

0.1 Dielectric Slab Waveguide

At high frequencies (especially optical frequencies) the loss associated with the induced current in the metal walls is too high. A transmission line filled with dielectric material but without conducting walls is another structure that may be used to guide electromagnetic waves. This dielectric slab waveguide eliminates the metallic absorption loss.

Consider a dielectric slab that is surrounded by another dielectric material that has a lower permittivity. A representative slab is shown in Fig. 1. The waveguide thickness is $2d$ and the center region (core) has a higher permittivity than the two outer regions (cladding) ($\epsilon_1 > \epsilon_2$). We assume propagation in the z direction and no variation of fields in the y direction ($W \gg 2d$). This makes the problem simpler, because it reduces to a two-dimensional analysis.

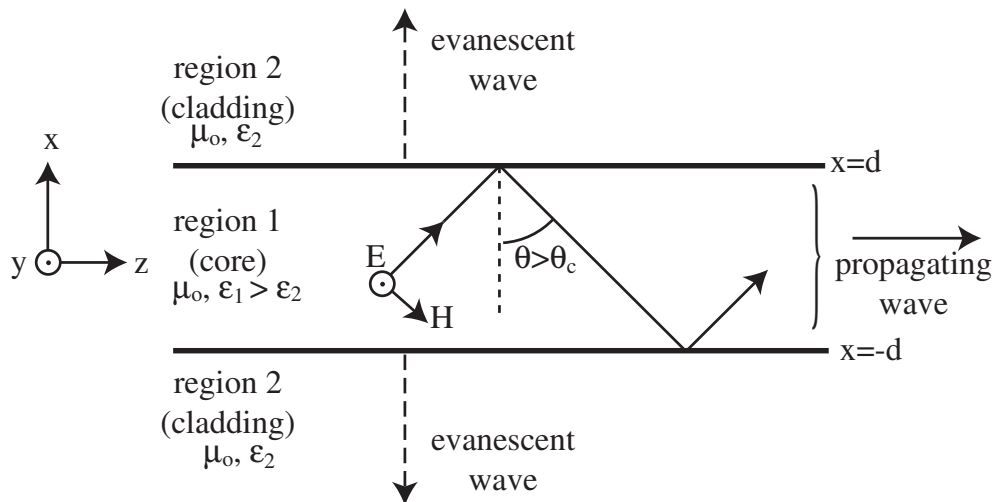


Figure 1: Dielectric slab waveguide of width $2d$.

Since the dielectric waveguide is intended to guide the light, the fields in the cladding region should be evanescent or decay in amplitude away from the slab. This guiding property requires the ray angle to be past the critical angle $\theta > \theta_c = \sin^{-1} \left(\sqrt{\frac{\epsilon_2}{\epsilon_1}} \right) = \sin^{-1} \left(\frac{n_2}{n_1} \right)$. This requires the propagation constant to be in the range

$$n_1 k_o \sin(\theta_c) < \beta < n_1 k_o \sin(90^\circ) \quad (1)$$

$$n_2 k_o < \beta < n_1 k_o. \quad (2)$$

We could use the concept of computing E_z or H_z and then applying Maxwell's equations to obtain the remaining field components just like we did in the metallic waveguides. However, the fact that we are now in two dimensions makes things a little easier. We will still use TE and TM to indicate that the field has no longitudinal electric or magnetic field, respectively. In the case of TE modes, the electric field will only have a \hat{y} component (why?). We can then easily find the magnetic field using Faraday's law. A similar argument holds for TM modes, as we will see shortly.

Two other key concepts concerning dielectric waveguides deserve attention. The first is that, due to the symmetry of the geometry, the fields will either be symmetric or anti-symmetric about the y - z plane. The second is that in order for the field to be guided by the high-permittivity dielectric slab, the fields outside the

slab must be evanescent, i.e. they decay in the x direction. We will use these observations in the formulations that follow.

0.1.1 TE Modes

The electric field for the TE modes must satisfy our wave equation as given by

$$\frac{d^2 E_y}{dx^2} + (\omega^2 \mu \epsilon - \beta^2) E_y = 0 \quad (3)$$

This wave equation is valid in all regions. However, remember that the permittivity ϵ is different in the two regions.

