Chapter 1

Radiation

We are now going to look at the properties of radiating currents. Ultimately, this understanding will lead us into the basics of antenna design and analysis.

1.1 Scalar Green's Function

The problem we are examining is: given a current distribution, what are the fields radiated. In statics, fields have a simple enough behavior that we can write down the form of the field and solve for unknown constants using Ampere's law or Gauss's law. With time-varying fields, the solutions are more complicated, so we need a more powerful approach.

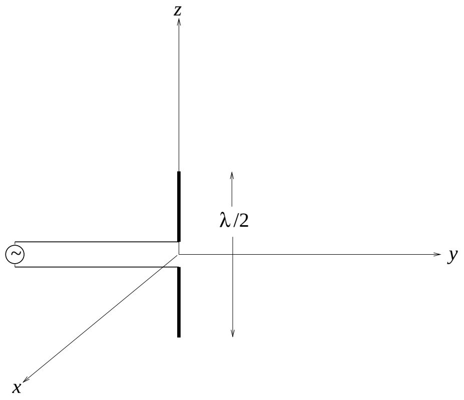

Figure 1.1: A half-wavelength dipole antenna.

As an example, consider a dipole antenna that is $\lambda/2$ long and where the wire is in the $\hat{z}$ direction, as shown in Fig. 1.1. We can approximately express the current on the wire as

$$\overline{J}(\overline{r}) = \delta(x)\delta(y)\hat{z}J_o \cos kz \tag{1.1}$$

where k is our wavenumber. (In general, we don't know the current on a wire antenna exactly, so the $\cos kz$ dependence is an approximation. We will look at some other approximations as well...) Note that this current goes to zero at $z = \pm\lambda/4$ which is at the ends of our wire. We now wish to compute the electric and magnetic fields radiated by this dipole antenna.

To simplify this analysis, we appeal to our knowledge of linear system theory. Recall that if you have a system, and you inject into this system an impulse at time $t = 0$ of the form $\delta(t)$, then the output $y(t) = h(t)$ of this system will be the system impulse response. Then, if I wish to know the response of the system to an arbitrary excitation $x(t)$, I simply perform the convolution

$$y(t) = h(t) * x(t) = \int_{-\infty}^{\infty} x(t-\tau)h(\tau)d\tau \tag{1.2}$$

The point is that the temporal impulse response completely characterizes the system, and therefore I can use it to determine the output for *any* potential input.

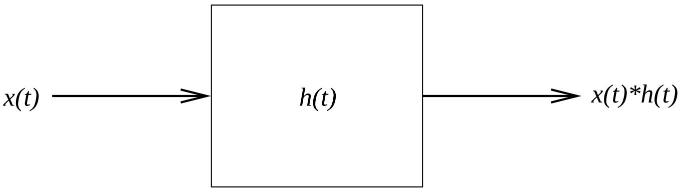

Figure 1.2: Linear system with impulse response $h(t)$. The output is obtained by convolving the input with the impulse response.

In electromagnetic analysis, we have been working the in the frequency domain (sinusoidal steady state), and therefore we are not so interested in the temporal impulse response. In other words, our excitations at this point are a sinusoid, and so we only need to know the system response for a sinusoid. However, in our case, a current distribution has a *spatial* characteristic. Just like in the analysis of temporal signal characteristics, an arbitrary temporal function varies in time, in our case our excitation varies in *space*. Therefore, we would like to construct a *spatial* impulse response for the system. This should provide us with a mechanism for determining the radiation for any current distribution.

This spatial impulse response is named the **Green's Function**. We will first solve for this function, and subsequently use it to determine the radiation from arbitrary currents.

1.1.1 Magnetic Vector Potential

One way to find this impulse response would be to work directly with Maxwell's equations. This can be done, but it is rather difficult. An easier approach is to use the simpler wave equation,

$$\nabla^2\overline{E} + k^2\overline{E} = j\omega\mu\overline{J} \tag{1.3}$$

The catch to this is that the wave equation has nonphysical solutions like $\hat{x}e^{-jkx}$ which do not satisfy Maxwell's equations (longitudinal waves). To sidestep this problem, we will come up with another vector quantity which also satisfies (1.3) and from which we can derive $\overline{E}$. This quantity is the magnetic vector potential $\overline{A}$, which can be thought of as a mathematical construct to assist in the analysis of electromagnetic fields.

The magnetic vector potential is defined based upon Gauss' law $\nabla \cdot \overline{B} = 0$. Since there is a vector identity stating that

$$\nabla \cdot (\nabla \times \overline{A}) = 0 \tag{1.4}$$

for any arbitrary vector $\overline{A}$, then we can express $\overline{B}$ as

$$\overline{B} = \nabla \times \overline{A} \tag{1.5}$$

In other words, if the divergence of a vector field is zero, then we can express it as the curl of another vector field.

Now, we can solve for $\overline{A}$ from Maxwell's equations. From Faraday's law

$$\begin{aligned} \nabla \times \overline{E} &= -j\omega\overline{B} = -j\omega\nabla \times \overline{A} && (1.6)\\ \nabla \times (\overline{E} + j\omega\overline{A}) &= 0 && (1.7) \end{aligned}$$

Now, we also have an identity that states

$$\nabla \times \nabla\phi = 0 \tag{1.8}$$

for any scalar function ϕ, so that if the curl of a vector field is zero, then it can be expressed as the gradient of some scalar. Therefore, we will rewrite the right hand side as

$$\begin{aligned} \nabla \times (\overline{E} + j\omega\overline{A}) &= -\nabla \times \nabla\phi && (1.9)\\ \overline{E} + j\omega\overline{A} &= -\nabla\phi && (1.10)\\ \overline{E} &= -j\omega\overline{A} - \nabla\phi && (1.11) \end{aligned}$$

The function ϕ here is the scalar electric potential that you encountered in beginning electromagnetics. The difference here is that we are looking at dynamic fields (whereas you mainly used this function for static fields previously).

