Chapter 1

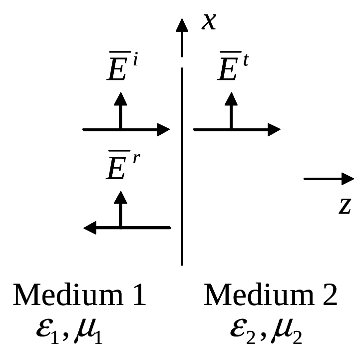
Reflection and Refraction

We are now interested in exploring what happens when a plane wave traveling in one medium encounters an interface (boundary) with another medium. Understanding this phenomenon allows us to understand things like:

1. How an optical lens works
2. Why we get glare off a pane of glass (and how to design anti-glare windows)
3. How buildings and walls influence cellular-phone signals
4. Why light bends when it enters water
5. Optical fibers

1.1 Normal Incidence

We will begin by looking at a wave normally incident on the interface between lossless media.



1.1.1 Fields

In general, we must allow for three different waves in the system:

1. Incident wave

$$\begin{aligned}
\overline{E}^i(z) &= \hat{x}E_o^i e^{-jk_1 z} \\
\overline{H}^i(z) &= \frac{1}{\eta_1}\hat{k}^i \times \overline{E}^i = \hat{y}\frac{E_o^i}{\eta_1}e^{-jk_1 z} \qquad (\hat{k}^i = \hat{z}) \\
k_1 &= \omega\sqrt{\mu_1\epsilon_1} \qquad \eta_1 = \sqrt{\frac{\mu_1}{\epsilon_1}}
\end{aligned}$$

2. Reflected wave

$$\begin{aligned}
\overline{E}^r(z) &= \hat{x}E_o^r e^{+jk_1 z} \\
\overline{H}^r(z) &= \frac{1}{\eta_1}\hat{k}^r \times \overline{E}^r = -\hat{y}\frac{E_o^r}{\eta_1}e^{+jk_1 z} \qquad (\hat{k}^r = -\hat{z})
\end{aligned}$$

3. Transmitted wave

$$\begin{aligned}
\overline{E}^t(z) &= \hat{x}E_o^t e^{-jk_2 z} \\
\overline{H}^t(z) &= \frac{1}{\eta_2}\hat{k}^t \times \overline{E}^t = \hat{y}\frac{E_o^t}{\eta_2}e^{-jk_2 z} \qquad (\hat{k}^t = \hat{z}) \\
k_2 &= \omega\sqrt{\mu_2\epsilon_2} \qquad \eta_2 = \sqrt{\frac{\mu_2}{\epsilon_2}}
\end{aligned}$$

We assume we know E_o^i. We must determine E_o^r and E_o^t.

1.1.2 Boundary Conditions

We finally get to use those boundary conditions for something useful. Since there is no surface current on the boundary between two dielectrics:

$$\hat{n} = \hat{z}: \qquad \begin{aligned} \hat{z} \times (\overline{E}_2 - \overline{E}_1)\big|_{z=0} &= 0 \\ \hat{z} \times (\overline{H}_2 - \overline{H}_1)\big|_{z=0} &= 0 \end{aligned}$$

At $z = 0$ we have:

$$\begin{aligned}
\overline{E}_1 &= \overline{E}^i(0) + \overline{E}^r(0) = \hat{x}(E_o^i + E_o^r) \\
\overline{H}_1 &= \overline{H}^i(0) + \overline{H}^r(0) = \hat{y}\frac{(E_o^i - E_o^r)}{\eta_1} \\
\overline{E}_2 &= \overline{E}^t(0) = \hat{x}E_o^t \\
\overline{H}_2 &= \overline{H}^t(0) = \hat{y}\frac{E_o^t}{\eta_2}
\end{aligned}$$

Therefore, application of the boundary conditions gives:

$$\hat{z} \times \left[\hat{x}E_o^t - \hat{x}(E_o^i + E_o^r)\right] = 0 \quad \rightarrow \quad E_o^t - (E_o^i + E_o^r) = 0 \tag{1.1}$$

$$\hat{z} \times \left[\hat{y}\frac{E_o^t}{\eta_2} - \hat{y}\frac{(E_o^i - E_o^r)}{\eta_1}\right] = 0 \quad \rightarrow \quad \frac{E_o^t}{\eta_2} - \frac{(E_o^i - E_o^r)}{\eta_1} = 0 \tag{1.2}$$

