Chapter 2

Waveguides

A waveguide guides or directs electromagnetic waves from one region of space to another.

There are two primary purposes of waveguides:

- Guide power from one point to another (i.e. a power line)
- Transmit information from one point to another

In this class we are more interested in the second purpose of transmitting information.

What properties of waveguides do we care about most?

- The propagation rate of information
- Possible distortions in transmitted information
- Attenaution of the EM wave
- What frequencies propagate in particular waveguides?
- What waveguide dimensions guide EM waves of a particular frequency?
- Why would it be important to support only a single mode?
- What happens if multiple modes exist within a waveguide?
- What determines the excitation of the various modes?
- And of course, a fundamental understanding of how waveguides work and how to analyze them.

We will be covering all of these concepts except for waveguide attenuation.

What is the general approach that we will be using:

- Start with a general solution to Maxwells equations for an arbitrary waveguide
- Determine the parameters of interest (field profiles and propagation constant)

- Get the parameters of interest for parallel plate metallic waveguides. This is from EE360 so the derivation will not be covered but we will just get the end results.
- Apply these parameters to answer some of the questions listed above
- Derive the solutions for more complicated waveguides (rectangular metallic, dielectric slab)

2.1 Parallel Plates

We first consider a plane wave reflecting off a conducting surface.

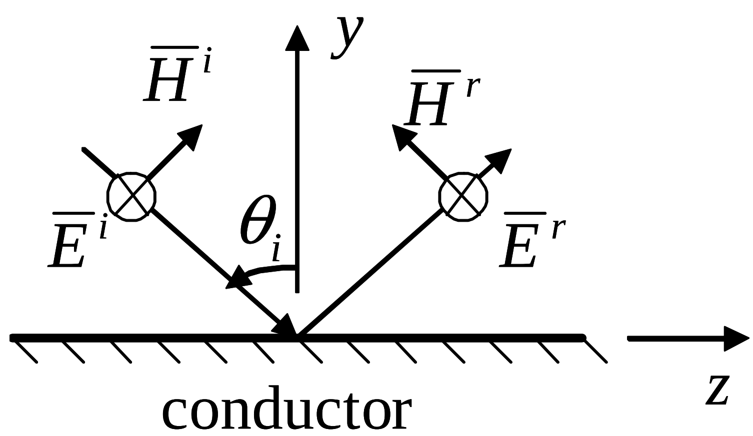

Figure 2.1: Plane wave reflecting off of a boundary.

2.1.1 TE (Transverse Electric)

$$
\begin{aligned}
\overline{E}^i &= \hat{x} E_o^i e^{-jk_0(-y\cos\theta_i + z\sin\theta_i)} \\
\overline{H}^i &= (\hat{y}\sin\theta_i + \hat{z}\cos\theta_i)\frac{E_o^i}{\eta_0} e^{-jk_0(-y\cos\theta_i + z\sin\theta_i)} \\
\overline{E}^r &= \hat{x}\Gamma_\perp E_o^i e^{-jk_0(y\cos\theta_i + z\sin\theta_i)} \\
\overline{H}^r &= (\hat{y}\sin\theta_i - \hat{z}\cos\theta_i)\Gamma_\perp \frac{E_o^i}{\eta_0} e^{-jk_0(y\cos\theta_i + z\sin\theta_i)}
\end{aligned}
$$

where we have used that $\theta_r = \theta_i$. Since $\Gamma_\perp = -1$, the total fields are:

$$\begin{aligned}
\overline{E} = \overline{E}^i + \overline{E}^r &= \hat{x} E_o^i \left(e^{jk_0 y \cos\theta_i} - e^{-jk_0 y \cos\theta_i} \right) e^{-jk_0 z \sin\theta_i} \\
&= \hat{x} j 2 E_o^i \sin\left(k_0 y \cos\theta_i\right) e^{-jk_0 z \sin\theta_i} \\
\overline{H} = \overline{H}^i + \overline{H}^r &= \frac{E_o^i}{\eta_0} \left[\hat{y} \sin\theta_i \left(e^{jk_0 y \cos\theta_i} - e^{-jk_0 y \cos\theta_i} \right) \right. \\
&\quad \left. + \hat{z} \cos\theta_i \left(e^{jk_0 y \cos\theta_i} + e^{-jk_0 y \cos\theta_i} \right) \right] e^{-jk_0 z \sin\theta_i} \\
&= 2 \frac{E_o^i}{\eta_0} \left[\hat{y} j \sin\theta_i \sin\left(k_0 y \cos\theta_i\right) + \hat{z} \cos\theta_i \cos\left(k_0 y \cos\theta_i\right) \right] e^{-jk_0 z \sin\theta_i}
\end{aligned}$$

Note that $E_x = H_y = 0$ at $k_0 y \cos\theta_i = m\pi$, $m = 1, 2, \ldots$. Since tangential $\overline{E}$ and normal $\overline{H}$ are zero here, we could slide a conductor in at these points without disturbing the field. If we could get the field into this structure, it would have the same form as we have observed here.

