ECEn380
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The One-Armed Driver

Consider a car driving down a road which meanders slightly left and right while maintaining generally the same direction, as illustrated in Figure 1.

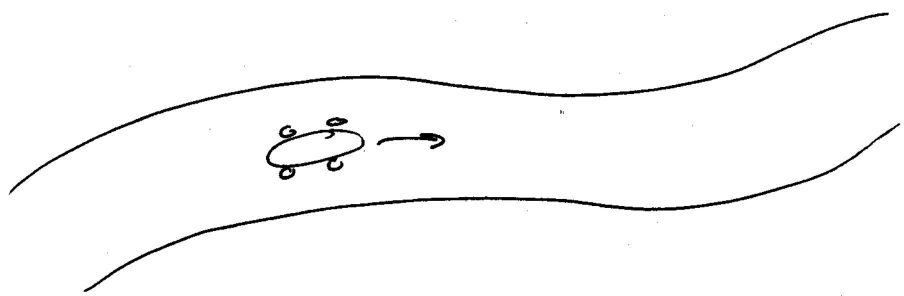

Figure 1: Car on a Meandering Road

Let the dominant direction of travel be the abscissa in the plot provided in Figure 2, and take the right-left deviations of the road and car as the ordinate. The solid line is the desired path, or the deviation the road makes from the nominal direction, and the dashed like represents the actual path taken by the car. The difference between these two plots is the error. Assuming that the car is proceeding at a constant speed, this graph is also a plot of deviations versus time.

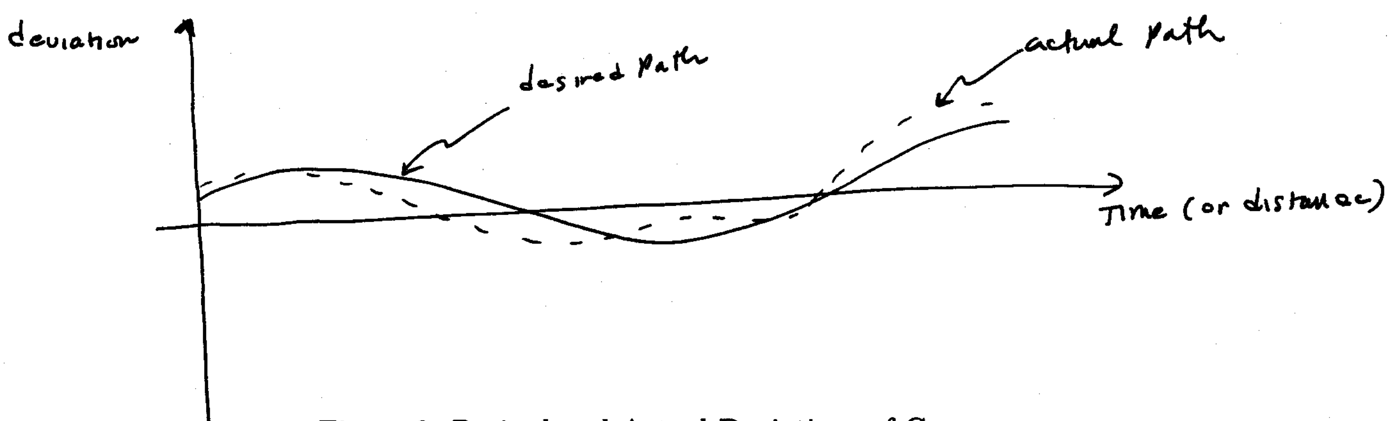

Figure 2: Desired and Actual Deviations of Car

The desired path-time function (solid line) can be considered as the input, $x[n]$, to some system whose output, $y[n]$, is the actual path taken by the vehicle (dashed line). The car-driver, man-machine combination is the system which is attempting to reproduce the road deflection, $x[n]$, in the motion of the car, $y[n]$. This kind of a system is called a control system or servomechanism.

System Model

Due to circumstances beyond the control of the driver, it manages to get a glimpse of the road only at discrete times. For simplicity we assume that these glimpses are obtained at regular intervals. In effect, then, the driver acts on sampled data. We shall only desire to know the position of the car at these same times, so that for our purposes the problem consists of relating the output samples, $y[n]$, to the input samples, $x[n]$.

