma\overline{E}$, then in a conductor since $\overline{E}$ decays rapidly, the current is concentrated near the conductor surface. In a perfect conductor, the current becomes a surface current density.

1.4.3 Loss Tangent

The loss tangent is simply a commonly-used parameter to describe the loss of a medium. It is defined as:

$$\text{Loss Tangent} = \tan\delta = \frac{\epsilon''}{\epsilon'} \tag{1.109}$$

Often, materials are specified by ϵ' and $\tan\delta$ at a certain frequency:

Polystyrene Foam:	$\epsilon' = 1.03\epsilon_0$	$\tan\delta = 0.3 \times 10^{-4}$	$f = 3\text{GHz}$
Fresh Snow:	$\epsilon' = 1.20\epsilon_0$	$\tan\delta = 3 \times 10^{-4}$	$f = 3\text{GHz}$
Round Steak:	$\epsilon' = 40\epsilon_0$	$\tan\delta = 0.3$	$f = 3\text{GHz}$

So, let's put the round steak in the microwave oven (not my favorite way to prepare steak). The complex permittivity is:

$$\epsilon_c = \epsilon' - j\epsilon'' = \epsilon'\left(1 - j\frac{\epsilon''}{\epsilon'}\right) = \epsilon'\left(1 - j\tan\delta\right) \tag{1.110}$$

$$= 40\left(1 - j0.3\right)\epsilon_0 \tag{1.111}$$

$$\gamma = j\omega\sqrt{\mu_0\epsilon_c} = j\omega\sqrt{\mu_0\epsilon_0}\sqrt{40(1-j0.3)} = j\frac{2\pi}{\lambda_0}\sqrt{40(1-j0.3)} \tag{1.112}$$

At $f = 3\text{GHz}$, $\lambda_0 = 10\text{cm} = 0.1\text{m}$:

$$\gamma = \alpha + j\beta = 59 + j402\text{m}^{-1} \tag{1.113}$$

$$\delta_s = \frac{1}{\alpha} = 0.017\text{m} = 1.7\text{cm} \tag{1.114}$$

So, the microwave over heats the surface more rapidly that it heats the center (contrary to popular belief). However, it is true that a microwave immediately starts heating the center (not all heat arrives at the center through heat conduction). For polystyrene foam:

$$\epsilon_c = 1.03\left(1 - j0.3 \times 10^{-4}\right)\epsilon_0 \tag{1.115}$$

$$\gamma = 9.6 \times 10^{-4} + j63.8\text{m}^{-1} \tag{1.116}$$

Since α is so small, very little wave attenuation (and therefore heating) occurs. This is why you can reheat your meat in a styrofoam box in the microwave without the box getting hot.

1.5 Parameter Simplifications

The purpose of simplifying the expressions for α and β is to get a more qualitative understanding of how the various material properties (σ, f, ϵ_r) parameters affect plane wave propagation is different types of materials. The approximations are: (1) if $\epsilon'' \ll \epsilon'$ the material is a low loss medium, and (2) if $\epsilon'' \gg \epsilon'$ the material is a good conductor.

For a low-loss dielectric, the expression for γ that is given by

$$\gamma = j\omega\sqrt{\mu\epsilon'}\left(1 - j\frac{\epsilon''}{\epsilon'}\right)^{1/2} \tag{1.117}$$

can be approximated using the first two terms of a binomial expansion ($\sqrt{1+\Delta} = 1 - \Delta/2$). The resulting expression for γ is given by

$$\gamma \simeq j\omega\sqrt{\mu\epsilon'}\left(1 - j\frac{\epsilon''}{2\epsilon'}\right) \tag{1.118}$$

The real and imaginary part are

$$\alpha \simeq \frac{\omega\epsilon''}{2}\sqrt{\frac{\mu}{\epsilon'}} = \frac{\sigma}{2}\sqrt{\frac{\mu}{\epsilon}} \tag{1.119}$$

$$\beta \simeq \omega\sqrt{\mu\epsilon'} = \omega\sqrt{\mu\epsilon}. \tag{1.120}$$

This expression shows that β is exactly the same as the lossless case. The plane propagation behavior for a low-loss medium is the same with the addition of a loss term.