If we look at this another way, if we put a plate at $y = d$, then this field will exist in the structure. The value of θ_i at which the wave will travel will be given by:

$$k_0 d \cos\theta_i = m\pi, \qquad m = 1, 2, 3, \ldots$$

Let's generalize this a little more. Now that we see what the waves look like, we recognize that the angle θ_i is not very important in general. Let's take our wavenumber k_0 and make it into a vector.

$$\begin{aligned}
\hat{k} &= = \hat{y} k_y + \hat{z} k_z \\
k_y &= k_0 \cos\theta_i \\
k_z &= k_0 \sin\theta_i \\
k_y^2 + k_z^2 = k_0^2 &
\end{aligned}$$

Now

$$\begin{aligned}
\overline{E} &= \hat{x} j 2 E_o \sin(k_y y) e^{-jk_z z} \\
\overline{H} &= \frac{2E_o}{\eta_0 k_0} \left[\hat{y} j k_z \sin(k_y y) + \hat{z} k_y \cos(k_y y) \right] e^{-jk_z z}
\end{aligned}$$

The Poynting vector is:

$$\begin{aligned}
\overline{S} = \overline{E} \times \overline{H}^* &= \left[j 2 E_o \sin(k_y y) e^{-jk_z z} \right] \left[\frac{2E_o^*}{\eta_0 k_0} e^{jk_z z} \right] \left[-\hat{z} j k_z \sin(k_y y) - \hat{y} k_y \cos(k_y y) \right] \\
&= \frac{4|E_o|^2}{\eta_0 k_0} \sin(k_y y) \left[\hat{z} k_z \sin(k_y y) - \hat{y} j k_y \cos(k_y y) \right]
\end{aligned}$$

Notice that all real power flows in the z direction!! So, our structure is guiding power along the z direction.

So, the physical picture is that we have two plates which confine the electromagnetic wave. Because we know that

$$k_y d = m\pi$$

there is a countably infinite set of field configurations possible. We call each possible configuration a **mode**. In this case, we call the modes TE$_m$ modes. Note that if $m = 0$, $\overline{E} = 0$, which means no field exists. Therefore, m ranges from 1 to ∞.

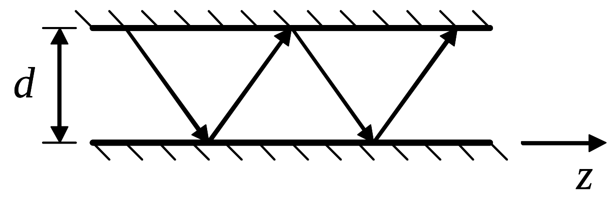

Figure 2.2: Plane wave reflecting off of a boundary.

2.1.2 TM (Transverse Magnetic)

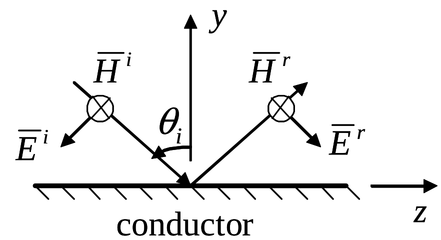

Figure 2.3: Plane wave reflecting off of a boundary.

$$
\begin{aligned}
\overline{H}^i &= \hat{x}\frac{E_o}{\eta_0}e^{-j(-k_y y + k_z z)} \\
\overline{E}^i &= (-\hat{y}k_z - \hat{z}k_y)\frac{E_o}{k_0}e^{-j(-k_y y + k_z z)} \\
\overline{H}^r &= \hat{x}\frac{E_o}{\eta_0}e^{-j(k_y y + k_z z)} \\
\overline{E}^r &= (-\hat{y}k_z + \hat{z}k_y)\frac{E_o}{k_0}e^{-j(k_y y + k_z z)}
\end{aligned}
$$