Now, using Ampere's law, we have

$$\begin{aligned} \nabla \times \overline{H} &= \nabla \times \frac{1}{\mu}\nabla \times \overline{A} = \overline{J} + j\omega\epsilon\overline{E} && (1.12)\\ \nabla \times \nabla \times \overline{A} &= \mu\overline{J} + j\omega\mu\epsilon\overline{E} && (1.13)\\ \nabla(\nabla \cdot \overline{A}) - \nabla^2\overline{A} &= \mu\overline{J} + j\omega\mu\epsilon(-j\omega\overline{A} - \nabla\phi) && (1.14)\\ &= \mu\overline{J} + \omega^2\mu\epsilon\overline{A} - j\omega\mu\epsilon\nabla\phi && (1.15) \end{aligned}$$

Now, we have specified that the curl of $\overline{A}$ must equal $\overline{B}$. However, this does not completely specify $\overline{A}$. There is a theorem in vector calculus that indicates that to completely specify a vector, we must specify its curl *and* divergence. In looking at (1.15), we see that our result would be significantly simplified if we choose

$$\nabla(\nabla \cdot \overline{A}) = -j\omega\mu\epsilon\nabla\phi \tag{1.16}$$

or

$$\nabla \cdot \overline{A} = -j\omega\mu\epsilon\phi \tag{1.17}$$

We call this condition the *Lorentz Gauge*. There are other ways to specify the divergence of $\overline{A}$ (other gauges), but this is the best for our current problem. Using this gauge in (1.15) leads to a Helmholtz equation for $\overline{A}$ in the form

$$\boxed{\nabla^2\overline{A} + k^2\overline{A} = -\mu\overline{J} \qquad (1.18)}$$

Notice that this equation is similar to the wave equation (1.3).

1.1.2 Spatial Impulse Response Solution

Let's assume that the solution to this differential equation can be written as a convolution form, that is

$$\overline{A}(\overline{r}) = \int \overline{J}(\overline{r}')g(\overline{r}-\overline{r}')\,d\overline{r}'. \tag{1.19}$$

In this convolution the term $g(\overline{r})$ is the impulse response of the system. This impulse response relates the magnetic potential $\overline{A}$ to the current density $\overline{J}$ of the antenna. As shown in Fig. 1.3 the primed coordinates correspond to the source terms and the un-primed coordinates to the observation point.

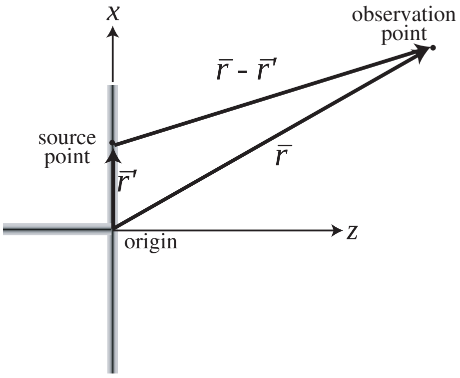

Figure 1.3: The coordinant system.

The equation that we need to solve becomes

$$(\nabla^2 + k^2)\int \overline{J}(\overline{r}')g(\overline{r}-\overline{r}')\,d\overline{r}' = -\mu\overline{J}(\overline{r}) \tag{1.20}$$

Brining the terms inside of the integral results in

$$\int (\nabla^2 + k^2)\overline{J}(\overline{r}')g(\overline{r}-\overline{r}')\,d\overline{r}' = -\mu\overline{J}(\overline{r}) \tag{1.21}$$

Since the term $\overline{J}(\overline{r}')$ does not vary with $\overline{r}$,

$$\int \overline{J}(\overline{r}')(\nabla^2 + k^2)g(\overline{r}-\overline{r}')\,d\overline{r}' = -\mu\overline{J}(\overline{r}) \tag{1.22}$$

We can create the same integral on the right hand side by using a delta function as given by

$$\int \overline{J}(\overline{r}')(\nabla^2 + k^2)g(\overline{r}-\overline{r}')\,d\overline{r}' = \int -\mu\overline{J}(\overline{r}')\delta(\overline{r}-\overline{r}')\,d\overline{r}' \tag{1.23}$$

Eliminating the integrals and dividing both sides by $\overline{J}$ results in

$$\boxed{(\nabla^2 + k^2)g(\overline{r}-\overline{r}') = -\mu\delta(\overline{r}-\overline{r}')} \tag{1.24}$$

We call $g(\overline{r}-\overline{r}')$ the **Scalar Green's Function**. It is essentially the impulse response of the antenna that relates the spatial magnetic vector potential $\overline{A}$ to an impulse current density.

It is important to notice that the functions depend on $\overline{r} - \overline{r}'$. By symmetry, $g(\overline{r} - \overline{r}')$ really must depend only on the distance between the *source point* $\overline{r}'$ and the *observation point* $\overline{r}$. We can therefore first solve the equation for $\overline{r}' = 0$, and then replace $\overline{r} \to \overline{r} - \overline{r}'$ after the final solution. In spherical coordinates, then, we have

$$g(\overline{r} - \overline{r}')\big|_{\overline{r}'=0} = g(\overline{r}) = g(r) \tag{1.25}$$

Now, expanding our Laplacian in spherical coordinates and recognizing that $g(r)$ is not a function of the coordinates θ or ϕ, we obtain

$$\frac{1}{r}\frac{d^2}{dr^2}\left[rg(r)\right] + k^2 g(r) = -\mu\delta(r) \tag{1.26}$$

If $r > 0$, the right hand side is zero, so

$$\frac{d^2}{dr^2}\left[rg(r)\right] + k^2\left[rg(r)\right] = 0 \tag{1.27}$$

which leads to the solution

$$rg(r) = Ce^{-jkr} + Fe^{jkr} \tag{1.28}$$

For an outgoing wave, we must have $F = 0$. Therefore, our Green's function is expressed as

$$\boxed{g(r) = C\frac{e^{-jkr}}{r}} \tag{1.29}$$

where we still do not know the value of the constant C. This wave is simply a spherical wave that is eminating from a point source.