Our experience with rectangular waveguides tells us what the solutions must be (before application of the boundary conditions). However, our argument about symmetry makes it so that within the slab, the variation will either be

$$E_y = A \cos k_{1x} x \quad (4)$$

or

$$E_y = B \sin k_{1x} x. \quad (5)$$

Note that we used the sin and cos form rather than the exp form. This is because we know that the fields form standing waves within the waveguide region. By substituting these into the wave equation (Eq. 3), it can be shown that

$$k_{1x}^2 = \omega^2 \mu \epsilon_1 - \beta^2 = k_1^2 - \beta^2. \quad (6)$$

The fields outside of the slab (in the cladding regions) are also of the same basic form.

$$\begin{aligned} E_y &= C e^{j k_{2x} x} & x \geq d \\ E_y &= D e^{j k_{2x} x} & x \leq -d \end{aligned} \quad (7)$$

These field solutions use the exponential form because we know that they will not form standing waves. By substituting these in Eq. (3), it can be shown that

$$k_{2x}^2 = \omega^2 \mu \epsilon_2 - \beta^2 = k_2^2 - \beta^2. \quad (8)$$

In order to maintain guiding, the fields in the cladding must be evanescent or decay in amplitude with distance away from the slab. This requirement caused the propagation constant to be in the range of $n_2 k_o < \beta$. Therefore, the propagation constant in the cladding regions is complex as given by

$$k_{2x} = \pm j \alpha_2, \quad (9)$$

where

$$\alpha_2 = \pm \sqrt{\beta^2 - k_2^2}. \quad (10)$$

The sign of k_{2x} is chosen such that the fields decay with distance away from the waveguide. The resulting fields in the cladding regions are then given by

$$\begin{aligned} E_y &= C e^{-\alpha_2 x} & x \geq d \\ E_y &= D e^{\alpha_2 x} & x \leq -d \end{aligned} \quad (11)$$

Therefore, the electric field in the various regions is given by

$$\bar{E} = \hat{y} \begin{cases} E_2 e^{-\alpha x - j\beta z} & x \geq d \\ E_1 \begin{Bmatrix} \sin k_x x \\ \cos k_x x \end{Bmatrix} e^{-j\beta z} & |x| \leq d \\ \begin{Bmatrix} - \\ + \end{Bmatrix} E_2 e^{+\alpha x - j\beta z} & x \leq -d \end{cases} \quad (12)$$

where the top and bottom lines in the braces refer to the antisymmetric and symmetric modes, respectively. The solution that uses the cos is called the symmetric solution and the solution that uses the sin is called the anti-symmetric solution.

Using Faraday's law, we can now compute the magnetic fields,

$$\bar{H} = \frac{-1}{j\omega\mu} \nabla \times \bar{E} = \begin{cases} \frac{E_2}{\omega\mu_0} (-\hat{x}\beta - \hat{z}j\alpha) e^{-\alpha x - j\beta z} & x \geq d \\ \frac{E_1}{\omega\mu} \left(-\hat{x}\beta \begin{Bmatrix} \sin k_x x \\ \cos k_x x \end{Bmatrix} - \hat{z}j k_x \begin{Bmatrix} -\cos k_x x \\ \sin k_x x \end{Bmatrix} \right) e^{-j\beta z} & |x| \leq d \\ \begin{Bmatrix} - \\ + \end{Bmatrix} \frac{E_2}{\omega\mu_0} (-\hat{x}\beta + \hat{z}j\alpha) e^{+\alpha x - j\beta z} & x \leq -d \end{cases} \quad (13)$$

Note that in these field expressions, we have used the fact that the z variation is of the form $e^{-j\beta z}$ both inside and outside the slab. Why do we know that this propagation constant is the same in both regions? Note also that we have 4 unknowns: E_2/E_1 , k_x , α , and β .

Since these fields must obey the wave equation (with $\partial^2/\partial y^2 = 0$), we know that

$$k_x^2 + \beta^2 = k_1^2 = \omega^2 \mu \epsilon_1 \quad (14)$$

$$-\alpha^2 + \beta^2 = k_2^2 = \omega^2 \mu_0 \epsilon_2 \quad (15)$$

which gives us two constraints for determining our unknowns. We need two additional constraints in order to find all 4 unknowns.