Substitution of the result $E_o^t = (E_o^i + E_o^r)$ into (**??**) gives

$$
\begin{aligned}
\frac{(E_o^i + E_o^r)}{\eta_2} - \frac{(E_o^i - E_o^r)}{\eta_1} = 0 \quad \rightarrow \quad & E_o^r\left(\frac{1}{\eta_1} + \frac{1}{\eta_2}\right) = E_o^i\left(\frac{1}{\eta_1} - \frac{1}{\eta_2}\right) \\
& E_o^r = \left(\frac{\eta_2 - \eta_1}{\eta_2 + \eta_1}\right) E_o^i = \Gamma E_o^i \\
E_o^t = E_o^i + \left(\frac{\eta_2 - \eta_1}{\eta_2 + \eta_1}\right) E_o^i \quad = \quad & \left(\frac{\eta_2 + \eta_1 + \eta_2 - \eta_1}{\eta_2 + \eta_1}\right) E_o^i \\
& E_o^t = \left(\frac{2\eta_2}{\eta_2 + \eta_1}\right) E_o^i = \tau E_o^i
\end{aligned}
$$

So, for **Normal Incidence**:

$$
\begin{aligned}
\Gamma &= \frac{\eta_2 - \eta_1}{\eta_2 + \eta_1} = \text{reflection coefficient} \\
\tau &= \frac{2\eta_2}{\eta_2 + \eta_1} = \text{transmission coefficient}
\end{aligned}
$$

Notice that we can write: $E_o^t = E_o^i + \Gamma E_o^i = (1+\Gamma)E_o^i$ as well as $E_o^t = \tau E_o^i$. Therefore, for normal incidence:

$$\tau = 1 + \Gamma$$

We also observe that if region 2 is a perfect conductor, $\sigma_2 \rightarrow \infty$, $\epsilon_{c2} \rightarrow \infty$, and $\eta_2 = 0$. This results in $\Gamma = -1$ which is consistent with the boundary condition that the tangential electric field must go to zero at the boundary of the conductor (short circuit).

Transmission Line Analogue

The expression for Γ is very similar to what we found in transmission lines, with the intrinsic impedance replacing the characteristic impedance. In fact, this system behaves very much like a transmission line system, and we have the same standing wave concept in region 1 that we have on a transmission line.

Electromagnetic Wave	Electrical Transmission Line
$\overline{E}$	V
$\overline{H}$	I
η	Z_o
Γ	Γ_r
$\overline{E}_1 = \hat{x}\overline{E}_o^i\left(e^{-jk_1z} + \Gamma e^{jk_1z}\right)$	$V = V_o^+\left(e^{-jkz} + \Gamma e^{jkz}\right)$

$$\overline{E}_1 = \hat{x}E_o^i\left(e^{-jk_1z} + \Gamma e^{jk_1z}\right)$$

which leads to the standing wave pattern:

$$
\begin{aligned}
|\overline{E}_1| &= |E_o^i|\left|1 + \Gamma e^{j2k_1z}\right| \\
|\overline{E}_1|_{\max} &= |E_o^i|\left(1 + |\Gamma|\right) \\
|\overline{E}_1|_{\min} &= |E_o^i|\left(1 - |\Gamma|\right)
\end{aligned}
$$

The standing wave ratio (SWR) is:

$$S = \frac{|\overline{E}_1|_{\max}}{|\overline{E}_1|_{\min}} = \frac{1+|\Gamma|}{1-|\Gamma|}$$

This is exactly the same expression we obtained for transmission lines.

As in transmission lines, the standing wave pattern is periodic with period ℓ.

$$\begin{aligned} e^{j2k_1\ell} = e^{j2\pi} \quad \rightarrow \quad & k_1\ell = \pi \\ & \frac{2\pi}{\lambda_1}\ell = \pi \\ & \ell = \frac{\lambda_1}{2} \end{aligned}$$

Again we see that the distance between adjacent maxima (or minima) is a half wavelength.

Continuing on with the transmission line analogue, the characteristic impedance of air is given by

$$\eta_{air} = \sqrt{\frac{\mu_o}{\epsilon_o}} = 377\ \Omega. \tag{1.3}$$

The characteristic impedance of a dielectric is

$$\eta = \sqrt{\frac{\mu_o}{\epsilon_r\ \epsilon_o}} = \frac{377}{n}\ \Omega, \tag{1.4}$$

where n is the index of refraction of the material. The characteristic impedance of a conductor is

$$\eta = (1+j)\sqrt{\frac{\pi f \mu}{\sigma}} \sim 0. \tag{1.5}$$

So a conductor acts as a short circuit.