To compute C, we need to integrate our differential equation (1.26) over a sphere of radius Δ in order to include the effect of the delta function. For simplicity, however, we will go back to our Laplacian form so that

$$\int_V \nabla^2 g(r)\,dv + k^2\int_V g(r)\,dv = -\mu\int_V \delta(r)\,dv = -\mu \tag{1.30}$$

Notice that we can write the first term in this equation as

$$\int_V \nabla^2 g(r)\,dv = \int_V \nabla\cdot\nabla g(r)\,dv = \oint_S \nabla g(r)\cdot\hat{n}dS \tag{1.31}$$

where we have used the Divergence Theorem, and $\hat{n}$ is the outward unit normal to our spherical integration surface ($\hat{n} = \hat{r}$). Using an expansion for the gradient in spherical coordinates leads to

$$\nabla g(r)\cdot\hat{n} = \nabla g(r)\cdot\hat{r} = -C\frac{e^{-jkr}}{r^2}(jkr+1) \tag{1.32}$$

Also, in spherical coordinates we have

$$\begin{aligned} dv &= r^2\sin\theta\,drd\theta d\phi &(1.33)\\ ds &= r^2\sin\theta\,d\theta d\phi &(1.34)\\ \int_0^{2\pi}\int_0^{\pi}\sin\theta\,d\theta d\phi &= 4\pi &(1.35)\end{aligned}$$

Finally, our integrated differential equation becomes

$$\begin{aligned} -\int_0^{2\pi}\int_0^{\pi} C\frac{e^{-jkr}}{r^2}(jkr+1)r^2\sin\theta\,d\theta d\phi + k^2\int_0^{\Delta}\int_0^{2\pi}\int_0^{\pi} C\frac{e^{-jkr}}{r}r^2\sin\theta\,d\theta d\phi dr &= -\mu &(1.36)\\ -4\pi Ce^{-jkr}(jkr+1)\Big|_{r=\Delta} + 4\pi k^2\int_0^{\Delta} Ce^{-jkr}rdr &= -\mu &(1.37)\end{aligned}$$

If we let $\Delta \to 0$, we obtain

$$\begin{aligned} -4\pi C &= -\mu && (1.38)\\ C &= \frac{\mu}{4\pi} && (1.39) \end{aligned}$$

so that

$$g(r) = \frac{\mu}{4\pi}\frac{e^{-jkr}}{r} \qquad (1.40)$$

If we then let $\overline{r}' \neq 0$, we obtain

$$\boxed{\begin{aligned} g(\overline{r}-\overline{r}') &= \frac{\mu}{4\pi}\frac{e^{-jk|\overline{r}-\overline{r}'|}}{|\overline{r}-\overline{r}'|} && (1.41)\\ \overline{A}(\overline{r}) &= \frac{\mu}{4\pi}\int \overline{J}(\overline{r}')\frac{e^{-jk|\overline{r}-\overline{r}'|}}{|\overline{r}-\overline{r}'|}\,d\overline{r}' && (1.42) \end{aligned}}$$

In these equations the $\overline{r}'$ term is the source (the current) and the $\overline{r}$ term is the observation point (see Fig. 1.3).

As an example, consider an Infinitesimal Dipole (Hertzian Dipole). The current is given as

$$\overline{J}(\overline{r}) = \hat{z}Id\delta(\overline{r}) \qquad (1.43)$$

This corresponds to a wire antenna with a constant current distribution on each side of the antenna and a short length relative to the electromagnetic wavelength. The magnetic vector potential is then easy to find, resulting in

$$\begin{aligned} \overline{A}(\overline{r}) &= \frac{\mu}{4\pi}\int \hat{z}Id\delta(\overline{r})\frac{e^{-jk|\overline{r}-\overline{r}'|}}{|\overline{r}-\overline{r}'|}\,d\overline{r}' && (1.44)\\ &= \hat{z}\frac{\mu Id}{4\pi r}e^{-jkr} = (\hat{r}\cos\theta - \hat{\theta}\sin\theta)\frac{\mu Id}{4\pi r}e^{-jkr} && (1.45) \end{aligned}$$

The fields and Poynting vector are

$$\overline{H}(\overline{r}) = \frac{1}{\mu}\nabla\times\overline{A}(\overline{r}) = \frac{jkId}{4\pi r}\hat{\phi}\left[1+\frac{1}{jkr}\right]\sin\theta e^{-jkr} \qquad (1.46)$$

$$\begin{aligned} \overline{E}(\overline{r}) &= \frac{1}{j\omega\epsilon}\nabla\times\overline{H}(\overline{r}) = \frac{jk\eta_o Id}{4\pi r}\left\{\hat{r}\left[\frac{1}{jkr}+\frac{1}{(jkr)^2}\right]2\cos\theta\right. && (1.47)\\ &\quad\left. +\,\hat{\theta}\left[1+\frac{1}{jkr}+\frac{1}{(jkr)^2}\right]\sin\theta\right\}e^{-jkr} && (1.48) \end{aligned}$$

$$\begin{aligned} \overline{S}(\overline{r}) &= \eta_o\left|\frac{kId}{4\pi r}\right|^2\left\{\hat{r}\left[1+\frac{1}{(jkr)^3}\right]\sin^2\theta\right. && (1.49)\\ &\quad\left. -\,\hat{\theta}\left[\frac{1}{jkr}-\frac{1}{(jkr)^3}\right]\sin 2\theta\right\} && (1.50) \end{aligned}$$

1.1.3 Far-Field Approximation

We should note that we generally cannot perform the integration to compute $\overline{A}$ given a current $\overline{J}$ due to the complexity of the Green's function inside the integrand. Furthermore, in most applications, we are interested in the fields far from the antenna. We therefore can use an approximation to simplify the mathematics.

Let $\hat{r}$ be the unit vector in the direction of the observation vector $\overline{r}$. For a point $\overline{r}$ very far from the source point $\overline{r}'$, we can approximate the value

$$|\overline{r} - \overline{r}'| \approx r - \hat{r} \cdot \overline{r}' \tag{1.51}$$

So, for the phase term in our Green's function, we can write

$$e^{-jk|\overline{r}-\overline{r}'|} \approx e^{-jkr} e^{jk\hat{r}\cdot\overline{r}'} \tag{1.52}$$

For the magnitude, we can simplify this expression even further by neglecting the term $\hat{r} \cdot \overline{r}'$ to write

$$\frac{1}{|\overline{r} - \overline{r}'|} \approx \frac{1}{r} \tag{1.53}$$

We therefore have our far-field approximate form of the magnetic vector potential given as

$$\boxed{\overline{A}_{\text{ff}}(\overline{r}) = \frac{\mu}{4\pi} \frac{e^{-jkr}}{r} \int \overline{J}(\overline{r}') e^{jk\hat{r}\cdot\overline{r}'} \, d\overline{r}' \qquad (1.54)}$$

