Chapter 1

Introduction and Review

ECE360 is a prerequisite for ECE 462. Here is a list of the topics covered in ECE360 that you will be using in this class.

- Maxwell's Equations
- Time Harmonic Form
- Phasors
- Plane Waves
- Polarization
- Boundary Conditions

These topic will be briefly reviewed in these notes. If you are unfamiliar with these topics or the brief description is unclear you will need to go back to your ECE 360 book to review these topics.

1.1 Maxwell's Equations

Maxwell's equations are a system of partial differential equations that relate electric and magnetic fields to sources. Static fields are not time-varying, and all terms in Maxwell's equations with time derivatives vanish. For dynamic problems, time-derivative terms are nonzero. If all sources are is sinusoidal with frequency $\omega = 2\pi f$ rad/s, then Maxwell's equations can be put in phasor or time harmonic-form.

Time Domain	*Time-Harmonic*
$\nabla \times \overline{\mathcal{E}}(r,t) = -\frac{\partial}{\partial t}\overline{\mathcal{B}}(r,t)$	$\nabla \times \overline{E}(r) = -j\omega\overline{B}(r)$
$\nabla \times \overline{\mathcal{H}}(r,t) = \frac{\partial}{\partial t}\overline{\mathcal{D}}(r,t) + \overline{\mathcal{J}}(r,t)$	$\nabla \times \overline{H}(r) = j\omega\overline{D}(r) + \overline{J}(r)$
$\nabla \cdot \overline{\mathcal{D}}(r,t) = \rho(r,t)$	$\nabla \cdot \overline{D}(r) = \rho(r)$
$\nabla \cdot \overline{\mathcal{B}}(r,t) = 0$	$\nabla \cdot \overline{B}(r) = 0$
$\overline{\mathcal{D}}(r,t) = \epsilon(r,t) * \overline{\mathcal{E}}(r,t)$	$\overline{D}(r) = \epsilon(r)\overline{E}$
$\overline{\mathcal{B}}(v,t) = \mu(r,t) * \overline{\mathcal{H}}(r,t)$	$\overline{B}(r) = \mu(r)\overline{H}$

where the "del" operator is

$$\nabla = \frac{\partial}{\partial x}\hat{x} + \frac{\partial}{\partial y}\hat{y} + \frac{\partial}{\partial z}\hat{z}. \tag{1.1}$$

and $r = (x, y, z)$ represents a point in space.

In these notes, we will generally deal with the time-harmonic form. The relationship between the time-varying fields and phasor fields is

$$\overline{\mathcal{E}}(r,t) = \text{Re}\{\overline{E}(r)e^{j\omega t}\} \tag{1.2}$$

Some literature uses the time convention $e^{-i\omega t}$.

Definition of Quantities

The physical quantities appearing in Maxwell's equations can be placed in three groups: sources, field intensities, and flux densities. The electric field, for example, is represented by a field intensity ($\overline{E}$) and a flux density ($\overline{D}$). The two quantities represent different viewpoints on the same physical field.

$\overline{E}$ = Electric field intensity (V/m)
$\overline{H}$ = Magnetic field intensity (A/m)
$\overline{D}$ = Electric flux density (C/m^2)
$\overline{B}$ = Magnetic flux density (Wb/m^2)

$\overline{J}$ = Electric current density (A/m^2)
ρ = Electric charge density (C/m^3)

ϵ = $\epsilon_0\epsilon_r$ = permittivity (F/m), free space: $\epsilon_0 \simeq 8.854 \times 10^{-12}$ F/m
μ = $\mu_0\mu_r$ = permeability (H/m), free space: $\mu_0 = 4\pi \times 10^{-7}$ H/m

Physically, only electric sources are known to exist. As a mathematical convenience, however, fictitious magnetic currents and charges are often added to Faraday's law and Gauss's law for the magnetic flux density.

As with most partial differential equations, boundary conditions must be specified in order to obtain a unique solution for the fields. Maxwell's equations, given sources J and ρ, together with an appropriate boundary conditions, constitute a problem that can be uniquely solved for unknown field quantities.