1.1.3 Power Flow in Region 1

In region 1, we have the fields:

$$\begin{aligned} \overline{E}_1 &= \hat{x}E_o^i\left(e^{-jk_1z} + \Gamma e^{jk_1z}\right) \\ \overline{H}_1 &= \hat{y}\frac{E_o^i}{\eta_1}\left(e^{-jk_1z} - \Gamma e^{jk_1z}\right) \end{aligned}$$

The time-average Poynting vector is:

$$\begin{aligned} \overline{S}_{av,1} = \frac{1}{2}\ \mathrm{Re}\left\{\overline{E}_1 \times \overline{H}_1^*\right\} &= \frac{1}{2}\ \mathrm{Re}\left\{E_o^i\left(e^{-jk_1z} + \Gamma e^{jk_1z}\right)\frac{E_o^{i*}}{\eta_1}\left(e^{jk_1z} - \Gamma^* e^{-jk_1z}\right)\hat{x}\times\hat{y}\right\} \\ &= \frac{1}{2}\ \mathrm{Re}\left\{\frac{|E_o^i|^2}{\eta_1}\left[1 - \Gamma^* e^{-j2k_1z} + \Gamma e^{j2k_1z} - |\Gamma|^2\right]\right\}\hat{z} \end{aligned}$$

Recall that $A - A^* = (a+jb) - (a-jb) = j2b = 2\ \mathrm{Im}\{A\}$.

$$\begin{aligned} \overline{S}_{av,1} &= \frac{1}{2}\ \mathrm{Re}\left\{\frac{|E_o^i|^2}{\eta_1}\left[1 - |\Gamma|^2 + 2\ \mathrm{Im}\left\{\Gamma e^{j2k_1z}\right\}\right]\right\}\hat{z} \\ &= \hat{z}\frac{|E_o^i|^2}{2\eta_1}\left[1 - |\Gamma|^2\right] \end{aligned}$$

Note that

$$\overline{S}_{av}^{i} = \hat{z}\frac{|E_o^i|^2}{2\eta_1}$$
$$\overline{S}_{av}^{r} = -\hat{z}\frac{|E_o^i|^2}{2\eta_1}|\Gamma|^2 = -|\Gamma|^2\overline{S}_{av}^{i}$$

So, $\overline{S}_{av,1}$ is simply the sum of the power densities in the incident and reflected waves.

The average power density in region 2 is:

$$\overline{S}_{av,2} = \hat{z}\frac{|\overline{E}_2|^2}{2\eta_2} = \hat{z}|\tau|^2\frac{|E_o^i|^2}{2\eta_2}$$

For lossless media (which we are working with here), $|\Gamma|^2 = \Gamma^2$ and $|\tau|^2 = \tau^2$. Therefore:

$$\begin{aligned}
\frac{1-\Gamma^2}{\eta_1} = \frac{1-\left(\frac{\eta_2-\eta_1}{\eta_2+\eta_1}\right)^2}{\eta_1} &= \frac{(\eta_2+\eta_1)^2-(\eta_2-\eta_1)^2}{\eta_1(\eta_2-\eta_1)^2} \\
&= \frac{\eta_2^2+\eta_1^2+2\eta_1\eta_2-\eta_2^2-\eta_1^2+2\eta_1\eta_2}{\eta_1(\eta_2-\eta_1)^2} \\
&= \frac{4\eta_2}{(\eta_2-\eta_1)^2} \\
&= \frac{1}{\eta_2}\left[\frac{2\eta_2}{\eta_2-\eta_1}\right]^2 = \frac{\tau^2}{\eta_2}
\end{aligned}$$

So:

$$\frac{1-\Gamma^2}{\eta_1} = \frac{\tau^2}{\eta_2}$$

This means:

$$\overline{S}_{av,1} = \hat{z}\frac{|E_o^i|^2}{2\eta_1}[1-\Gamma^2] = \hat{z}\frac{|E_o^i|^2}{2}\frac{\tau^2}{\eta_2} = \overline{S}_{av,2}$$

1.1.4 Lossy Media

We can use our developments above, and simply make the substitutions:

$$\begin{aligned}
jk_1 &\rightarrow \gamma_1 \\
jk_2 &\rightarrow \gamma_2 \\
\eta_1 &\rightarrow \eta_{c1} \\
\eta_2 &\rightarrow \eta_{c2}
\end{aligned}$$

Then, all of the equations will work fine for lossy media.