Furthermore, when we take the curl of $\overline{A}_{\text{ff}}$ to obtain the magnetic and electric fields, we neglect any terms that come from this expression that decay faster than $1/r$ (i.e. terms that behave as $1/r^2$, $1/r^3$, etc.):

$$\begin{aligned}
\overline{B}_{\text{ff}} &= \nabla \times \overline{A}_{\text{ff}} \\
&= \frac{\hat{r}}{r\sin\theta}\left[\frac{\partial}{\partial\theta}A_\phi \sin\theta - \frac{\partial A_\theta}{\partial\phi}\right] + \frac{\hat{\theta}}{r}\left[\frac{1}{\sin\theta}\frac{\partial A_r}{\partial\phi} - \underbrace{\frac{\partial}{\partial r}rA_\phi}\right] + \frac{\hat{\phi}}{r}\left[\underbrace{\frac{\partial}{\partial r}rA_\theta} - \frac{\partial A_r}{\partial\theta}\right] \\
&\simeq -\frac{\hat{\theta}}{r}\frac{\partial}{\partial r}rA_\phi + \frac{\hat{\phi}}{r}\frac{\partial}{\partial r}rA_\theta \\
&\simeq \hat{\theta}jkA_\phi - \hat{\phi}jkA_\theta \\
&= -jk\hat{r} \times \overline{A}_{\text{ff}} \qquad (1.55)
\end{aligned}$$

Similarly, the electric field is

$$\begin{aligned}
\overline{E}_{\text{ff}} &= \frac{1}{j\omega\epsilon}\nabla \times \overline{H}_{\text{ff}} \approx -\frac{jk}{j\omega\epsilon}\hat{r} \times \overline{H}_{\text{ff}} \approx j\omega\hat{r} \times (\hat{r} \times \overline{A}_{\text{ff}}) \qquad (1.56) \\
& \qquad (1.57)
\end{aligned}$$

Consider again the Hertzian dipole. If we neglect all terms that decay faster than $1/r$ we obtain

$$\overline{H}_{\text{ff}}(\overline{r}) = \frac{jkId}{4\pi r}\hat{\phi}\sin\theta e^{-jkr} \tag{1.58}$$

$$\overline{E}_{\text{ff}}(\overline{r}) = \frac{jk\eta_o Id}{4\pi r}\hat{\theta}\sin\theta e^{-jkr} \tag{1.59}$$

$$\langle\overline{S}_{\text{ff}}\rangle = \frac{1}{2}\text{Re}\left\{\overline{S}\right\} = \hat{r}\frac{\eta_o}{2}\left|\frac{kId}{4\pi r}\right|^2 \sin^2\theta \tag{1.60}$$

It is important to note that only the real power gets to the far-field. All of the reactive power falls off faster than $1/r$, so it is confined to the region near the antenna and goes to zero in the far field.

Also, notice that $\overline{E}$ and $\overline{H}$ are orthogonal in the far field, much as with a plane wave. This type of wave is known as a spherical wave, because the constant phase fronts of the wave are spheres.

1.2 Antenna Definitions

There are several quantities that are important in defining the radiation behavior of antennas.

1. **Power Radiated to the Far-Field:**

$$P_{\text{rad}} = \oint_S \langle \overline{S}_{\text{ff}} \rangle \cdot \hat{n} ds \tag{1.61}$$

   where the surface S encloses the antenna. The vector $\hat{n}$ is the outward normal to the surface. Typically, we use a sphere for this surface.

2. **Radiation Resistance:**

$$R_{\text{rad}} = \frac{P_{\text{rad}}}{\frac{1}{2}|I|^2} \tag{1.62}$$

   where I is the current injected into the antenna terminals. This is the effective load seen by the antenna's feed transmission line.

3. **Radiation Efficiency:**

$$\eta_{\text{rad}} = \frac{P_{\text{rad}}}{P_{\text{in}}} \tag{1.63}$$

   where P_{in} is the power delivered to the antenna terminals. If the antenna is made of lossy materials, then this quantity is less than one. For many antennas, conductive losses are small, so that $\eta_{\text{rad}} \simeq 1$.

4. **Directivity:**

$$D(\theta,\phi) = \frac{\text{Power density at } (\theta,\phi)}{\text{Power density of an isotropic radiator}} = \frac{\langle \overline{S}_{\text{ff}} \rangle \cdot \hat{r}}{P_{\text{rad}}/4\pi r^2} = \frac{G(\theta,\phi)}{\eta_{\text{rad}}} \tag{1.64}$$

   Large directivity in a given direction means that the antenna radiation is tightly focused. An ideal isotropic (omnidirectional) antenna has a directivity of one in all directions. For a highly directive antenna, the maximum value of the directivity in dB is often specified,

5. **Gain:**

$$G(\theta,\phi) = \frac{\text{Power density at } (\theta,\phi)}{\text{Power density of an isotropic, lossless radiator}} = \frac{\langle \overline{S}_{\text{ff}} \rangle \cdot \hat{r}}{P_{\text{in}}/4\pi r^2} \tag{1.65}$$

   Gain differs from directivity only if the radiation efficiency is not equal to one.

6. **Radiation Pattern:**

$$p(\theta,\phi) = \frac{G(\theta,\phi)}{\max\{G(\theta,\phi)\}} = \frac{D(\theta,\phi)}{\max\{D(\theta,\phi)\}} \tag{1.66}$$

7. **Beamwidth**

   The beamwidth is the angular width of the main lobe (largest directivity peak). There are several ways of quantifying this: half power (3 dB point) full width, half power half width, and null to null width.

8. **Sidelobe levels**

   For a highly directive antenna, low sidelobes are generally desirable, so a common specification is maximum sidelobe level relative to the maximum directivity in dB.

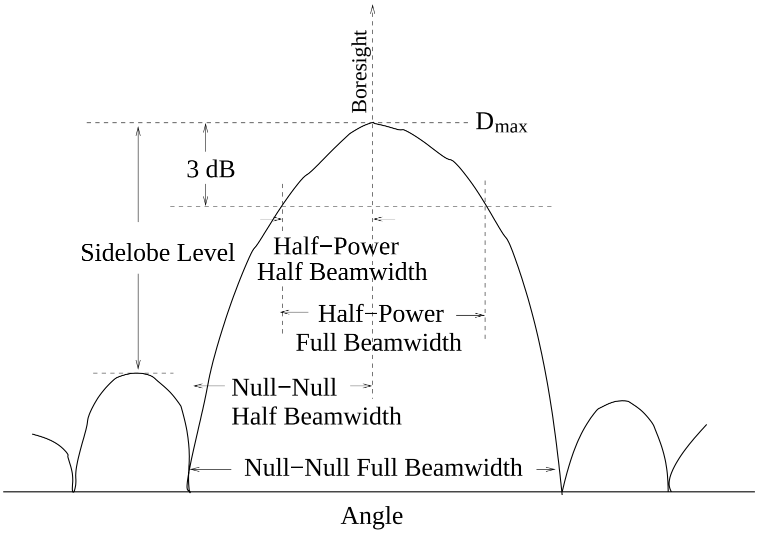

Figure 1.4: Antenna pattern definitions. The curve shown is a slice through the directivity pattern $D(\theta, \phi)$.