Now consider the system mechanism. A reasonable model for this system is shown in the following block diagram:

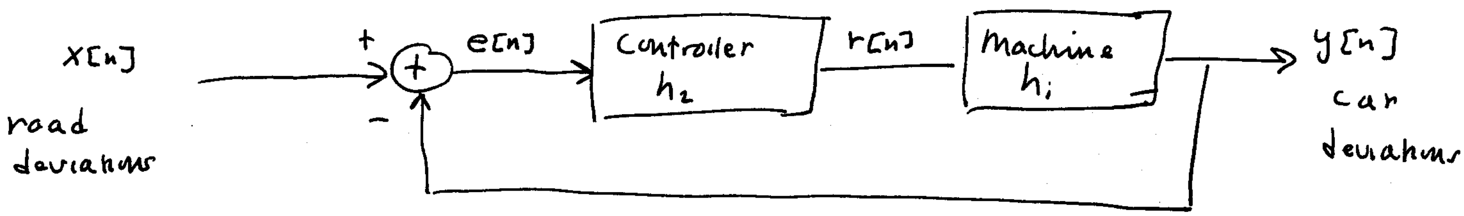

Figure 3: Block Diagram of Man-Machine System

This diagram simply states that the driver observes the difference between the desired positions, $x[n]$, and the actual positions, $y[n]$, that he processes these differences in some way to determine how much to turn the steering wheel of the car, and that the car responds to produce the output motion $y[n]$. We assume that the steering wheel is moved abruptly at the sampling times so that the car changes direction quickly at these times. Between samples the car proceeds in a straight line. The steering wheel motion is then a momentary right or left twist, the value of which we call the signal $r[n]$. We take $r[n]$ positive for left turns, negative for right turns.

To describe the machine portion of the system we require the relation between car motion $y[n]$ and wheel motion $r[n]$. Evidently a unit sample input ($r[n] = \delta[n]$) to the quiescent machine system results in some changes in direction to the left which produces a linearly increasing deflection, $h_1[n] = Cnu[n]$, as shown in Figure 4.

By conveniently defining $r[n]$ physically we can insure that $r[n] = a\delta[n]$ produces a linear ramp, $y[n] = ah_1[n]$, with a times the rate of increase of the response to a $\delta[n]$. Moreover, we see that since we are considering only small deflections, the response to $r[n] = a\delta[n]+b\delta[n-1]$ will nearly be $y[n] = ah_1[n] + bh_1[n-1]$. Generalizing this result, we see that the machine superposes the partial responses to separate input samples to obtain the over-all response. In other words the machine system is linear (also time-invariant) and is, therefore, characterized by the impulse response $h_1[n]$. The corresponding transfer function is

$$H_1(z) = \frac{Cz}{(1-z)^2},$$

where C is some positive constant depending on the vehicle.

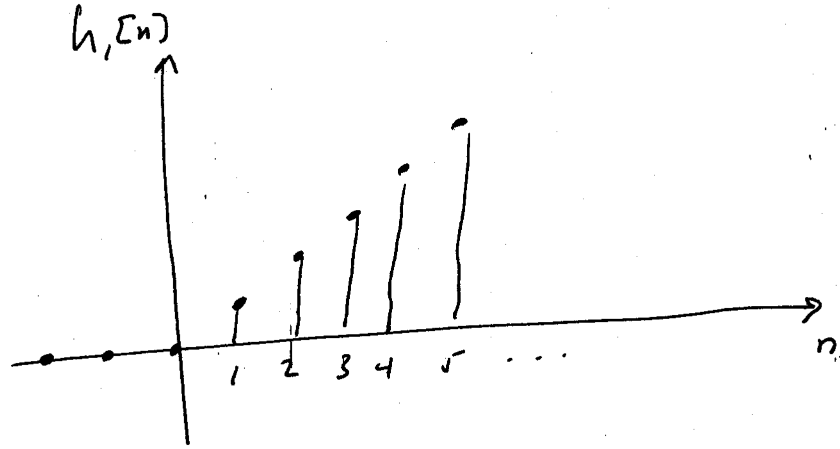

Figure 4: Impulse Response of Machine Portion of System

We postulate that the controller portion of the system is also a linear, time-invariant system. Under these restrictions, what is the best way for the man to generate turning signals, $r[n]$, from the observed errors, $e[n] = x[n] - y[n]$? To make this problem simple, we consider only logics of restricted form. Thus we shall permit the controller to make $r[n]$ proportional to the error $e[n]$, to the change of error $e[n] - e[n-1]$, or more generally to a linear combination of these. Therefore,

$$r[n] = C_1 e[n] + C_2(e[n] - e[n-1]).$$

The transfer function of the driver portion of the system is then:

$$H_2(z) = C_1 + C_2(1 - z^{-1}).$$

In other words the controller makes use of at most the current and immediately preceding errors to compute a turn signal. We should expect C_1 and C_2 to be positive; then an errors or changing error would bring about the correct compensating action.