$$
\begin{aligned}
\overline{H} = \overline{H}^i + \overline{H}^r &= \hat{x}\frac{E_o}{\eta_0}\left(e^{jk_y y} + e^{-jk_y y}\right)e^{-jk_z z} \\
&= \hat{x}\frac{2E_o}{\eta_0}\cos(k_y y)e^{-jk_z z} \\
\overline{E} = \overline{E}^i + \overline{E}^r &= \frac{E_o}{k_0}\left[-\hat{y}k_z\left(e^{jk_y y} + e^{-jk_y y}\right) - \hat{z}k_y\left(e^{jk_y y} - e^{-jk_y y}\right)\right]e^{-jk_z z} \\
&= \frac{2E_o}{k_0}\left[-\hat{y}k_z\cos(k_y y) - \hat{z}jk_y\sin(k_y y)\right]e^{-jk_z z}
\end{aligned}
$$

Once again, if $k_y d = m\pi$ (i.e. plate placed at $y = d$), there will be no impact on the fields since $E_z(y = d) = 0$.

In this case, if $k_y = m\pi/d$ and $m = 0$ we have that $k_y = 0$. Since $k_y^2 + k_z^2 = k_0^2$, then $k_z = k_0$.

$$
\begin{aligned}
\overline{E} &= -\hat{y}\frac{2E_o}{k_0}k_z e^{-jk_z z} = \hat{y}2E_o e^{-jk_0 z} \\
\overline{H} &= \hat{x}\frac{2E_o}{\eta_0}e^{-jk_0 z}
\end{aligned}
$$

This is just a plane wave traveling in the $+z$ direction.

Therefore, we have TM_m modes for $m = 0, 1, 2, \ldots$. The TM_0 mode is a TEM (transverse electromagnetic) mode.

2.2 Waveguide Parameters

2.2.1 Ray Picture

Let's look at what the transverse wavevector is.

We know that the field solution for an arbitrary parallel plate waveguide modes is not a plane wave, because a plane wave cannot satisfy the appropriate boundary conditions at the waveguide walls. But the modes can be expressed as a sum of plane waves. Viewing the solution this way leads to some interesting additional insight.

Let's consider a TE mode. The electric field can be expressed as

$$\begin{aligned}
\overline{E} &= \hat{x} j 2 E_o \sin(k_y y) e^{-j\beta z} && (2.1)\\
&= \hat{x} j 2 E_o \left(\frac{e^{+jk_y y} - e^{-jk_y y}}{2j} \right) e^{-j\beta z} && (2.2)\\
&= \hat{x} E_o e^{-j(-k_y y + \beta z)} - \hat{x} E_o e^{-j(k_y y + \beta z)} && (2.3)\\
& && (2.4)
\end{aligned}$$

Notice that the two terms are plane waves, with wavevectors $\pm k_y \hat{x} + \beta \hat{z}$.

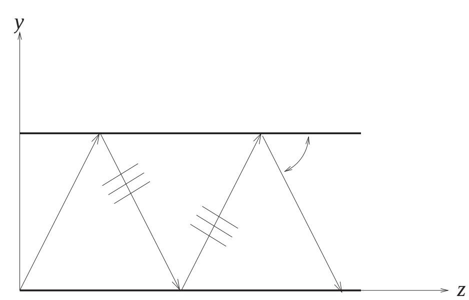

Figure 2.4: Ray picture of the dominant mode of a rectangular waveguide.

The two plane waves can be thought of as a single wave or ray that bounces back and forth between the right and left sides of the waveguide. The angle of the direction of propagation of one of the plane waves with respect to the z axis is

$$\tan \Psi = \frac{k_y}{\beta}$$

for the first mode ($m = 1$) this becomes

$$\begin{aligned}\tan\Psi &= \frac{\pi/d}{\sqrt{k^2 - k_y^2}} \\ &= \frac{\pi/d}{\sqrt{\omega^2\mu\epsilon - \pi^2/d^2}} \\ &= \frac{\pi/d}{\sqrt{\omega^2/c^2 - \pi^2/d^2}} \\ &= \frac{1}{\sqrt{(f/f_c)^2 - 1}} \end{aligned} \tag{2.5}$$

where we have used the fact that $f_c = c/(2d)$ in the last step. As the operating frequency goes to infinity, the angle Ψ goes to zero, so the field propagates essentially straight down the waveguide. As $f \to f_c$, the angle goes to $\pi/2$, so the field is zig-zagging back and forth between the waveguide walls and makes little forward progress. This shows that operating near cutoff is undesirable, because the group velocity (velocity at which a signal can be transmitted) is small relative to the speed of light.