1.2 Plane Waves

One approach to solving Maxwell's equations is to find a field that satisfies the equations for a given source, such as a dipole. Another method is to set the sources to zero, and find a general form for the solution in a source-free region. This second method is useful because by the principle of superposition, the field in that region can be represented as a combination of fields of that general form, regardless of what the sources are outside the region. The most commonly used form for this general solution is the plane wave.

1.2.1 Wave Equation

Maxwell's equations are a system of coupled first order partial differential equations (PDEs). By eliminating all field quantities but one, we can obtain a single, uncoupled second order PDE. This second order equation is known as the wave equation.

To accomplish this, we first take the curl of Faraday's Law and plug Ampere's Law in:

$$\begin{aligned}\nabla \times \nabla \times \overline{E} = -j\omega\mu\nabla \times H &= -j\omega\mu\left[j\omega\epsilon\overline{E} + \overline{J}\right] \\ &= \omega^2\mu\epsilon\overline{E} - j\omega\mu\overline{J} \end{aligned} \tag{1.3}$$

We assume that $\overline{J} = 0$ (source-free region). Also, we use the vector identity

$$\nabla \times \nabla \times \overline{E} = \nabla(\nabla \cdot \overline{E}) - \nabla^2\overline{E} \tag{1.4}$$

In a source-free environment, $\rho = 0$ as well, so that

$$\nabla \cdot \overline{D} = \nabla \cdot \epsilon\overline{E} = \epsilon(\nabla \cdot \overline{E}) = \rho = 0 \tag{1.5}$$

What have we assumed in this step? Combining the previous three equations leads to the wave equation

$$\nabla^2\overline{E} + k^2\overline{E} = 0 \tag{1.6}$$

where $k^2 = \omega^2\mu\epsilon$. The quantity $k = \omega\sqrt{\mu\epsilon} = \omega/c = 2\pi/\lambda$ is the wavenumber (rad/m).

1.2.2 Wave Equation Solution

In Cartesian coordinates, $\nabla^2\overline{E} = \hat{x}\nabla^2 E_x + \hat{y}\nabla^2 E_y + \hat{z}\nabla^2 E_z$ where

$$\nabla^2 E_x(x, y, z) = \frac{\partial^2 E_x}{\partial x^2} + \frac{\partial^2 E_x}{\partial y^2} + \frac{\partial^2 E_x}{\partial z^2} \tag{1.7}$$

For the moment, assume that $E_y = E_z = 0$ and that $\frac{\partial^2 E_x}{\partial x^2} = \frac{\partial^2 E_x}{\partial y^2} = 0$. This leads to the simplest possible form of (1.6):

$$\frac{\partial^2 E_x}{\partial z^2} + k^2 E_x = 0 \tag{1.8}$$

with solution

$$E_x(z) = E^+e^{-jkz} + E^-e^{+jkz} \tag{1.9}$$

In order to make physical sense of these solutions, we need to take them to the time domain:

$$\begin{aligned}\mathcal{E}_x(z,t) &= \text{Re}\left\{E^+e^{-jkz}e^{j\omega t} + E^-e^{-+kz}e^{j\omega t}\right\} \\ &= E^+\cos(\omega t - kz) + E^-\cos(\omega t + kz) \qquad (1.10)\end{aligned}$$

As t increases, how would one have to run in order to keep the argument of the cosine the same?

1.2.3 Phase Velocity

Let's consider the forward-traveling wave. In order to remain at a point of constant phase, we must have

$$\omega t - kz = \phi = \text{constant} \qquad (1.11)$$

This means that the point z must move in time according to

$$z(t) = \frac{\omega t - \phi}{k} \qquad (1.12)$$

The velocity of the point is

$$v_p = \frac{\partial z}{\partial t} = \frac{\omega}{k} = \frac{1}{\sqrt{\mu\epsilon}} \qquad (1.13)$$

Also, the wavelength is the spatial period:

$$k(z+\lambda) - kz = 2\pi \qquad \rightarrow \qquad k = \frac{2\pi}{\lambda} \qquad (1.14)$$