Let's start by enforcing continuity of tangential electric fields at the core-cladding interface. We will first consider the symmetric modes. Therefore, at $x = d$

$$E_1 \cos(k_x d) e^{-j\beta z} = E_2 e^{-\alpha d} e^{-j\beta z} \quad \rightarrow \quad \cos(k_x d) E_1 - e^{-\alpha d} E_2 = 0 \quad (16)$$

Note that applying continuity at $x = -d$ results in an identical equation, so this does not help us. This stems from the symmetry of the problem, and in reality we have already used this symmetry to break the problem into symmetric and antisymmetric modes.

Since we need one more equation, we will apply continuity of tangential (i.e. \hat{z} component) magnetic field at the boundary. At $x = d$ we have

$$-jk_x \frac{E_1}{\omega\mu} \sin(k_x d) e^{-j\beta z} = -j\alpha \frac{E_2}{\omega\mu} e^{-\alpha d} e^{-j\beta z} \quad \rightarrow \quad k_x \sin(k_x d) E_1 - \alpha e^{-\alpha d} E_2 = 0 \quad (17)$$

and again, we get the exact same equation at $x = -d$. The easiest thing to do is to divide these two equations to simplify them.

$$\frac{k_x \sin(k_x d) E_1}{\cos(k_x d) E_1} = \frac{\alpha e^{-\alpha d} E_2}{e^{-\alpha d} E_2} \quad (18)$$

$$\alpha = k_x \tan(k_x d) \quad (19)$$

which can be re-written as

$(\alpha d) = (k_x d) \tan(k_x d) \quad \text{symmetric TE modes} \quad (20)$

Subtracting (14) and (15) yields

$$k_x^2 + \alpha^2 = \omega^2 \mu_0 \epsilon_0 (\epsilon_1 - \epsilon_2) \quad (21)$$

or as

$$(k_x d)^2 + (\alpha d)^2 = \omega^2 \mu_0 \epsilon_0 (\epsilon_1 - \epsilon_2) d^2 \quad (22)$$

We can actually solve this graphically, as will be shown later. Alternately, we can combine these two equations to obtain

$$\begin{aligned} \alpha^2 + k_x^2 &= \omega^2 \mu_o \epsilon_o (n_1^2 - n_2^2) \\ k_x^2 \tan^2(k_x d) + k_x^2 &= \omega^2 \mu_o \epsilon_o (n_1^2 - n_2^2) \\ \tan^2(k_x d) + 1 &= \frac{\omega^2 \mu_o \epsilon_o (n_1^2 - n_2^2)}{k_x^2} \\ \tan(k_x d) &= \sqrt{\frac{\omega^2 \mu_o \epsilon_o (n_1 - n_2)}{k_x^2} - 1} \end{aligned} \quad (23)$$

and solve this with a nonlinear solver on a calculator or computer.

1. Solutions in the range $(m - 1)\pi/2 \leq k_x d \leq m\pi/2$ $m = 1, 3, 5, \dots$ we will call TE_m modes. These correspond to the symmetric TE modes.
2. Cutoff occurs when the mode is no longer guided, which occurs as soon as α becomes negative. So, we define cutoff as the frequency at which $\alpha = 0$. Using (20), this implies that $\tan(k_x d) = 0$, so that $k_x d = (m - 1)\pi/2$, $m = 1, 3, 5, \dots$ Using (22) with $\alpha = 0$ leads to

$$f_{c,m} = \frac{c(m - 1)}{4d\sqrt{\mu_r \epsilon_r - 1}} \quad (24)$$

for the cutoff frequencies of the TE modes. Note that $f_{c,1} = 0$, so the lowest order mode propagates at any frequency. Furthermore, since at cutoff $\beta = k_2$ and $\beta^2 + k_x^2 = k_1^2$, the angle of incidence of the wave on the dielectric boundary can be expressed as

$$\theta_i = \sin^{-1} \frac{\beta}{\sqrt{\beta^2 + k_x^2}} = \sin^{-1} \frac{k_2}{k_1} = \sin^{-1} \sqrt{\frac{\epsilon_2}{\epsilon_1}} = \theta_c \quad (25)$$

which you may recognize as the critical angle. So, cutoff occurs when the angle of incidence on the boundary is smaller than the critical angle. Makes sense, doesn't it?

Observe also that the cutoff condition of $\beta = k_2$ means that the propagation constant becomes that of the surrounding medium.