2.2.2 Dispersion and Wave Velocities

Dispersion relates to the frequency dependence of propagation speed in a waveguide or material. So far, we have considered only single frequency signals. You can't send information using a single frequency tone. In order to analyze the propagation of a multifrequency signal such as a modulated carrier tone, we must break the signal into single-frequency components using the Fourier transform. Each Fourier component can then be propagated separately through the system (waveguide, material, etc.) and then recombined to determine the output signal.

Consider a signal of the form

$$p(t) = f(t)\cos\omega_o t = \text{Re}\left\{f(t)e^{j\omega_o t}\right\} = \text{Re}\left\{s(t)\right\} \tag{2.6}$$

where $f(t)$ is a relatively narrow band and slowly varying (low frequency) envelope that modulates the carrier amplitude. The Fourier transform of $s(t)$ is

$$S(\omega) = \int_{-\infty}^{\infty} s(t)e^{-j\omega t}\,dt = \int_{-\infty}^{\infty} f(t)e^{j\omega_o t}e^{-j\omega t}\,dt = \int_{-\infty}^{\infty} f(t)e^{-j(\omega-\omega_o)t}\,dt = F(\omega - \omega_o) \tag{2.7}$$

where

$$F(\omega) = \int_{-\infty}^{\infty} f(t)e^{-j\omega t}\,dt \tag{2.8}$$

is the Fourier transform of the envelope function.

Now, this wave travels as $A(z)e^{-j\beta z}$, where $A(z)$ represents attenuation of the signal due to loss (or possibly amplification). The signal at a point z is given by

$$S_o(\omega) = A(z)F(\omega-\omega_o)e^{-j\beta z} \tag{2.9}$$

$$s_o(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} A(z)F(\omega-\omega_o)e^{-j\beta z}e^{j\omega t}\,d\omega \tag{2.10}$$

$$= \frac{A(z)}{2\pi}\int_{-\infty}^{\infty} F(\omega-\omega_o)e^{j(\omega t-\beta z)}\,d\omega \tag{2.11}$$

Let's look at a few special cases.

1. Suppose $\beta = \omega/c$, which is what we observe for a plane wave in a dispersionless medium.

$$s_o(t) = \frac{A(z)}{2\pi} \int_{-\infty}^{\infty} F(\omega - \omega_o) e^{j\omega(t-z/c)} \, d\omega = \frac{A(z)}{2\pi} f(t - z/c) e^{j\omega_o(t-z/c)} \tag{2.12}$$

   So, the signal is simply delayed in time by z/c, which represents the propagation time. No other signal distortion occurs.

2. Since $f(t)$ is narrowband, $F(\omega - \omega_o)$ is zero except for small values of $(\omega - \omega_o)$. Then if β depends on frequency, we only need a few terms of the Taylor expansion of $\beta(\omega)$:

$$\beta(\omega) = \beta(\omega_o) + (\omega - \omega_o) \left.\frac{d\beta}{d\omega}\right|_{\omega=\omega_o} + \frac{(\omega-\omega_o)^2}{2} \left.\frac{d^2\beta}{d\omega^2}\right|_{\omega=\omega_o} + \cdots \tag{2.13}$$

$$\approx \beta(\omega_o) + (\omega - \omega_o)\beta'(\omega_o) \tag{2.14}$$

$$\approx \beta_o + (\omega - \omega_o)\beta'_o \tag{2.15}$$

   In this approximation, the output signal is

$$s_o(t) = \frac{A(z)}{2\pi} \int_{-\infty}^{\infty} F(\omega - \omega_o) e^{j[\omega t - \beta_o z - (\omega - \omega_o)\beta'_o z]} \, d\omega \tag{2.16}$$

$$= \frac{A(z)}{2\pi} \int_{-\infty}^{\infty} F(y) e^{j[yt + \omega_o t - \beta_o z - y\beta'_o z]} \, dy \tag{2.17}$$

$$= A(z) e^{j(\omega_o t - \beta_o z)} \frac{1}{2\pi} \int_{-\infty}^{\infty} F(y) e^{jy[t - \beta'_o z]} \, dy \tag{2.18}$$

$$= A(z) e^{j(\omega_o t - \beta_o z)} f(t - \beta'_o z) \tag{2.19}$$

   where we have used $y = \omega - \omega_o$ and $dy = d\omega$. Then, the output time-domain waveform is

$$p_o(t) = \text{Re}\{s_o(t)\} = A(z) f(t - \beta'_o z) \cos(\omega_o t - \beta_o z) \tag{2.20}$$