1.2.4 Magnetic Field

From Faraday's Law,

$$\overline{H} = \frac{1}{-j\omega\mu}\nabla\times\overline{E} = \frac{\hat{y}}{\eta}\left[E^+e^{-jkz} - E^-e^{jkz}\right] \qquad (1.15)$$

where $\eta = \sqrt{\mu/\epsilon}$ is the intrinsic impedance. Note that $\overline{E}\times\overline{H}$ is in the $\hat{z}$ direction (propagation direction). In general, $\overline{E} = \overline{E}_0 e^{-j\overline{k}\cdot\overline{r}}$, where

$$\overline{k} = k_x\hat{x} + k_y\hat{y} + k_z\hat{z} \qquad \overline{r} = x\hat{x} + y\hat{y} + z\hat{z} \qquad (1.16)$$

Since

$$\nabla\cdot\overline{E} = \nabla\cdot\left(\overline{E}_0 e^{-j\overline{k}\cdot\overline{r}}\right) = \overline{E}_0\cdot\nabla e^{-j\overline{k}\cdot\overline{r}} = -j\overline{k}\cdot\overline{E}_0 e^{-j\overline{k}\cdot\overline{r}} = 0 \qquad (1.17)$$

we conclude that $\overline{k}\cdot\overline{E}_0 = 0$, which means that $\overline{k}\perp\overline{E}$. Also

$$\overline{H} = \frac{1}{-j\omega\mu}\nabla\times\overline{E} = \frac{-j\overline{k}}{-j\omega\mu}\times\overline{E} \qquad (1.18)$$

$$\overline{H} = \frac{1}{\eta}\overline{k}\times\overline{E} \qquad (1.19)$$

which means that $\overline{H}\perp\overline{k}$ and $\overline{H}\perp\overline{E}$.

1.2.5 Dispersion Relation

For a general plane wave solution to Maxwell's equations, either from the separation of variables solution of Eq. (1.6) or by plugging $\overline{E} = \overline{E}_0 e^{-j\overline{k}\cdot\overline{r}}$ into (1.6), we obtain the dispersion relation

$$k_x^2 + k_y^2 + k_z^2 = k^2 = \omega^2 \mu\epsilon \tag{1.20}$$

If we define a unit vector $\hat{k} = \overline{k}/k$, then $\hat{k}$ is the direction of travel of the wave. This equation requires that no matter the direction of travel $\hat{k}$, the magnitude of the wave vector (or wavenumber of the plane wave) is always constant. The magnitude of the wave vector can be arranged into several differnt forms as given by

$$\begin{aligned} k &= \omega\sqrt{\mu\epsilon} && (1.21)\\ &= \omega\sqrt{\mu_o\epsilon_o}\sqrt{\mu_r\epsilon_r} && (1.22) \end{aligned}$$

for nonmagnetic materials $\mu_r = 1$

$$\begin{aligned} k &= \omega\sqrt{\mu_o\epsilon_o}\sqrt{\epsilon_r} && (1.23)\\ &= \omega\sqrt{\mu_o\epsilon_o}n && (1.24)\\ &= n\frac{\omega}{c} && (1.25)\\ &= \frac{2\pi}{\lambda} && (1.26) \end{aligned}$$

1.2.6 Summary of Plane Waves

If the plane is traveling in an arbitrary direction then the wave propagation direction is $\overline{k}$ and the general plane wave field equations are given by

$$\overline{E}(r) = \overline{E}_o e^{j\overline{k}\cdot\overline{r}}, \tag{1.27}$$

$$\overline{H}(r) = \overline{H}_o e^{j\overline{k}\cdot\overline{r}}, \tag{1.28}$$

where $\overline{r} = x\hat{x} + y\hat{y} + z\hat{z}$. The field amplitudes are related by

$$\begin{aligned} \overline{E}_o &\perp \overline{H}_o && (1.29)\\ \overline{E}_o &\perp \overline{k} && (1.30)\\ \overline{H}_o &\perp \overline{k} && (1.31)\\ |\overline{H}_o| &= \frac{|\overline{E}_o|}{\eta} && (1.32) \end{aligned}$$

$$\begin{aligned} \overline{E}_o &= -\eta\hat{k} \times \overline{H} && (1.33)\\ \overline{H}_o &= \frac{1}{\eta}\hat{k} \times \overline{H} && (1.34) \end{aligned}$$