1.3 Wire Antennas

Let's consider a wire antenna of the type shown in Fig. 1.1. We can express the current distribution on the antenna as

$$\overline{J}(\overline{r}) = \delta(x)\delta(y)\hat{z}I(z) \tag{1.67}$$

where $I(z) = 0$ for $|z| > d/2$, and d is the length of the antenna.

We now want to compute the fields radiated by this antenna. We will do this using the magnetic vector potential in the far field, which from Eq. (1.54) is given by

$$\overline{A}_{\text{ff}}(\overline{r}) = \frac{\mu}{4\pi}\frac{e^{-jkr}}{r}\int \overline{J}(\overline{r}')e^{jk\hat{r}\cdot\overline{r}'}d\overline{r}'$$

In the far field, it is convenient to express the components of $\hat{r}$ in terms of the angles θ and ϕ of the spherical coordinate system (we will need to take a dot product with $\overline{r}'$, so we leave the unit vectors in the same coordinate system as $\overline{r}'$). Since the point $\overline{r}'$ will be integrated over the antenna, we express $\overline{r}'$ in a coordinate system that best matches the antenna (in this case, rectangular):

$$\begin{aligned}
\hat{r} &= \hat{x}\sin\theta\cos\phi + \hat{y}\sin\theta\sin\phi + \hat{z}\cos\theta && (1.68)\\
\overline{r}' &= x'\hat{x} + y'\hat{y} + z'\hat{z} \Rightarrow z'\hat{z} && (1.69)\\
\hat{r}\cdot\overline{r}' &= z'\cos\theta && (1.70)
\end{aligned}$$

Using this result,

$$\begin{aligned}
\overline{A}_{\text{ff}}(\overline{r}) &= \frac{\mu}{4\pi}\frac{e^{-jkr}}{r}\int \delta(x')\delta(y')\hat{z}I(z')e^{jk\hat{r}\cdot\overline{r}'}dx'\,dy'\,dz'\\
&= \hat{z}\frac{\mu}{4\pi}\frac{e^{-jkr}}{r}\int_{-d/2}^{d/2} I(z')e^{jkz'\cos\theta}dz' && (1.71)
\end{aligned}$$

For the Hertzian dipole example, $I(z') = I$ is constant and the length d is much shorter than the wavelength $(d \ll \lambda)$. Because the length is small, the exponential term in Eq. (1.71) is approximately equal to one, and the integral evaluates to Id. The magnetic vector potential becomes

$$\overline{A}_{\text{ff}}(\overline{r}) = \hat{z}\frac{\mu I d}{4\pi}\frac{e^{-jkr}}{r} = (\hat{r}\cos\theta - \hat{\theta}\sin\theta)\frac{\mu I d}{4\pi}\frac{e^{-jkr}}{r} \tag{1.72}$$

If we use $\overline{H}_{\text{ff}} = -(j\omega/\eta)\hat{r}\times\overline{A}_{\text{ff}}$ and $\overline{E}_{\text{ff}} = j\omega\hat{r}\times(\hat{r}\times\overline{A}_{\text{ff}})$ to find the far magnetic and electric fields, we obtain the results found in Eqs. (1.58) and (1.59). From the radiated fields, we can compute all of the antenna parameters (power radiated to the far field, radiation resistance, directivity, pattern, beamwidth, and so forth).

The constant current assumed in the Hertzian dipole example is merely an approximation to the true current on a wire antenna. A better approximation is found in Eq. (1.1). The radiated fields and antenna parameters for this current distribution can be computed by using (1.71). The directivity and radiation resistance found using this more exact current model will be different from that of the Hertzian dipole, and closer to actual measured values for a real wire antenna.

1.4 Aperture Diffraction

We now are prepared to use our theory to examine aperture antennas. An aperture is nothing more than an opening in an otherwise inpenetrable sheet or *screen*, which is illuminated from behind (Fig. 1.5). The radiated electromagnetic wave emanates from this opening or aperture.

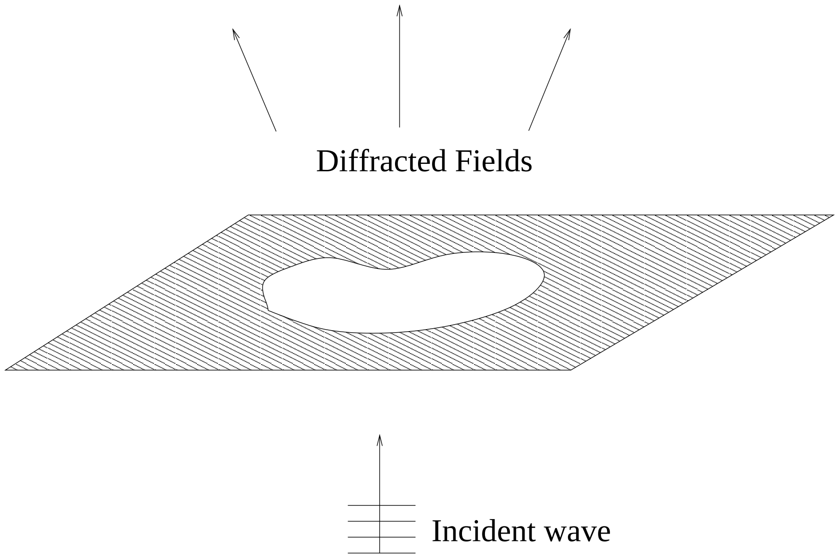

Figure 1.5: An aperture antenna, consisting of a hole in a conducting screen illuminated by an incident wave. The fields above the screen are diffracted by the aperture.

We will express the incident field as a nonuniform plane wave traveling in the $+z$ direction, so that

$$\overline{E}^i = \hat{x}E_o(x,y)e^{-jkz} \tag{1.73}$$

Note that we allow this plane wave to have variations in the x-y plane, and it is therefore not a *uniform* plane wave like we have looked at in the past. An example of this type of wave is a mode of a rectangular waveguide. This wave is incident on an aperture in an infinite ground plane lying in the x-y plane at $z = 0$. We wish to determine the fields radiated by the aperture.