Since H_1 and H_2 are in series they may immediately be combined into an overall transfer function

$$H_3(z) = H_1(z)H_2(z) = \frac{Cz}{(1-z)^2}[C_1 + C_2(1 - z^{-1})].$$

By letting $A = C(C_1 + C_2)$ and $B = CC_2$ we simplify the expressions to

$$H_3(z) = \frac{Az - B}{(1-z)^2},$$

so the system becomes as illustrated in Figure 5.

We can simplify this graph further by noting that $E(z) = X(z) - Y(z)$ and thus $Y(z) = H_3(z)E(z) = H_3(z)[X(z) - Y(z)]$. Consequently, we can solve this latter expression for the ratio $\frac{Y(z)}{X(z)}$, yielding the so-called *closed-loop* transfer function

$$\begin{aligned} H_y(z) &= \frac{Y(z)}{X(z)} = \frac{H_3(z)}{1 + H_3(z)} \\ &= \frac{Az - B}{z^2 + (A-2)z + 1 - B}. \end{aligned}$$

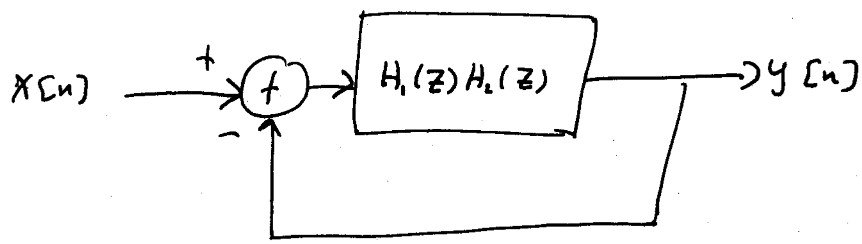

Figure 5: Feedback Control System

This transfer function is appropriate for considering $x[n]$ as the input and $y[n]$ as the output. But we might, alternatively, wish to view $e[n]$ as the output, in which case we define the transfer function

$$H_e(z) = \frac{E(z)}{X(z)} = \frac{E(z)}{Y(z)}\frac{Y(z)}{X(z)}.$$

But $\frac{E(z)}{Y(z)}$ is simply the reciprocal of $\frac{Y(z)}{E(z)} = H_3(z)$ so

$$H_e(z) = \frac{Az-B}{z^2+(A-2)z+1-B}\cdot\frac{(1-z)^2}{Az-B} = \frac{(1-z)^2}{z^2+(A-2)z+1-B}.$$

Note that both of these transfer functions have the same denominator polynomial

$$D(z) = z^2 + (A-2)z + 1 - B,$$

whose roots are given by

$$\begin{aligned} z_1 &= \frac{-(A-2)+\sqrt{(A-2)^2-4(1-B)}}{2} \\ z_2 &= \frac{-(A-2)-\sqrt{(A-2)^2-4(1-B)}}{2}. \end{aligned}$$

All of the subsystems (controller, h_2, and machine, h_1), of the above system are physically realizable, and do not respond until excited. Therefore, each of the transfer functions H_1, H_2, H_3, H_e, and H_y has a region of convergence outside a circle centered at $z = 0$. In their respective regions of convergence, the power series expansions represent the time functions. This means we can expand these transfer functions by straight long division, since only descending powers are z are required.

Stability Analysis

In control problems of this type, it is essential that the closed-loop system be stable. Otherwise, the system may oscillate wildly because the errors are overcorrected or may simply compound the error by corrections in the wrong direction.