3. Note that k_x is frequency dependent, unlike in the rectangular waveguide. From the two dispersion relations, we can see that $\omega \sqrt{\mu_0 \epsilon_2} \leq \beta \leq \omega \sqrt{\mu_0 \epsilon_1}$, which should be intuitive.
4. As the frequency gets larger, $\alpha \rightarrow \infty$ which means that the field decays very rapidly outside the dielectric. The behavior of the mode becomes like that of a parallel plate waveguide filled with a dielectric.

Note that we could repeat the entire procedure for the antisymmetric TE modes. The dispersion relation (22) remains the same. The guidance condition becomes

$$\boxed{(\alpha d) = -(k_x d) \cot(k_x d) \quad \text{antisymmetric TE modes} \quad (26)}$$

Again, cutoff occurs for $k_x d = (m - 1)\pi/2$, $m = 2, 4, 6, \dots$. These are therefore the even order TE modes.

0.1.2 TM Modes

We can repeat the whole process for TM modes. In this case, we have

$$\overline{H} = \hat{y} \begin{cases} H_2 e^{-\alpha x - j\beta z} & x \geq d \\ H_1 \begin{Bmatrix} \sin k_x x \\ \cos k_x x \end{Bmatrix} e^{-j\beta z} & |x| \leq d \\ \begin{Bmatrix} - \\ + \end{Bmatrix} H_2 e^{+\alpha x - j\beta z} & x \leq -d \end{cases} \quad (27)$$

where the top and bottom lines in the braces refer to the antisymmetric and symmetric modes, respectively. Using Ampere's law, we can now compute the electric fields

$$\overline{E} = \frac{1}{j\omega\epsilon} \nabla \times \overline{H} = \begin{cases} \frac{H_2}{\omega\epsilon_0} (\hat{x}\beta + \hat{z}j\alpha) e^{-\alpha x - j\beta z} & x \geq d \\ \frac{H_1}{\omega\epsilon} \left(\hat{x}\beta \begin{Bmatrix} \sin k_x x \\ \cos k_x x \end{Bmatrix} + \hat{z}jk_x \begin{Bmatrix} -\cos k_x x \\ \sin k_x x \end{Bmatrix} \right) e^{-j\beta z} & |x| \leq d \\ \begin{Bmatrix} - \\ + \end{Bmatrix} \frac{H_2}{\omega\epsilon_0} (\hat{x}\beta - \hat{z}j\alpha) e^{+\alpha x - j\beta z} & x \leq -d \end{cases} \quad (28)$$

We go through the exact same sequence of steps for this case. The dispersion relations remain the same. The guidance conditions become

$$\boxed{(\alpha d) = \frac{\epsilon_0}{\epsilon} (k_x d) \tan(k_x d) \quad \text{symmetric TM modes} \quad (29)}$$

$$\boxed{(\alpha d) = -\frac{\epsilon_0}{\epsilon} (k_x d) \cot(k_x d) \quad \text{antisymmetric TM modes} \quad (30)}$$

0.1.3 Graphical Mode Solution

Solving the simultaneous equations (22) and (20) (symmetric modes) or (26) (antisymmetric modes) for the values of k_x and α is difficult, because the equations are transcendental. But it is easy to find an approximate solution using a graphical technique. This also provides valuable physical insight.

Each solution involves a guidance equation and a dispersion equation. These equations are:

$$(\alpha d) = (k_x d) \tan(k_x d) \quad (31)$$

$$(\alpha d) = -(k_x d) \cot(k_x d) \quad (32)$$

$$(k_x d)^2 + (\alpha d)^2 = \omega^2 \mu_0 \epsilon_0 (\epsilon_1 - \epsilon_2) d^2 \quad (33)$$

Notice that we have multiplied each of the equations by d to make them dimensionless, which allows us to use one plot for any value of the frequency or slab thickness. The idea is to plot these equations on the same axes. Think of $k_x d$ as x and αd as y . These equations then become

$$(\alpha d) = (k_x d) \tan(k_x d) \quad \rightarrow \quad y = x \tan(x) \quad (34)$$

$$(\alpha d) = -(k_x d) \cot(k_x d) \quad \rightarrow \quad y = -x \cot(x) \quad (35)$$

$$(k_x d)^2 + (\alpha d)^2 = \omega^2 \mu_0 \epsilon_0 (\epsilon_1 - \epsilon_2) d^2 \quad \rightarrow \quad x^2 + y^2 = \omega^2 \mu_0 \epsilon_0 (\epsilon_1 - \epsilon_2) d^2 \quad (36)$$

- The first two equations are periodic functions. These functions are independent of both the permittivities and the thickness of the waveguide. Figure 2 shows these plots.
- The second equation is a circle with a radius of $R = \omega \sqrt{\mu_0 \epsilon_0 (\epsilon_1 - \epsilon_2)} d$. Thus, the radius of this circle changes with waveguide parameters.