1.4.1 Equivalent Current

The difficulty is that the radiation integral derived previously is only valid for radiation by currents in free space. We need to transform the aperture problem from a screen and incident wave to an equivalent current in free space (which radiates approximately the same fields in our region of interest above the screen) as shown in Fig. 1.6

To solve this problem, we make 3 key assumptions or observations:

1. The fields in the aperture are the same as the incident field distribution over the aperture
2. The currents induced on the conducting ground plane don't contribute significantly to the radiation
3. We are not concerned with the *reflection* behavior in the region $z < 0$.

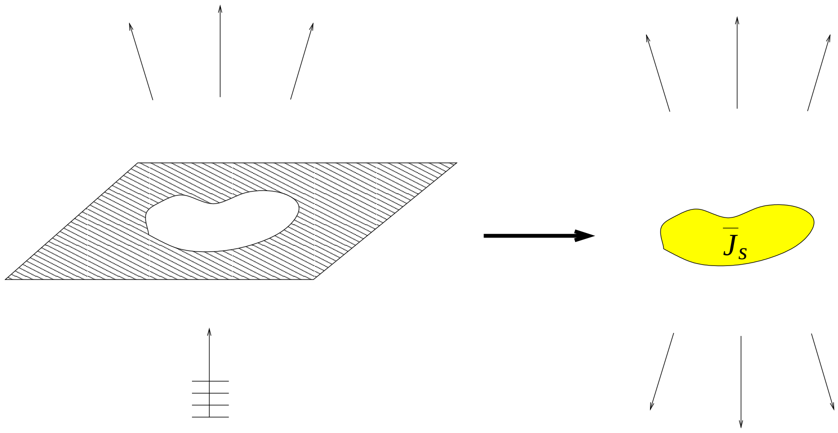

Figure 1.6: Equivalent current for an aperture antenna. The fields radiated by the current are approximately the same as the diffracted fields above the screen.

Item 3 above allows us to set up an equivalent problem that will create a field for $z > 0$ consistent with our assumptions, even if it does not provide the correct field for $z < 0$. Using Item 2, we will therefore create an equivalent problem of a current sitting in space within the aperture. We can then remove the ground plane, and work with the current distribution that has replaced the aperture. To determine the form of this surface current, we recognize from Item 1 that it must create fields just beyond the aperture ($z = 0^+$) that are the same as the plane wave fields. In other words:

$$\overline{E}(z=0^+) = \hat{x}E_o(x,y) \tag{1.74}$$

$$\overline{H}(z=0^+) = \hat{y}\frac{E_o(x,y)}{\eta_0} \tag{1.75}$$

Since we have replaced the aperture with a current, it must also radiate into the space $z < 0$. Furthermore, it will radiate symmetrically in both directions, with the exception that the fields will travel in the opposite direction. Therefore,

$$\overline{E}(z=0^-) = \hat{x}E_o(x,y) \tag{1.76}$$

$$\overline{H}(z=0^-) = -\hat{y}\frac{E_o(x,y)}{\eta_0} \tag{1.77}$$

The equivalent current density in the aperture can now be determined using the continuity conditions on $\overline{H}$ at $z = 0$, or

$$\begin{aligned}\overline{J}_s(x,y) &= \hat{z}\times\left\{\overline{H}(z=0^+) - \overline{H}(z=0^-)\right\} \\ &= -\hat{x}\left\{\frac{E_o(x,y)}{\eta_0} + \frac{E_o(x,y)}{\eta_0}\right\} \\ &= -\hat{x}\frac{2E_o(x,y)}{\eta_0}\end{aligned} \tag{1.78}$$

where x, y are restricted to the region defined by the aperture. From now on, we will denote this region by the symbol S.

1.4.2 Far-Field Radiation

Using (1.54) leads to

$$\overline{A}_{\text{ff}}(\overline{r}) = \frac{\mu}{4\pi}\frac{e^{-jkr}}{r}\int_S \overline{J}_s(x',y')e^{jk\hat{r}\cdot\overline{r}'}\,dx'\,dy' \tag{1.79}$$

For this problem, we can expand our vectors as

$$\begin{aligned}
\hat{r} &= \hat{x}\sin\theta\cos\phi + \hat{y}\sin\theta\sin\phi + \hat{z}\cos\theta && (1.80)\\
\overline{r}' &= x'\hat{x} + y'\hat{y} && (1.81)\\
\hat{r}\cdot\overline{r}' &= x'\sin\theta\cos\phi + y'\sin\theta\sin\phi = \sin\theta(x'\cos\phi + y'\sin\phi) && (1.82)
\end{aligned}$$

For notational simplicity, let us define

$$\begin{aligned}
k_x &= k\sin\theta\cos\phi && (1.83)\\
k_y &= k\sin\theta\sin\phi && (1.84)\\
k\hat{r}\cdot\overline{r}' &= k_x x' + k_y y' && (1.85)
\end{aligned}$$

We can therefore write

$$\overline{A}_{\text{ff}}(\overline{r}) = -\hat{x}\frac{\mu}{4\pi}\frac{e^{-jkr}}{r}\int_S \frac{2E_o(x',y')}{\eta_0}e^{j(k_x x' + k_y y')}\,dx'\,dy' \tag{1.86}$$

Finally, using (1.56) leads to

$$\overline{E}_{\text{ff}} = j\omega\hat{r}\times(\hat{r}\times\overline{A}_{\text{ff}}) = \overline{\mathcal{E}}(r,\theta,\phi)F(\theta,\phi) \tag{1.87}$$

where

$$\begin{aligned}
\overline{\mathcal{E}}(r,\theta,\phi) &= -\hat{r}\times\hat{r}\times\hat{x}\frac{j\omega\mu}{4\pi}\frac{e^{-jkr}}{r} && (1.88)\\
F(\theta,\phi) &= \int_S \frac{2E_o(x',y')}{\eta_0}e^{j(k_x x' + k_y y')}\,dx'\,dy' && (1.89)
\end{aligned}$$

Clearly, the first term results from the $\hat{x}$ polarization of the incident wave, but is otherwise independent of the incident wave. The second term, $F(\theta,\phi)$, represents the influence of the incident wave. What mathematical transform does this term look like?