To test the system for stability we simply shock it by applying a unit impulse at time $n = 0$ and sit back and watch it to see whether the resulting error eventually decays to zero or whether it builds up with increasing time. In other words, we inspect H_e to see whether the time signal, $h_e[n]$, tends to zero. $H_e(z)$ can be expanded in a partial fraction expansion of the form

$$\begin{aligned} H_e(z) &= a_0 + \frac{a_1 z}{z - z_1} + \frac{a_2 z}{z - z_2} \\ &= a_0 + a_1(1 + z_1 z^{-1} + z_1 z^{-2} + \ldots) \\ &\quad + a_2(1 + z_2 z^{-1} + z_2 z^{-2} + \ldots). \end{aligned}$$

The condition that $h_e[n] \to 0$ as $n \to \infty$ is just that $|z_1| < 1$ and $|z_2| < 1$. Thus, the condition for stability is that the poles of $H_e(z)$ all lie inside the unit circle. Since $H_y(z)$ and $H_e(z)$ have the same poles, we may have just as well used $H_y(z)$ to determine stability.

By way of review, we can look at stability another way to get the same result. From our definition of stability, we know that the impulse response must be absolutely summable: $\sum_{n=0}^{\infty} |h_e[n]| < \infty$. But this means that the Fourier transform of $h_e[n]$ exists, which means that the ROC must contain the unit circle, which means that all of the poles must lie strictly inside the unit circle.

Now we want to determine those values of A and B in our system which result in stable operation. We must find values of A and B for which both roots of

$$D(z) = z^2 + (A - 2)z + (1 - B)$$

lie inside the unit circle. Although there are general methods of doing this for high-order systems, we will pursue a special approach for this particular second-order system. We have two real parameters to choose: A, and B. We will proceed by defining regions of the $A - B$ plane such that the roots of $D(z)$ are inside the unit circle. We will first determine the values of A and B that result in complex roots to $D(z)$.

We note that the roots of $D(z)$ are complex if the quantity

$$d = (A - 2)^2 - 4(1 - B) < 0$$

or, after some simple reduction, the condition

$$B < A(1 - A/4)$$

obtains.

Thus, the boundary between real and complex roots is a parabola in the $A - B$ plane, $B = p(A) = A(1 - A/4)$ (see Figure 6). If the roots are complex, then we may use the results of Example 10.24 of Oppenheim and Wilsky, and rewrite the transfer function as

$$H_e(z) = \frac{(1 - z)^2 z^{-2}}{1 + (A - 2)z^{-1} + (1 - B)z^{-2}},$$

and observe, from Equation (10.101) of Oppenheim and Wilsky, that $1 - B$ corresponds to the radius of the roots. For stability, we require that $r < 1$, and this means that we must have $0 < B < 1$ for this to be a stable system with complex roots.

We next consider the case of real roots to $D(z)$. Let

$$d = (A-2)^2 - 4(1-B).$$

Then $z_1 = \frac{2-A+\sqrt{d}}{2}$, and to insure that $|z_1| < 1$, we must have $2 - A + \sqrt{d} < 2$ or $\sqrt{d} < A$. Substituting for d, squaring, and rearranging, we obtain the condition $(A-2)^2 - 4(1-B) < A^2$, or $B < A$. Thus a necessary condition for stability is that B lie below the line $l_1(A) = A$, as plotted in Figure 6. By a similar analysis, we may see that stability also requires that B lie below the line $l_2(A) = -A + 4$. Thus, the complete stability plot is evident in Figure 6.