1.2 Oblique (Non-normal) Incidence

We need to first become familiar with the idea of waves traveling in a direction other than $\pm z$. In order for the phase progression to occur along the $\hat{k}$ direction, we need to use the component of the position vector $\overline{r}$ that points along the $\hat{k}$ direction. So, our distance variable is:

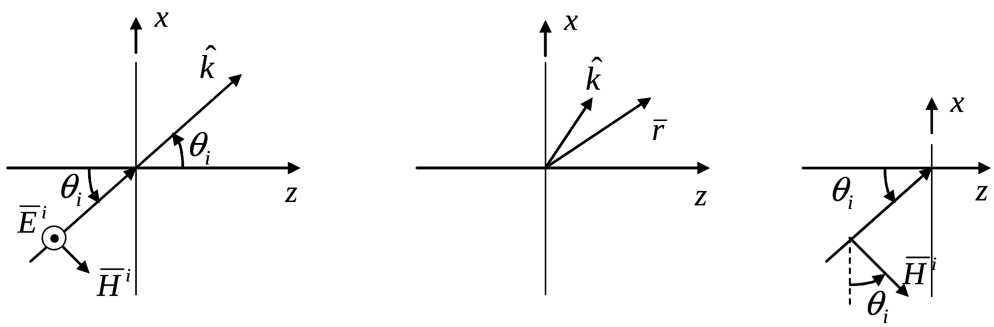
$$\begin{aligned} \xi^i = \hat{k}^i \cdot \overline{r} &= (\hat{x}\sin\theta_i + \hat{z}\cos\theta_i) \cdot (\hat{x}x + \hat{z}z) \\ &= x\sin\theta_i + z\cos\theta_i \end{aligned}$$

The plane wave exponential is therefore

$$e^{-jk_1\hat{k}^i\cdot\overline{r}} = e^{-jk_1(x\sin\theta_i + z\cos\theta_i)}$$

creating the fields:

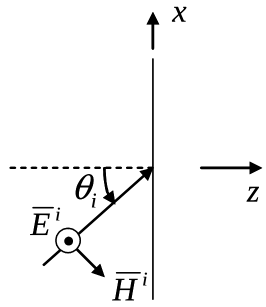
$$\begin{aligned} \overline{E}^i &= \hat{y}E_o^i e^{-jk_1(x\sin\theta_i + z\cos\theta_i)} \\ \overline{H}^i &= \frac{1}{\eta_1}\hat{k}^i \times \overline{E}^i \\ &= \frac{1}{\eta_1}(\hat{x}\sin\theta_i + \hat{z}\cos\theta_i) \times \hat{y}E_o^i e^{-jk_1(x\sin\theta_i + z\cos\theta_i)} \\ &= \frac{E_o^i}{\eta_1}(-\hat{x}\cos\theta_i + \hat{z}\sin\theta_i) e^{-jk_1(x\sin\theta_i + z\cos\theta_i)} \end{aligned}$$



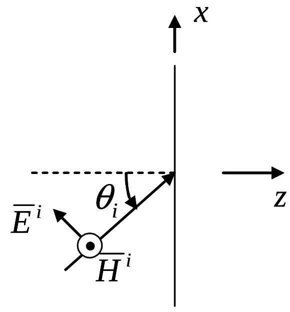
Note that you can also deduce $\overline{H}^i$ using trigonometry.

Finally, we must make some definitions:

Plane of Incidence:	Plane containing normal to boundary and $\hat{k}$ vector
Perpendicular Polarization:	$\overline{E}$ perpendicular to the plane of incidence. Also called *Transverse Electric* (TE) polarization.
Parallel Polarization:	$\overline{E}$ parallel to the plane of incidence. Also called the *Transverse Magnetic* (TM) polarization.

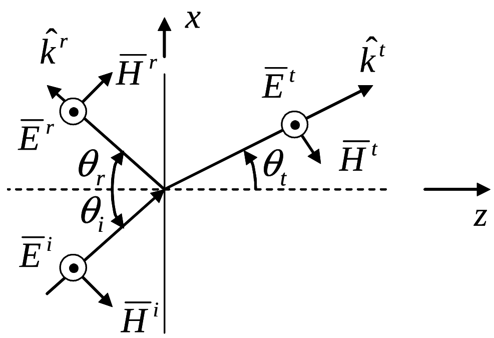


Perpendicular



Parallel

1.2.1 Perpendicular (TE) Polarization



1.2.2 Fields

1. Incident wave

$$\begin{aligned}\overline{E}^i &= \hat{y}E_o^i e^{-jk_1(x\sin\theta_i + z\cos\theta_i)}\\ \overline{H}^i &= (-\hat{x}\cos\theta_i + \hat{z}\sin\theta_i)\frac{E_o^i}{\eta_1}e^{-jk_1(x\sin\theta_i+z\cos\theta_i)}\end{aligned}$$