1.3 Boundary Conditions on Fields

At a boundary or two-dimensional surface, the electric and magnetic fields on either side of the boundary must satisty the following conditions:

Field Quantity	*Relation*
Tangential $\overline{E}$	$\hat{n} \times \left(\overline{E}_1 - \overline{E}_2\right) = 0$
Tangential $\overline{H}$	$\hat{n} \times \left(\overline{H}_1 - \overline{H}_2\right) = \overline{J}_s$
Normal $\overline{D}$	$\hat{n} \cdot \left(\overline{D}_1 - \overline{D}_2\right) = \rho_s$
Normal $\overline{B}$	$\hat{n} \cdot \left(\overline{B}_1 - \overline{B}_2\right) = 0$

These conditions follow from Maxwell's equations. Note: $\hat{n}$ points from $2 \rightarrow 1$. The subscript 's' represents a *surface* quantity, so that the units of the sources are $J_s\ (A/m)$ and $\rho_s\ (C/m^2)$.

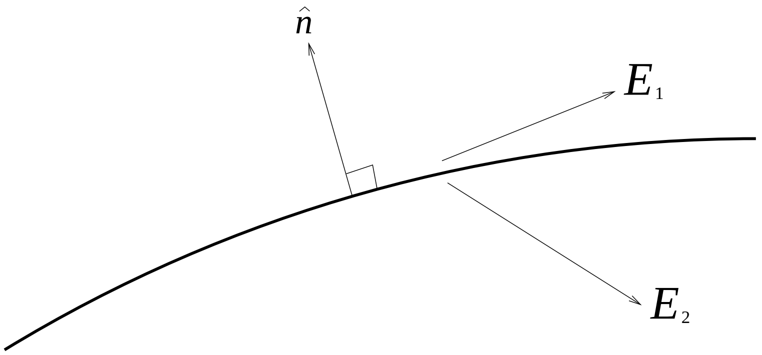

Figure 1.1: Boundary condition on the electric field. Above and below the interface, the tangential components of the electric field intensity vector must be equal. The normal components may differ, either if the permittivity changes at the boundary, or if there is a surface charge on the boundary.

1.4 Solution Methods

There are four main ways to solve Maxwell's equations:

Canonical sources. For a source with a simple symmetry, such as a point, line, or plane charge,

1. Guess the form of the field solution.
2. Plug back into the integral forms of Maxwell's equations and solve for unknown constants.

This approach is important but only works for a limited number of "canonical" sources. More complicated problems must be solved in other ways, although much physical understanding can be gained by relating complicated problems to these basic solutions.

Modal Method. This approach is generally used for problems with a canonical type of boundary condition, such as the rectangular waveguide.

1. Solve Maxwell's equations in free space with no sources (homogeneous solutions), *e.g.*, plane waves, cylindrical waves, or spherical waves.
2. Apply boundary conditions, usually at perfect electric conductors (PEC) or dielectric interfaces. This usually results in a countable subset of allowable field solutions, or modes. No matter what source is driving the system, the field must always be a linear combination of these modes.
3. Match the fields to the driving source to obtain a unique linear combination of modes.

Green's Functions. The Green's function method uses the field radiated by a point source to construct the field radiated by an arbitrary source:

1. Solve Maxwell's equations + boundary conditions for the field radiated by a point or "delta function" source. This solution is the Green's function $g(\overline{r}, \overline{r}')$, which gives the field at the point $\overline{r}$ due to a source located at the point $\overline{r}'$.
2. Obtain the field for an arbitrary source by convolving the Green's function with the source. Physically, this means using the principle of superposition to add the field due to many small point sources that together make up the actual distributed source.

Numerical. Maxwell's equations can be transformed into an approximate set of diference equations or a linear system, which can be solved by computer. This approach is used for complicated problems that arise in applications. Analytical solutions obtained using the other approaches are often very useful in gaining physical insight into numerical solutions.

In order to obtain a unique solution to Maxwell's equations using any of these methods, we must have three ingredients: Maxwell's equations, a driving source, and boundary conditions. If the source or boundary conditions are not specified, then we obtain a family of possible solutions.

In the second solution method, explicit examples of boundary conditions are given. What types of boundary conditions figure in the first and third approaches?