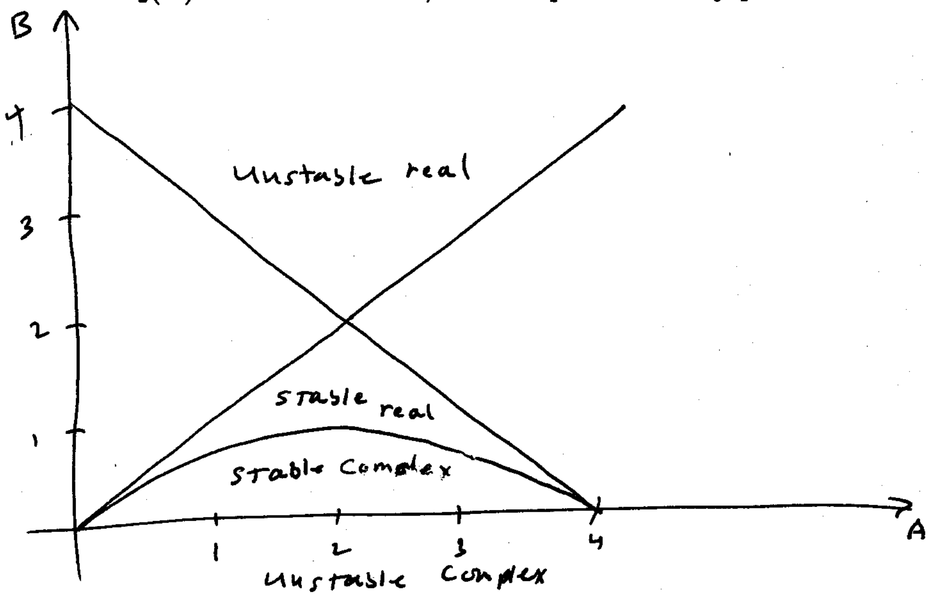

Figure 6: Stability Diagram

In more complicated problems than this second-order system we can still delineate the regions of stability and instability by algebraic expressions involving the coefficients of the transfer function denominator polynomial.

Interpretations

We are now able to draw some conclusions about the effectiveness of proposed procedures that the controller could use to control the car. Suppose first that $C_2 = 0$ (thus, $B = 0$). This condition means that the driver generates a turn at time n which is proportional to the error, $e[n]$ observed at that time. The velocity estimate, $e[n] - e[n-1]$, is not used. As might be expected, the stability diagram shows that the system is unstable if A is negative.

Now consider the case $0 < A < 4$, $B = 0$. Here the controller acts in the correct sense to compensate for the error, but the degree of correction is held within moderate bounds. This procedure is the one that naturally first comes to mind as a reasonable strategy. But as the stability diagram shows, the roots of $D(z)$ are complex and lie on the unit circle in the z-plane (since the system is on the border between stability with roots inside and instability with roots outside). The result is that under this type of control the car weaves from side-to-side after a disturbance, with the magnitude of the oscillations neither decreasing or increasing with time. For example, take $A = 2$, $B = 0$. Then

$$\begin{aligned} H_e(z) &= \frac{(1-z)^2}{1+z^2} \\ &= 1 + 2z^{-1} + 2z^{-3} - 2z^{-5} + 2z^{-7} - \ldots. \end{aligned}$$

Here, the oscillations are twice the amplitude of the original disturbance and have a period between maxima of four sampling periods.

We observe from the stability diagram that if we choose $A = B$, the system will be on the verge of stability, at best. This choice corresponds to setting $C_1 = 0$, and using only the difference $e[n] - e[n-1]$ to generate turning corrections. That is, only the rate of change of error is used to provide corrections.

It is apparent, then, that both error and velocity of error will be needed to properly stabilize the system, and A and B must be chosen from the interior of the stability region.

System Design

To make specific choices for A and B, we will apply some test inputs to the system and attempt to gauge the quality of the responses. These test inputs are chosen to be simple but also to strain the mathematical mechanism of the system in order that weaknesses should become apparent. A particularly appropriate set of test inputs consists of the unit impulse, the unit step, the unit ramp, and the unit parabola.

Input	Time Function	z-Transform
Unit Impulse	$\delta[n]$	1
Unit Step	$u[n]$	$\frac{z}{z-1} = \frac{1}{1-z^{-1}} = 1 + z^{-1} + z^{-2} + \cdots$
Unit Ramp	$nu[n]$	$\frac{z}{(z-1)^2} = \frac{z^{-1}}{(1-z^{-1})^2} = z^{-1} + 2z^{-2} + 3z^{-3} + \cdots$
Unit Parabola	$n^2u[n]$	$\frac{z(z+1)}{(z-2)^3} = \frac{z^{-1}(1+z^{-1})}{(1-z^{-1})^3} = z^{-1} + 4z^{-2} + 9z^{-3} + \cdots$

The unit impulse results from a roadway that contains a momentary jog in the road, then returns to it's original path. The unit step results from a roadway that is suddenly displaced to the left. The unit ramp results from a sharp change of direction in the road. The unit parabola serves as an approximation to a turn of constant radius.

Applying the unit step to the system, the error is

$$E(z) = \frac{z}{z-1} H_e(z) = \frac{z(z-1)}{D(z)}.$$

The error for a unit ramp is

$$E(z) = \frac{z}{(z-1)^2} H_e(z) = \frac{z}{D(z)}.$$

Th error for a unit parabola is

$$E(z) = \frac{z(z+1)}{(z-1)^3} H_e(z) = \frac{z(z+1)}{(z-1)D(z)}.$$

If the system is stable the factors of $D(z)$ will contribute only decaying transient components to each of these $e[n]$ responses. Hence for the unit step and unit ramp, the error tends to zero with increasing time and the car eventually gets back to the the proper path. However, for the parabola, there is an additional pole in the response at $z = 1$ contributed by the third power of the $(z-1)^3$ term in the input, which did not get canceled by the transfer function. This adds a term of the form $\frac{\epsilon}{1-z^{-1}}$ in the partial fraction expansion. Thus, after the transients have died out, there will remain a term of the form $e[n] = \epsilon u[n]$ for all time, which corresponds to a lag in the response. See Figure 7.