Example: Constant rectangular aperture distribution. Suppose that $E_o(x',y') = E_o =$ constant. Let's also assume that the aperture is rectangular, defined by $-L_x/2 \le x \le L_x/2$ and $-L_y/2 \le y \le L_y/2$. Then

$$\begin{aligned}
F(\theta,\phi) &= \frac{2E_o}{\eta_0}\int_{-L_y/2}^{L_y/2}\int_{-L_x/2}^{L_x/2} e^{j(k_x x' + k_y y')}\,dx'\,dy' && (1.90)\\
&= \frac{2E_o}{\eta_0}L_x L_y\frac{\sin(L_x k_x/2)}{L_x k_x/2}\frac{\sin(L_y k_y/2)}{L_y k_y/2} && (1.91)
\end{aligned}$$

The radiated power for this field can be computed as discussed above. However, in this case, we know the fields immediately in front of the aperture to be those of the incident plane wave. Since the real power density for a plane wave is given as $|E|^2/2\eta_0$, we can in this case simplify our computation as

$$P_{\text{rad}} = \int_S \frac{|E_o(x',y')|^2}{2\eta_0}\,dx'\,dy' = \frac{|E_o|^2 L_x L_y}{2\eta_0} \tag{1.92}$$

The gain is therefore given as

$$
\begin{aligned}
G(\theta,\phi) &= \frac{|\overline{E}_{\text{ff}}|^2/2\eta_0}{P_{\text{rad}}/4\pi r^2} && (1.93)\\
&= \frac{4\pi}{\lambda^2}L_xL_y(\cos^2\theta\cos^2\phi+\sin^2\phi)\left[\frac{\sin(L_xk_x/2)}{L_xk_x/2}\right]^2\left[\frac{\sin(L_yk_y/2)}{L_yk_y/2}\right]^2 && (1.94)\\
&= \frac{4\pi}{\lambda^2}L_xL_y(\cos^2\theta\cos^2\phi+\sin^2\phi)\left[\frac{\sin(kL_x\sin\theta\cos\phi/2)}{kL_x\sin\theta\cos\phi/2}\right]^2\left[\frac{\sin(kL_y\sin\theta\sin\phi/2)}{kL_y\sin\theta\sin\phi/2}\right]^2 && (1.95)
\end{aligned}
$$

In going from the first line to the second line, we have used the relationships

$$
\begin{aligned}
\hat{x} &= \hat{r}\sin\theta\cos\phi+\hat{\theta}\cos\theta\cos\phi-\hat{\phi}\sin\phi\\
|\hat{r}\times\hat{r}\times\hat{x}|^2 &= \cos^2\theta\cos^2\phi+\sin^2\phi
\end{aligned}
$$

Notice that the maximum gain is

$$G_{\text{max}} = \frac{4\pi}{\lambda^2}\text{Area} \tag{1.96}$$

This relationship is often used to define an effective aperture area for other types of antennas (for example, what is the effective area of a Hertzian dipole)? A theorem in antenna analysis is that the power received by a matched load for any antenna is equal to the product of the incident field power density and the effective area of the antenna in the direction of the incident field.

With the same graphical method that is used for plotting antenna array patterns (visible window method), it is relatively straightforward to sketch the gain in the planes $\phi = 0$ and $\phi = \pi/2$. Let's consider $\phi = 0$. Since $\sin 0 = 0$ and $\cos 0 = 1$, our gain expression becomes

$$G(\theta,0) = \frac{4\pi}{\lambda^2}L_xL_y\left[\frac{\sin(kL_x\sin\theta/2)}{kL_x\sin\theta/2}\right]^2 = \frac{4\pi}{\lambda^2}L_xL_y\left[\frac{\sin u_x}{u_x}\right]^2 \tag{1.97}$$

where $u_x = kL_x\sin\theta/2$. Note that the zeros of the function occur at $u_x = m\pi$, where $m = \pm1,\pm2,\ldots$, and the maximum is at $u_x = 0$. Therefore, it is easy to sketch G as a function of u_x.

Now, in order to transform from a plot of $G(u_x)$ to a plot of $G(\theta,0)$, we create another u_x axis the same length as the one used to plot the gain. Since $u_x = \pi L_x/\lambda\sin\theta$, by trigonometry we can draw a semi-circle of radius $\pi L_x/\lambda$ centered on your new u_x axis. The angle from the vertical axis will represent the value of θ. It is then straightforward to sketch the gain as a function of θ.

What happens as the size of the aperture is increased (or the operating frequency is raised)? From the equations above or the graphical pattern, it is easy to see that the main lobe of the pattern will grow narrower and higher. This means that a wider aperture in general has a narrower pattern and higher gain. This is why radiotelescopes are so large! Mathematically, this also follows from the behavior of the Fourier transform in Eq. (1.89), since the transform of a broad function is narrow.