The intrinsic impedance is also approximated using the binomial expansion as given by

$$\eta \simeq \sqrt{\frac{\mu}{\epsilon'}}\left(1 + j\frac{\epsilon''}{2\epsilon'}\right) \tag{1.121}$$

$$\simeq \sqrt{\frac{\mu}{\epsilon}}, \tag{1.122}$$

which is the same as it was for the lossless case.

For the good conductor $\frac{\epsilon''}{\epsilon'} \gg 1$

$$\gamma = j\omega\sqrt{\mu\epsilon'}\left(1 - j\frac{\epsilon''}{\epsilon'}\right)^{1/2} \tag{1.123}$$

$$\simeq j\omega\sqrt{\mu\epsilon'}\left(-j\frac{\epsilon''}{\epsilon'}\right)^{1/2}\left(1 + j\frac{\epsilon'}{\epsilon''}\right)^{1/2} \tag{1.124}$$

$$\simeq \sqrt{j\frac{\omega^2\mu\epsilon'\epsilon''}{\epsilon'}}\left(1 + j\frac{\epsilon'}{2\epsilon''}\right). \tag{1.125}$$

Substitute $\epsilon'' = \frac{\sigma}{\omega}$ and $\sqrt{j} = \frac{1+j}{\sqrt{2}}$ to get

$$\gamma = \sqrt{\frac{\omega\mu\sigma}{2}}\,(1+j)\left(1 + j\frac{\epsilon'}{2\epsilon''}\right). \tag{1.126}$$

Since $\frac{\epsilon'}{\epsilon''} \ll 1$

$$\gamma = \sqrt{\frac{\omega\mu\sigma}{2}}\,(1+j)\,, \tag{1.127}$$

resulting in

$$\alpha = \beta = \sqrt{\frac{\omega\mu\sigma}{2}} = \sqrt{\pi f\mu\sigma}. \tag{1.128}$$

In this case the propogation changes with frequency. The approimation for the intrinsic impedance follows a similar process resulting in

$$\eta_c = \sqrt{j\frac{\mu}{\epsilon''}} = (1+j)\sqrt{\frac{\pi f\mu}{\sigma}}. \tag{1.129}$$

With a complex η the electric and magnetic fields are no longer in phase.

Is it valid to assume that dielectrics are low-loss and metals ae good conductors?

$$\begin{array}{cc} \text{Dielectric} & \text{Conductor} \\ 1 \gg \frac{\epsilon''}{\epsilon'} & 1 \ll \frac{\epsilon''}{\epsilon'} \\ \frac{1}{100} > \frac{\epsilon''}{\epsilon'} & 100 < \frac{\epsilon''}{\epsilon'} \\ \frac{1}{100} > \frac{\sigma}{\omega\epsilon_r\epsilon_o} & 100 < \frac{\sigma}{\omega\epsilon_r\epsilon_o} \\ \omega > \frac{100\sigma}{\epsilon_r\epsilon_o} & \omega < \frac{\sigma}{100\epsilon_r\epsilon_o} \\ \omega > \frac{10010^{-12}}{(4)(8.854e{-}12)} & \omega < \frac{10^6}{(100)(8.854e{-}12)} \\ \omega > 2.8 \text{ rad/s} & \omega < 10^{12} \text{ rad/s} \end{array} \tag{1.130}$$

These are very valid approximations for dielectrics and conductors.

Sea Water Example

Let's look at plane wave propagation through sea water. From the appendix we get the following material parameters:

$\epsilon_r = 72 - 80$ (We will use $\epsilon_r = 80$)

$\sigma = 4$

What is the range for the good conductor approximation?

$$\frac{\epsilon''}{\epsilon'} > 100 \tag{1.131}$$

$$\frac{\sigma}{\omega\epsilon_r\epsilon} > 100 \tag{1.132}$$

$$\omega < 56\ MHz \tag{1.133}$$

What is the range for the low-loss dielectric approximation?

$$\frac{\epsilon''\epsilon'}{<}\frac{1}{100} \tag{1.134}$$

$$\frac{\sigma}{\omega\epsilon_r\epsilon} < \frac{1}{100} \tag{1.135}$$

$$\omega > 565\ GHz \tag{1.136}$$

We could use an optical frequency as given by

$$f = \frac{c}{\lambda} = \frac{3^8}{0.5^{-6}} = 6^{14}\ Hz \tag{1.137}$$

Which frequency is better to use?