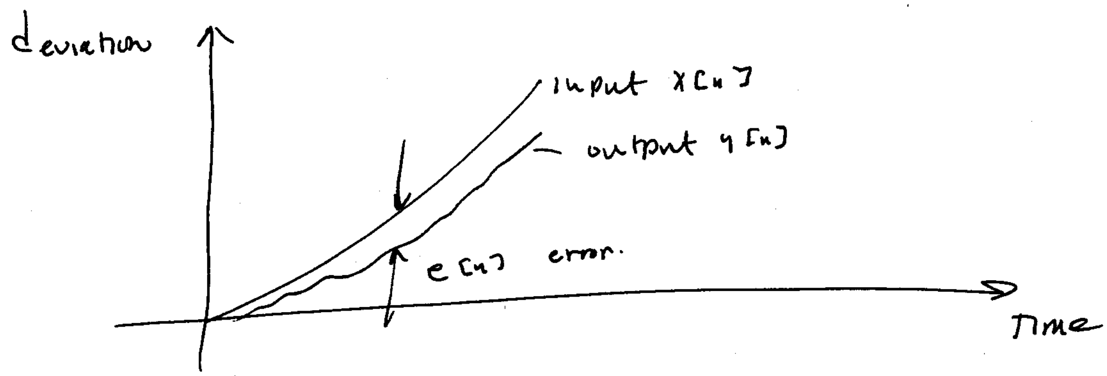

Figure 7: Error due to Parabolic Input

We may evaluate ϵ from the partial fraction expansion:

$$\epsilon = (z-1)E(z)\big|_{z=1} = \frac{2}{D(1)} = \frac{2}{A-B}.$$

To reduce this lag, we should make A large and B small. The stability diagram shows that the best choice, from this point of view, is the corner $A = 4$ and $B = 0$. However, we know that the response here is subject to undiminshed oscillations. Therefore, in adjusting A and B, we must strike a balance between loose control leading to big lag errors and tight control leading to pronounced oscillations.

This situation illuminates the fact that engineering design is an important problem. There will always be tradeoffs and there is no such thing as a globally best, perfect design. From this problem, we have seen that stability is critical and steady-state behavior and transient response are important. Another important design criteria would be the amount of power consumed in effecting the steering command. Large values of C_1 and C_2, for example, translate into energy that is required to effect the steering command. This requirement translates into power-steering requirements if the driver's arms are not sufficiently strong.

To conclude this problem, however, we ought to settle on some parameters. For a variety of reasons, we might consider letting $A = 2$ and $B = 1$. For one thing, this represents equal weighting of error and rate of change of error. The stability diagram indicates that this choice is on the boundary between complex and real roots to $D(z)$. The resulting transfer function is

$$H_e(z) = \frac{(1-z)^2}{z^2} = 1 - 2z^{-1} + z^{-2},$$

and the corresponding impulse response is

$$h_e[n] = \delta[n] - 2\delta[n-1] + \delta[n-2].$$

The response of this system to a step input is

$$E(z) = \frac{z(z-1)}{z^2} = \frac{z-1}{z} = 1 - z^{-1}$$

which corresponds to $e[n] = \delta[n] - \delta[n-1]$, so the system gets back on track after two sample times. The response to a ramp input is

$$E(z) = \frac{z}{z^2} = \frac{1}{z} = z^{-1},$$

corresponding to $e[n] = \delta[n-1]$, so the system recovers after just one move. The response to a parabolic input is

$$E(z) = \frac{z(z+1)}{(z-1)z^2} = \frac{z+1}{z(z-1)} = z^{-1} + 2z^{-2} + 2z^{-3} \cdots,$$

corresponding to

$$e[n] = -\delta[n-1] + 2u[n-1],$$

and the system reaches a steady state lag of two units of position after two sample times. Thus, this design should do quite well for roads that are not excessively narrow or windy.