2. Reflected wave

$$\begin{aligned}\overline{E}^r &= \hat{y}E_o^r e^{-jk_1(x\sin\theta_r - z\cos\theta_r)}\\ \overline{H}^r &= (\hat{x}\cos\theta_r + \hat{z}\sin\theta_r)\frac{E_o^r}{\eta_1}e^{-jk_1(x\sin\theta_r-z\cos\theta_r)}\end{aligned}$$

3. Transmitted wave

$$\begin{aligned}\overline{E}^t &= \hat{y}E_o^t e^{-jk_2(x\sin\theta_t + z\cos\theta_t)}\\ \overline{H}^t &= (-\hat{x}\cos\theta_t + \hat{z}\sin\theta_t)\frac{E_o^t}{\eta_2}e^{-jk_2(x\sin\theta_t+z\cos\theta_t)}\end{aligned}$$

We now have 4 unknowns: E_o^r, E_o^t, θ_r, θ_t.

1.2.3 Boundary Conditions

$$\begin{array}{ll} \text{Tangential } \overline{E} & \hat{n} \times (\overline{E}^i + \overline{E}^r)\Big|_{z=0} = \hat{n} \times \overline{E}^t\Big|_{z=0} \\ & E_o^i e^{-jk_1 x \sin\theta_i} + E_o^r e^{-jk_1 x \sin\theta_r} = E_o^t e^{-jk_2 x \sin\theta_t} \\ \text{Tangential } \overline{H} & \hat{n} \times (\overline{H}^i + \overline{H}^r)\Big|_{z=0} = \hat{n} \times \overline{H}^t\Big|_{z=0} \\ & -\frac{E_o^i}{\eta_1}\cos\theta_i e^{-jk_1 x \sin\theta_i} + \frac{E_o^r}{\eta_1}\cos\theta_r e^{-jk_1 x \sin\theta_r} = -\frac{E_o^t}{\eta_2}\cos\theta_t e^{-jk_2 x \sin\theta_t} \end{array}$$

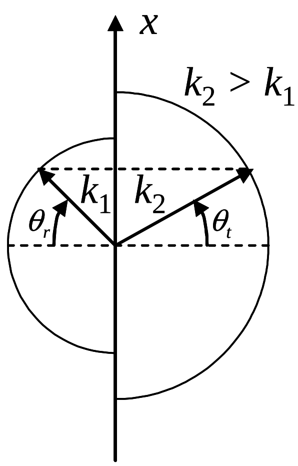
In order for the two equations to be satisfied for all x, the arguments of the exponentials must be equal. We call this the *phase matching condition*.

$$k_1 \sin\theta_i = k_1 \sin\theta_r = k_2 \sin\theta_t$$

This produces Snell's Laws:

$\theta_i = \theta_r$	Snell's Law of Reflection
$k_1 \sin\theta_i = k_2 \sin\theta_t$	Snell's Law of Refraction

We observe that this phase matching condition indicates the direction of propagation of the reflected and transmitted waves. A nice picture can be drawn of this:



$k_1 \sin\theta_i = k_1 \sin\theta_r = k_2 \sin\theta_t$
tangential component of $\hat{k}^r k_1$ equals tangential component of $\hat{k}^t k_2$
This implies that the phase is continous across the boundary

We can now proceed with application of the boundary conditions:

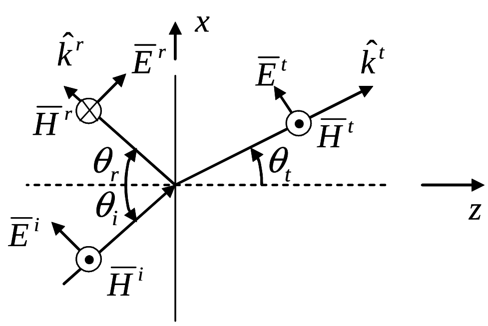
$$\begin{aligned} E_o^i + E_o^r &= E_o^t \\ \frac{E_o^i}{\eta_1}\cos\theta_i - \frac{E_o^r}{\eta_1}\cos\theta_i &= \frac{E_o^t}{\eta_2}\cos\theta_t \end{aligned}$$

$$\frac{\cos\theta_i}{\eta_1}(E_o^i - E_o^r) = \frac{\cos\theta_t}{\eta_2}(E_o^i + E_o^r)$$