$$\alpha\,(1\ kHz) = \sqrt{\pi 10^3 4\pi \times 10^{-7}} = 0.126\ np/m \tag{1.138}$$

$$\alpha\left(6 \times 10^{14}\ Hz\right) = \frac{4}{2}\sqrt{\frac{4\pi \times 10^{-7}}{80\ 8.854 \times 10^{-12}}} = 84\ np/m \tag{1.139}$$

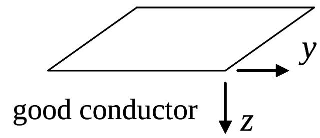
Is this accurate for optical frequencies? This would mean that after propagating through 6 inches of sea water the field would drop by $e^{-(84)\ (0.1524)} = 2.8 \times 10^{-6}$. This is not correct!

1.6 Current Flow in Good Conductors

If we have a DC current, the current will be uniformly distributed across the conductor cross section. However, in the AC case, the current is concentrated near the conductor surface.

Consider a semi-infinite slab of conducting material. A plane wave exists in the medium whose fields just below the top surface are expressed as:

$$\overline{E}(z=0^+) = \hat{x}E_0 \tag{1.140}$$
$$\overline{H}(z=0^+) = \hat{y}\frac{E_0}{\eta_c} \tag{1.141}$$



So, the plane waves are:

$$\overline{E}(z) = \hat{x}E_0 e^{-\alpha z}e^{-j\beta z} \tag{1.142}$$
$$\overline{H}(z) = \hat{y}\frac{E_0}{\eta_c}e^{-\alpha z}e^{-j\beta z} \tag{1.143}$$

Therefore, $\overline{J} = \sigma\overline{E} = \hat{x}\sigma E_0 e^{-\alpha z}e^{-j\beta z} = \hat{x}J_0 e^{-\alpha z}e^{-j\beta z}$. Let's again look at the expressions for α and β:

$$\alpha = \omega\left\{\frac{\mu\epsilon'}{2}\left[\sqrt{1+\left(\frac{\epsilon''}{\epsilon'}\right)^2}-1\right]\right\}^{1/2} \tag{1.144}$$

$$\beta = \omega\left\{\frac{\mu\epsilon'}{2}\left[\sqrt{1+\left(\frac{\epsilon''}{\epsilon'}\right)^2}+1\right]\right\}^{1/2} \tag{1.145}$$

If $\epsilon'' \gg \epsilon'$ (good conductor), then $\alpha = \beta = 1/\delta_s$. So,

$$\overline{J} = \hat{x}J_0 e^{-(1+j)z/\delta_s} \tag{1.146}$$

Finally, we explore the amount of current flowing through the region $0 \le y \le w$ and $0 \le z < \infty$.

$$\begin{aligned} I = \int_0^w \int_0^\infty J_0 e^{-(1+j)z/\delta_s} dz dy &= -J_0 w \frac{\delta_s}{1+j}\left[e^{-(1+j)\infty/\delta_s} - e^0\right] \\ &= J_0 w \frac{\delta_s}{1+j} \end{aligned} \tag{1.147}$$

Now, let's suppose we integrate in z only over the following ranges:

Integral in z over	**Error in calculating I is**
$0 \le z \le 3\delta_s$	5%
$0 \le z \le 5\delta_s$	1%

Therefore, we can treat the conductor as infinitely thick as long as the thickness is larger than about $5\delta_s$.

The basic principle is that the majority of the current flows within a few skin depths of the surface. Notice for example that for copper we have:

$$\begin{aligned} \sigma_c &= 5.8 \times 10^7 \text{ S/m} \\ \delta_s &= \frac{1}{\sqrt{\pi f \mu \sigma_c}} = 2.1\mu\text{m at } f = 1 \text{ GHz} \end{aligned}$$

Note that the expression for δ_s comes from Section 7-4 of your text. So, 99% of the current flows within 10μm of the surface.