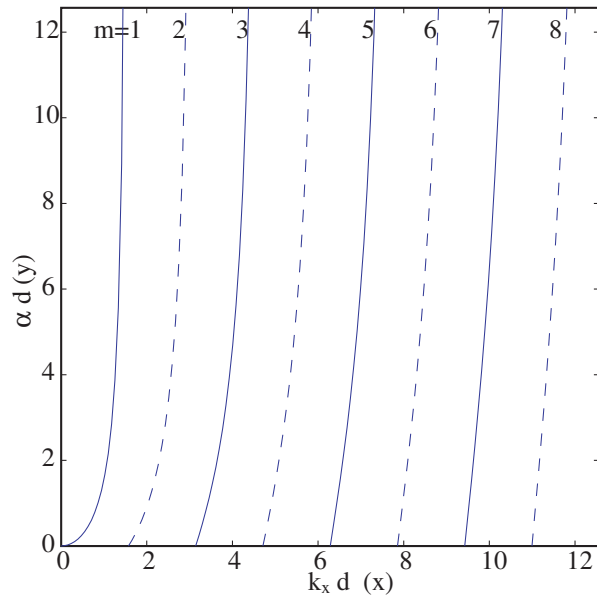


Figure 2: Plots of the guidance condition for a TE dielectric slab waveguide. The solid lines are for the symmetric modes and the dashed lines are for the antisymmetric modes.

The second equation can be easily plotted as a quarter circle onto the lines shown in Fig. 2. Each intersection of the curves corresponds to a mode. The coordinates of the point give us the value of $k_x d$ and αd , which can be easily divided by d to give k_x and α .

Dielectric Waveguide Example

How many modes exist in a dielectric waveguide that has the following parameters? index of refraction of the core $n_1 = 1.6$, index of refraction of the cladding $n_2 = 1.5$, wavelength $\lambda = 1.0\mu m$, waveguide core thickness $2d = 4\mu m$.

The equations are

$$\alpha d = k_y d \tan(k_y d) \quad (37)$$

$$\alpha d = -k_y d \cot(k_y d) \quad (38)$$

$$(k_y d)^2 + (\alpha d)^2 = (k_o d)^2 (n_1^2 - n_2^2) \quad (39)$$

Using $k_y d = x$ and $\alpha d = y$ these equations become

$$y = x \tan x \quad (40)$$

$$y = -x \cot x \quad (41)$$

$$x^2 + y^2 = (k_o d)^2 (n_1^2 - n_2^2) \quad (42)$$

For this example the radius of the circle is given by

$$r = \frac{2\pi}{1.0} \frac{4}{2} \sqrt{1.6^2 - 1.5^2} \quad (43)$$

$$r = 2.23\pi = 7.0 \quad (44)$$

The equation $x \tan x$ is equal to zero when $x = 0\pi, 2\pi, 3\pi, \dots, m\pi$ and is equal to ∞ when $x = \frac{\pi}{2}, \frac{3\pi}{2}, \frac{5\pi}{2}, \dots, \frac{\pi}{2} + m\pi$.

The equation $-x \cot x$ is equal to zero when $x = \frac{\pi}{2}, \frac{3\pi}{2}, \frac{5\pi}{2}, \dots, \frac{\pi}{2} + m\pi$ and is equal to ∞ when $x = \pi, 2\pi, 3\pi, \dots, m\pi$. And when $x = 0$ $-x \cot x = -1$.

The radius of the circle for this problem is $r = 7.0 = 2.23\pi$. There are 3 even modes ($0, \pi, 2\pi$) and 2 odd modes ($0.5\pi, 1.5\pi$).