Solving produces:

$$\begin{aligned} E_o^r &= \frac{\eta_2/\cos\theta_t - \eta_1/\cos\theta_i}{\eta_2/\cos\theta_t + \eta_1/\cos\theta_i} E_o^i = \Gamma_\perp E_o^i \\ E_o^t &= \frac{2\eta_2/\cos\theta_t}{\eta_2/\cos\theta_t + \eta_1/\cos\theta_i} E_o^i = \tau_\perp E_o^i = (1+\Gamma_\perp) E_o^i \end{aligned}$$

1.2.4 Parallel (TM) Polarization



1.2.5 Fields

1. Incident wave

$$\begin{aligned}\overline{E}^i &= (\hat{x}\cos\theta_i - \hat{z}\sin\theta_i)E_o^i e^{-jk_1(x\sin\theta_i + z\cos\theta_i)}\\ \overline{H}^i &= \hat{y}\frac{E_o^i}{\eta_1}e^{-jk_1(x\sin\theta_i + z\cos\theta_i)}\end{aligned}$$

2. Reflected wave

$$\begin{aligned}\overline{E}^r &= (\hat{x}\cos\theta_r + \hat{z}\sin\theta_r)E_o^r e^{-jk_1(x\sin\theta_r - z\cos\theta_r)}\\ \overline{H}^r &= -\hat{y}\frac{E_o^r}{\eta_1}e^{-jk_1(x\sin\theta_r - z\cos\theta_r)}\end{aligned}$$

3. Transmitted wave

$$\begin{aligned}\overline{E}^t &= (\hat{x}\cos\theta_t - \hat{z}\sin\theta_t)E_o^t e^{-jk_2(x\sin\theta_t + z\cos\theta_t)}\\ \overline{H}^t &= \hat{y}\frac{E_o^t}{\eta_2}e^{-jk_2(x\sin\theta_t + z\cos\theta_t)}\end{aligned}$$

1.2.6 Boundary Conditions

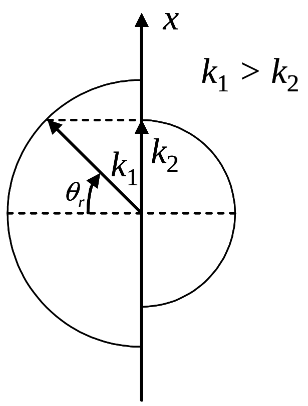
$$\begin{aligned}\text{Tangential } \overline{E} \qquad & \hat{n}\times(\overline{E}^i + \overline{E}^r)\Big|_{z=0} = \hat{n}\times\overline{E}^t\Big|_{z=0}\\ & E_o^i\cos\theta_i e^{-jk_1 x\sin\theta_i} + E_o^r\cos\theta_r e^{-jk_1 x\sin\theta_r} = E_o^t\cos\theta_t e^{-jk_2 x\sin\theta_t}\\ \text{Tangential } \overline{H} \qquad & \hat{n}\times(\overline{H}^i + \overline{H}^r)\Big|_{z=0} = \hat{n}\times\overline{H}^t\Big|_{z=0}\\ & \frac{E_o^i}{\eta_1}e^{-jk_1 x\sin\theta_i} - \frac{E_o^r}{\eta_1}e^{-jk_1 x\sin\theta_r} = \frac{E_o^t}{\eta_2}e^{-jk_2 x\sin\theta_t}\end{aligned}$$

Phase matching again produces Snell's Laws. Then, application of the boundary conditions produces

$$\begin{aligned}E_o^r &= \frac{\eta_2\cos\theta_t - \eta_1\cos\theta_i}{\eta_2\cos\theta_t + \eta_1\cos\theta_i}E_o^i = \Gamma_\parallel E_o^i\\ E_o^t &= \frac{2\eta_2\cos\theta_i}{\eta_2\cos\theta_t + \eta_1\cos\theta_i}E_o^i = \tau_\parallel E_o^i\end{aligned}$$

1.3 Total Internal Reflection

Suppose we have the case where $k_1 > k_2$ (generally $\epsilon_1 > \epsilon_2$). Then, our k-vector diagram looks like:



If $\theta_i = \theta_r$ gets too large, we have the case where $k_2 \sin\theta_t$ can't be large enough to equal $k_1 \sin\theta_i$.