Resistance

Remember that the impedance is the voltage divided by the total current. The voltage is given by

$$V = -\int \overline{E} \cdot dl \tag{1.148}$$

$$= El \tag{1.149}$$

The impedance is then given by

$$Z = (V)\left(\frac{1}{I}\right) \tag{1.150}$$

$$= (E_o\, l)\left(\frac{1+j}{\sigma E_o\, w\, \delta_s}\right) \tag{1.151}$$

$$= \frac{1+j}{\sigma\delta_s}\,\frac{l}{w} \tag{1.152}$$

So the surface resistance is given by

$$R_s = \frac{1}{\sigma\delta_s}\,\frac{l}{w} \tag{1.153}$$

$$= \sqrt{\frac{\pi f \mu}{\sigma}}\,\frac{l}{w} \tag{1.154}$$

Now what is the resistance per unit length of a coaxial transmission line. The width of the inner conductor is $w_{inner} = 2\pi a$ and of the outer conductor is $w_{outer} = 2\pi b$. The resulting resistance per unit length is then

$$R' = \sqrt{\frac{\pi f \mu}{\sigma}}\left(\frac{1}{w_{inner}} + \frac{1}{w_{outer}}\right) \tag{1.155}$$

$$= \sqrt{\frac{\pi f \mu}{\sigma}}\left(\frac{1}{2\pi}\right)\left(\frac{1}{a} + \frac{1}{b}\right) \tag{1.156}$$

This is the resistance given in Table 2-1 in the book.

1.7 Electromagnetic Power Density

1.7.1 Poynting Vector

Consider Maxwell's equations in the time domain modified as follows:

$$\overline{\mathcal{H}} \cdot \nabla \times \overline{\mathcal{E}} = -\overline{\mathcal{H}} \cdot \frac{\partial \overline{\mathcal{B}}}{\partial t} \tag{1.157}$$

$$\overline{\mathcal{E}} \cdot \nabla \times \overline{\mathcal{H}} = \overline{\mathcal{E}} \cdot \frac{\partial \overline{\mathcal{D}}}{\partial t} + \overline{\mathcal{E}} \cdot \overline{\mathcal{J}} \tag{1.158}$$

We have an identity: $\overline{\mathcal{H}} \cdot \nabla \times \overline{\mathcal{E}} - \overline{\mathcal{E}} \cdot \nabla \times \overline{\mathcal{H}} = \nabla \cdot (\overline{\mathcal{E}} \times \overline{\mathcal{H}})$. Therefore,

$$\nabla \cdot (\overline{\mathcal{E}} \times \overline{\mathcal{H}}) = -\overline{\mathcal{H}} \cdot \frac{\partial \overline{\mathcal{B}}}{\partial t} - \overline{\mathcal{E}} \cdot \frac{\partial \overline{\mathcal{D}}}{\partial t} - \overline{\mathcal{E}} \cdot \overline{\mathcal{J}} \tag{1.159}$$

Note that:

$$\frac{\partial}{\partial t}\left(\frac{1}{2}\mu \overline{\mathcal{H}} \cdot \overline{\mathcal{H}}\right) = \frac{1}{2}\mu\left(\frac{\partial \overline{\mathcal{H}}}{\partial t} \cdot \overline{\mathcal{H}} + \overline{\mathcal{H}} \cdot \frac{\partial \overline{\mathcal{H}}}{\partial t}\right) = \overline{\mathcal{H}} \cdot \frac{\partial(\mu \overline{\mathcal{H}})}{\partial t} = \overline{\mathcal{H}} \cdot \frac{\partial \overline{\mathcal{B}}}{\partial t}$$

Therefore,

$$\nabla \cdot (\overline{\mathcal{E}} \times \overline{\mathcal{H}}) + \frac{\partial}{\partial t}\left(\frac{1}{2}\mu \overline{\mathcal{H}} \cdot \overline{\mathcal{H}}\right) + \frac{\partial}{\partial t}\left(\frac{1}{2}\epsilon \overline{\mathcal{E}} \cdot \overline{\mathcal{E}}\right) + \sigma \overline{\mathcal{E}} \cdot \overline{\mathcal{E}} = 0$$

Let's integrate over a volume V and apply the divergence theorem to the first term:

$$\oint_S (\overline{\mathcal{E}} \times \overline{\mathcal{H}}) \cdot d\overline{s} + \frac{\partial}{\partial t} \int\int\int_V \left[\frac{1}{2}\mu \overline{\mathcal{H}} \cdot \overline{\mathcal{H}} + \frac{1}{2}\epsilon \overline{\mathcal{E}} \cdot \overline{\mathcal{E}}\right] dV + \int\int\int_V \sigma \overline{\mathcal{E}} \cdot \overline{\mathcal{E}} dV = 0 \tag{1.160}$$