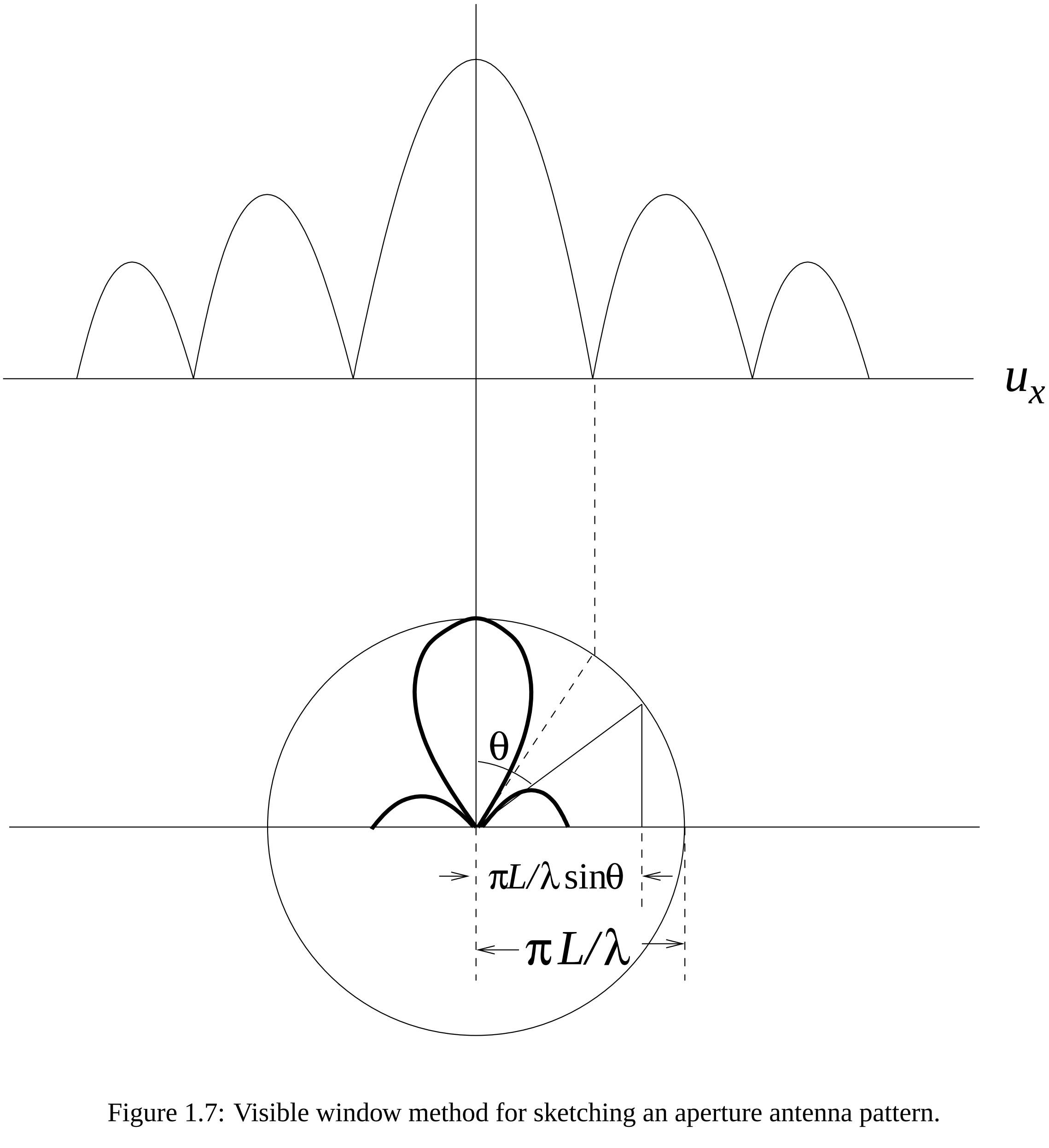

Figure 1.7: Visible window method for sketching an aperture antenna pattern.

1.5 Circular Aperture

Let's now consider that our aperture is circular with a radius of a instead of rectangular. In this case, we must integrate in cylindrical coordinates, which leads to the expressions

$$\begin{aligned} x' &= \rho' \cos\phi' && (1.98)\\ y' &= \rho' \sin\phi' && (1.99)\\ \hat{r}\cdot\overline{r}' &= x'\sin\theta\cos\phi + y'\sin\theta\sin\phi \\ &= \rho'\sin\theta(\cos\phi\cos\phi' + \sin\phi\sin\phi') = \rho'\sin\theta\cos(\phi-\phi') && (1.100) \end{aligned}$$

Eq. (1.89) then becomes

$$F(\theta,\phi) = \frac{2E_o}{\eta_0}\int_0^a \int_0^{2\pi} e^{jk\rho'\sin\theta\cos(\phi-\phi')}\,\rho'\,d\phi'\,d\rho' \tag{1.101}$$

We start with the ϕ' integration. There is an integration rule that states

$$\int_0^{2\pi} e^{jt\cos(\phi-\phi')}d\phi' = 2\pi J_0(t) \tag{1.102}$$

where $J_0(t)$ is the zeroth-order Bessel function of the first kind. We therefore have

$$F(\theta,\phi) = \frac{4\pi E_o}{\eta_0}\int_0^a \rho' J_0(k\rho'\sin\theta)\,d\rho' \tag{1.103}$$

Using $x = k\rho'\sin\theta$ and $dx = k\sin\theta d\rho'$, we obtain the integral

$$F(\theta,\phi) = \frac{4\pi E_o}{\eta_0}\frac{1}{(k\sin\theta)^2}\int_0^{ka\sin\theta} xJ_0(x)\,dx \tag{1.104}$$

We now use a second integration rule

$$\int xJ_0(x)dx = xJ_1(x) + C \tag{1.105}$$

to obtain

$$\begin{aligned} F(\theta,\phi) &= \frac{4\pi E_o}{\eta_0}\frac{1}{(k\sin\theta)^2}ka\sin\theta J_1(ka\sin\theta) && (1.106)\\ &= \frac{4\pi E_o}{\eta_0}\frac{a}{k\sin\theta}J_1(ka\sin\theta) && (1.107)\\ &= \frac{4\pi E_o a^2}{\eta_0}\frac{J_1(ka\sin\theta)}{ka\sin\theta} && (1.108) \end{aligned}$$

We also know that

$$\begin{aligned} P_{\text{rad}} &= \int_S \frac{|E_o(x',y')|^2}{2\eta_0}\,dS' = \frac{|E_o|^2\pi a^2}{2\eta_0} && (1.109)\\ \overline{E}_{\text{ff}} &= -\hat{r}\times\hat{r}\times\hat{x}\frac{j\omega\mu}{4\pi}\frac{e^{-jkr}}{r}\frac{4\pi E_o a^2}{\eta_0}\frac{J_1(ka\sin\theta)}{ka\sin\theta} && (1.110)\\ &= -\hat{r}\times\hat{r}\times\hat{x}jkE_o a^2\frac{e^{-jkr}}{r}\frac{J_1(ka\sin\theta)}{ka\sin\theta} && (1.111) \end{aligned}$$

The magnitude of the power density in the far field is

$$\frac{|\overline{E}_{\text{ff}}|^2}{2\eta_0} = \frac{k^2 a^4 |E_o|^2}{2\eta_0 r^2}(\cos^2\theta\cos^2\phi + \sin^2\phi)\left[\frac{J_1(ka\sin\theta)}{ka\sin\theta}\right]^2 \tag{1.112}$$

The gain (assuming a lossless aperture) is therefore given as

$$\begin{aligned} G(\theta,\phi) &= \frac{|\overline{E}_{\text{ff}}|^2/2\eta_0}{P_{\text{rad}}/4\pi r^2} && (1.113) \\ &= \frac{4\pi}{\lambda^2}\pi a^2(\cos^2\theta\cos^2\phi + \sin^2\phi)\left[\frac{2J_1(ka\sin\theta)}{ka\sin\theta}\right]^2 && (1.114) \end{aligned}$$