When the incidence angle θ_i is such that

$$k_1 \sin\theta_i = k_2 \text{ (so } \theta_t = 90^\circ)$$

we call $\theta_i = \theta_c$ the *critical angle*

In this case, the transmitted wave travels parallel to the interface so that the constant phase planes are parallel to the y-z plane.

Notice that if $\theta_i > \theta_c$:

$$k_2 \sin\theta_t = k_1 \sin\theta_i \rightarrow \sin\theta_t = \frac{k_1}{k_2}\sin\theta_i > 1$$

A real angle θ_t cannot satisfy this equation. In fact, θ_t becomes a complex number. Rather than determining θ_t, we can determine $\sin\theta_t$ and $\cos\theta_t$. $\sin\theta_t$ is determined by Snell's Law as above, which can be rewritten

$$\sin\theta_t = \frac{k_1}{k_2}\sin\theta_i = \frac{\beta}{k_2}$$

Then

$$\cos\theta_t = \sqrt{1-\sin^2\theta_t} = \sqrt{1-\left(\frac{k_1}{k_2}\right)^2\sin^2\theta_i} = \pm j\sqrt{\left(\frac{k_1}{k_2}\right)^2\sin^2\theta_i - 1} = \pm j\frac{\alpha}{k_2}$$

We will chose the negative root for reasons to be seen below.

The propagation term in the argument of the exponential for the transmitted wave becomes

$$k_2(x\sin\theta_t + z\cos\theta_t) = k_2\left(x\frac{\beta}{k_2} - jz\frac{\alpha}{k_2}\right) = \beta x - j\alpha z$$

Therefore, we can write the transmitted field as

$$\overline{E}^t = \hat{y}E_o^t e^{-jk_2(x\sin\theta_t + z\cos\theta_t)} = \hat{y}E_o^t e^{-\alpha z}e^{-j\beta x}$$

The wave decays in z, but phase progression (propagation) occurs in x. Note that at $\theta_i = \theta_c$, $\cos\theta_t = 0$ so $\alpha = 0$ (no decay).

We call such a wave an **evanescent wave**. The reflection coefficient for this case is

$$\begin{aligned}\Gamma_\perp &= \frac{\eta_2/\cos\theta_t - \eta_1/\cos\theta_i}{\eta_2/\cos\theta_t + \eta_1/\cos\theta_i} = \frac{j\eta_2/\alpha - \eta_1/\cos\theta_i}{j\eta_2/\alpha + \eta_1/\cos\theta_i} = -\frac{A^*}{A} \\ |\Gamma_\perp| &= \frac{|A^*|}{|A|} = 1\end{aligned}$$

The same is true for $\Gamma_{\|}$. Therefore, *no real power is transferred across the interface.* All power is reflected. We therefore call this *total internal reflection.*

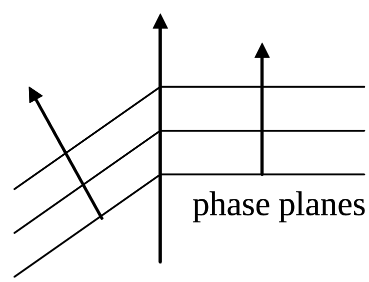
We can also write:

$$\begin{aligned}
\overline{H}^t &= (-\hat{x}\cos\theta_t + \hat{z}\sin\theta_t)\frac{E_o^t}{\eta_2}e^{-jk_2(x\sin\theta_t + z\cos\theta_t)} \\
&= (\hat{x}j\sqrt{\sin^2\theta_t - 1} + \hat{z}\sin\theta_t)\frac{E_o^t}{\eta_2}e^{-\alpha z}e^{-j\beta x} \\
\overline{S}^t &= \overline{E}^t \times \overline{H}^{t*} \\
&= E_o^t e^{-\alpha z}e^{-j\beta x}\frac{E_o^{t*}}{\eta_2}e^{-\alpha z}e^{j\beta x}\left[\hat{z}j\sqrt{\sin^2\theta_t - 1} + \hat{x}\sin\theta_t\right] \\
&= \frac{|E_o^t|^2}{\eta_2}e^{-2\alpha z}\left[\hat{x}\sin\theta_t + \hat{z}j\sqrt{\sin^2\theta_t - 1}\right]
\end{aligned}$$

This indicates that the power traveling in the $\hat{z}$ direction is reactive (stored energy). The real power is:

$$\overline{S}_{av}^t = \frac{1}{2}\frac{|E_o^t|^2}{\eta_2}e^{-2\alpha z}\hat{x}\sin\theta_t$$

which flows in the $\hat{x}$ direction. Note that this power flow simply represents a power flow related to phase progression in the x direction. It takes no power from the incident wave to sustain this.