This is **Poynting's Theorem** and represents a power balance for the fields. The units on each term (after integration) is Watts. We identify each term as:

$$\begin{aligned}
\oint_S (\overline{\mathcal{E}} \times \overline{\mathcal{H}}) \cdot d\overline{s} &= \text{total power leaving the volume (through the surface } S) \\
\frac{1}{2}\mu \overline{\mathcal{H}} \cdot \overline{\mathcal{H}} &= \text{stored magnetic energy density inside } V \\
\frac{1}{2}\epsilon \overline{\mathcal{E}} \cdot \overline{\mathcal{E}} &= \text{stored electric energy density inside } V \\
\frac{\partial}{\partial t} \int\int\int_V \left[\frac{1}{2}\mu \overline{\mathcal{H}} \cdot \overline{\mathcal{H}} + \frac{1}{2}\epsilon \overline{\mathcal{E}} \cdot \overline{\mathcal{E}}\right] dV &= \text{rate of } \textbf{increase} \text{ of stored energy} \\
\int\int\int_V \sigma \overline{\mathcal{E}} \cdot \overline{\mathcal{E}} dV &= \text{power lost to heat inside } V
\end{aligned} \tag{1.161}$$

Therefore, the theorem states that: *The power leaving the volume + the rate of increase in the stored energy + the power going into heat* = 0. Note that $\overline{\mathcal{E}} \times \overline{\mathcal{H}}$ has units of W/m^2. It represents a power density of the wave. We call it the **Poynting Vector**.

$$\overline{\mathcal{S}} = \overline{\mathcal{E}} \times \overline{\mathcal{H}} \tag{1.162}$$

Note that this is analogous to instantaneous power: $p(t) = v(t)i(t)$.

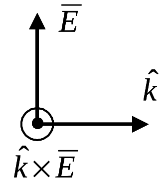
In the phasor domain, the average power delivered to a load in a circuit is

$$P = \frac{1}{2}\,\mathrm{Re}\{\tilde{V}\tilde{I}^*\}$$

Similarly, the time-average Poynting vector indicates the average real power density of a wave:

$$\overline{S}_{av} = \frac{1}{2}\,\mathrm{Re}\{\overline{E} \times \overline{H}^*\} \tag{1.163}$$

1.7.2 Poynting Vector for Plane Waves



We know that $\overline{H}(z) = \hat{k} \times \overline{E}(z)/\eta_c$. Therefore

$$\overline{E} \times \overline{H}^* = \frac{1}{\eta_c^*}\overline{E} \times (\overline{k} \times \overline{E})^* = \frac{1}{\eta_c^*}|\overline{E}|^2\overline{k}$$

So,

$$\overline{S}_{av} = \frac{1}{2}\ \mathrm{Re}\{\overline{E} \times \overline{H}^*\} = \frac{|\overline{E}|^2}{2}\ \mathrm{Re}\left\{\frac{1}{\eta_c^*}\right\}\hat{k}$$

For lossless media:

$$\overline{S}_{av} = \frac{|\overline{E}|^2}{2\eta}\hat{k} \tag{1.164}$$

Solar Illumination

$|\overline{S}_{av}| = 1$ kW/m^2 at the Earth's surface due to sun radiation.

$R_e = 6380$ km = earth radius

$R_s = 1.5 \times 10^8$ km = radius of earth's orbit around sun

1. Find the total power radiated by the sun

$$P_{sun} = S_{av}(4\pi R_s^2) = 2.8 \times 10^{26} W$$

2. Find the total power intercepted by the earth. The earth's cross sectional area is $A_e = \pi R_e^2$.

$$P_{earth} = S_{av}\pi R_e^2 = 1.28 \times 10^{17} W$$

3. Find the electric field strength at the earth (assuming single frequency)

$$S_{av} = \frac{|E_0|^2}{2\eta_0} \rightarrow |E_0| = \sqrt{2\eta_0 S_{av}} = 870 \text{ V/m}$$

Wireless Signals

How does the power from a wireless communications system antenna decrease with range?