What is the waveguide thickness for single mode operation? We need

$$r < 0.5\pi \quad (45)$$

$$\frac{2\pi}{1.0} d \sqrt{1.6^2 - 1.5^2} < \frac{\pi}{2} \quad (46)$$

$$d < 0.449 \quad (47)$$

or a slab thickness of $2d = 0.9\mu m$

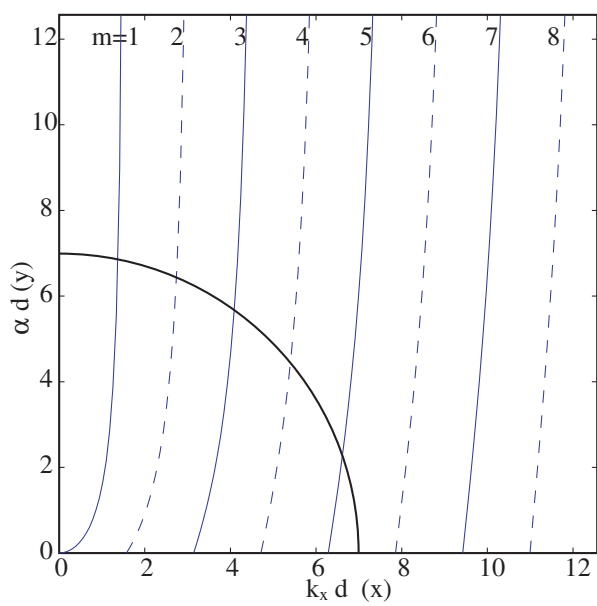


Figure 3: Graphical mode solution with $n_1 = 1.6$, $n_2 = 1.5$, $2d = 4\mu m$, and $\lambda = 1.0\mu m$.

0.1.4 Example - TM Modes

For the TM modes, the graphical solution method is complicated slightly by the fact that Eqs. (29) and (30) depend on the permittivity of the slab. This means that we need to plot curves for different values of ϵ_r . The required plot is shown below.

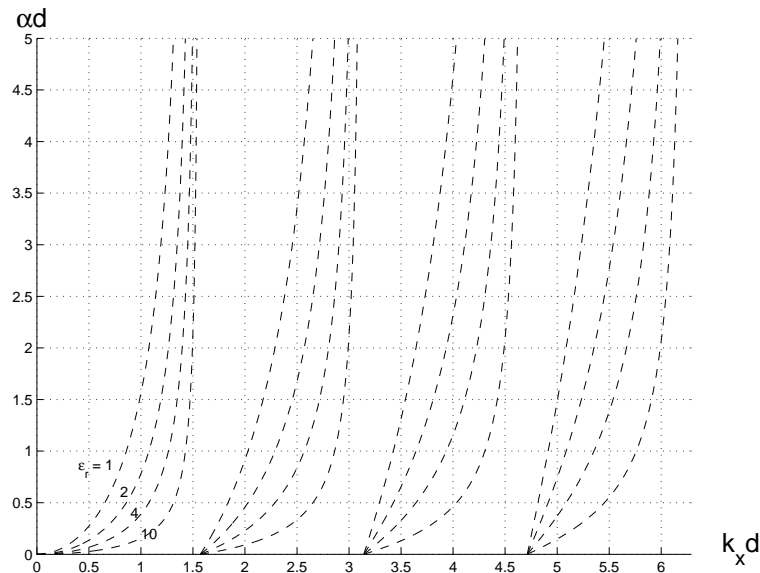


Figure 4: Graphical solution curves for TM modes of a dielectric slab waveguide for various values of ϵ_r . (Note that the $\epsilon_r = 1$ curve can also be used for TE modes.)

Consider an example with slab thickness 2 cm, relative permittivity 2, and operating frequency 20 GHz. The first few cutoff frequencies are

$$\begin{aligned} f_{c,1} &= 0 \\ f_{c,2} &= 7.5 \text{ GHz} \\ f_{c,3} &= 15 \text{ GHz} \\ f_{c,4} &= 22.5 \text{ GHz} \end{aligned}$$

From these values, we know that there are three propagating TM modes (and three TE modes as well). In order to find the values of k_x and α , we must find the intersections of a circle of radius $\omega d \sqrt{\epsilon_r - 1} / c \simeq 4.2$ with the curves in Fig. 4.

For the dominant mode, this intersection point lies at $\alpha d \simeq 4$, so that $\alpha = 4 \text{ Np/cm}$. This tells us that after $1/\alpha \simeq .25 \text{ cm}$, the fields outside the slab have decayed by a factor of $1/e$. The values of α as well as other constants can be found similarly for the higher order propagating modes. Will these modes be more or less tightly bound to the slab?