1.4 Total Transmission (Brewster's Angle)

Suppose we want $\Gamma = 0$ (no reflection)

$$\Gamma_\perp = \frac{\eta_2/\cos\theta_t - \eta_1/\cos\theta_i}{\eta_2/\cos\theta_t + \eta_1/\cos\theta_i} = 0$$

$$\begin{aligned}
\eta_2\cos\theta_i &= \eta_1\cos\theta_t \\
\frac{\mu_2}{\epsilon_2}\left[1 - \sin^2\theta_i\right] &= \frac{\mu_1}{\epsilon_1}\left[1 - \sin^2\theta_t\right] = \frac{\mu_1}{\epsilon_1}\left[1 - \left(\frac{k_1}{k_2}\right)^2\sin^2\theta_i\right] = \frac{\mu_1}{\epsilon_1}\left[1 - \frac{\mu_1\epsilon_1}{\mu_2\epsilon_2}\sin^2\theta_i\right]
\end{aligned}$$

We can then solve this to obtain

$$\sin\theta_i = \sqrt{\frac{1 - (\mu_1\epsilon_2/\mu_2\epsilon_1)}{1 - (\mu_1/\mu_2)^2}}$$

If $\mu_1 = \mu_2$, $\sin\theta_i = \infty$ which is impossible. So, we can't have total transmission for the perpendicular polarization in non-magnetic media.

For the parallel polarization:

$$\Gamma_{\|} = \frac{\eta_2\cos\theta_t - \eta_1\cos\theta_i}{\eta_2\cos\theta_t + \eta_1\cos\theta_i} = 0$$

$$\eta_2\cos\theta_t = \eta_1\cos\theta_i$$

The solution for this (following an analogous procedure) is:

$$\sin\theta_i = \sqrt{\frac{1-(\epsilon_1\mu_2/\epsilon_2\mu_1)}{1-(\epsilon_1/\epsilon_2)^2}}$$

Here, if $\mu_1 = \mu_2$:

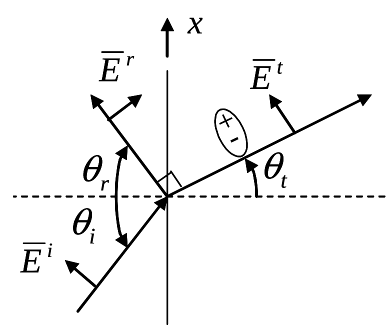
$$\begin{aligned}
\sin\theta_B &= \sqrt{\frac{1-\epsilon_1/\epsilon_2}{1-(\epsilon_1/\epsilon_2)^2}} = \sqrt{\frac{1}{1+\epsilon_1/\epsilon_2}} \\
\cos\theta_B &= \sqrt{1-\frac{1}{1+\epsilon_1/\epsilon_2}} = \sqrt{\frac{\epsilon_1/\epsilon_2}{1+\epsilon_1/\epsilon_2}} \\
\tan\theta_B &= \frac{\sin\theta_B}{\cos\theta_B} = \sqrt{\frac{\epsilon_2}{\epsilon_1}} \\
\theta_B &= \tan^{-1}\left(\sqrt{\frac{\epsilon_2}{\epsilon_1}}\right)
\end{aligned}$$

We call θ_B *Brewster's Angle.*

To explain Brewster's angle physically, we have to revisit the permittivity. Recall that materials are made of atoms whose charge will shift slightly under the influence of an electric field (the atoms or molecules will polarize, with positive and negative charge separating a bit to made a dipole). The permittivity ϵ represents the susceptibility of the medium to this alignment. When a time-varying field hits the medium, these dipoles oscillate. This charge motion represents a current which causes re-radiation of the electromagnetic field. This explains why a reflected field get's created.

We will later see that dipoles do not radiate along their polar axis. The oscillating dipole in material 2 won't radiate in the direction of the reflection. Therefore, $E^r = 0$.
It can be shown that

$$\theta_B + \theta_t = 90^\circ$$



Note that for the perpendicular polarization, the dipole oscillation is normal to the page, so we will always have radiation in the reflected direction. Therefore, there is no Brewster angle for this case.