Considering this second case, we see that this level of dispersion does not distort the waveform, but that the carrier wave and modulating signal "travel" at different velocities. For example, to ride on a constant phase front, we set

$$\phi = \omega_o t - \beta_o z = \text{constant} \tag{2.21}$$

$$z = \frac{\omega_o t - \phi}{\beta_o} \tag{2.22}$$

$$v_p = \frac{dz}{dt} = \frac{\omega_o}{\beta_o} \tag{2.23}$$

which indicates that the phase velocity is exactly what we are used to. To ride on top of the envelope, however:

$$\tau = t - \beta'_o z = \text{constant} \tag{2.24}$$

$$z = \frac{t - \tau}{\beta'_o} \tag{2.25}$$

$$v_g = \frac{dz}{dt} = \frac{1}{\beta'_o} = \left.\left(\frac{d\beta}{d\omega}\right)^{-1}\right|_{\omega=\omega_o} \tag{2.26}$$

v_g is called the group velocity. It is the velocity at which the energy or information travels.

In a waveguide, for example, we can evaluate the velocities:

$$\beta = \sqrt{k^2 - k_c^2} = k\sqrt{1 - (f_c/f)^2} \tag{2.27}$$

$$\frac{d\beta}{d\omega} = \frac{2k/c}{2\sqrt{k^2 - k_c^2}} = \frac{k}{c\beta} \tag{2.28}$$

$$v_g = \frac{c\beta}{k} < c \qquad (\text{since } \beta < k) \tag{2.29}$$

$$v_p = \frac{\omega}{\beta} = \frac{kc}{\beta} > c \tag{2.30}$$

$$v_g v_p = c^2 \tag{2.31}$$

2.3 Dielectric Slab Waveguide

At high frequencies (especially optical frequencies) the loss associated with the induced current in the metal walls is too high. A transmission line filled with dielectric material but without conducting walls is another structure that may be used to guide electromagnetic waves. This dielectric slab waveguide eliminates the metallic absorption loss.

Consider a dielectric slab that is surrounded by another dielectric material that has a lower permittivity. A representative slab is shown in Fig. 2.5. The waveguide thickness is $2d$ and the center region (core) has a higher permittivity than the two outer regions (cladding) ($\epsilon_1 > \epsilon_2$). We assume propagation in the z direction and no variation of fields in the y direction ($W \gg 2d$). This makes the problem simpler, because it reduces to a two-dimensional analysis.

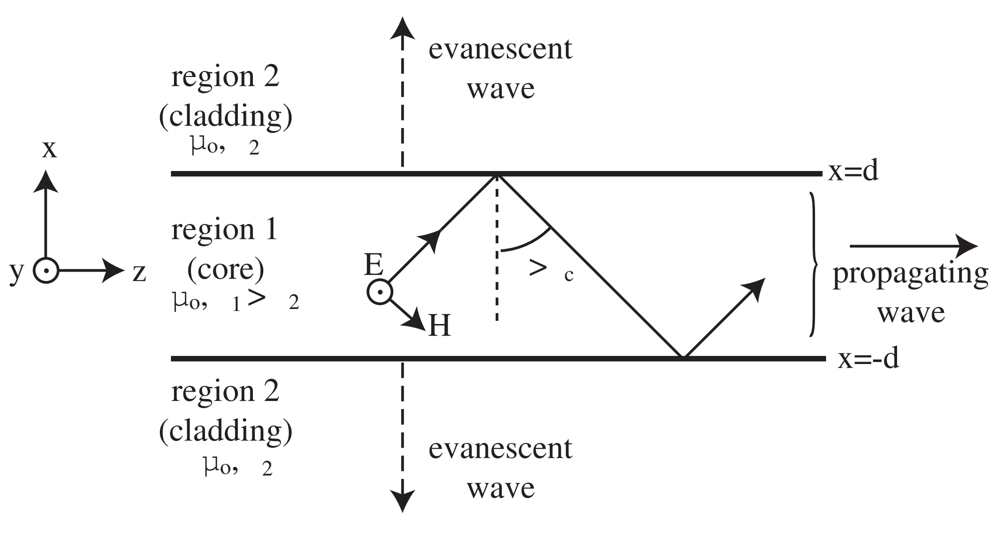

Figure 2.5: Dielectric slab waveguide of width $2d$.

Since the dielectric waveguide is intended to guide the light, the fields in the cladding region should be evanescent or decay in amplitude away from the slab. This guiding property requires the ray angle to be past the critical angle $\theta > \theta_c = \sin^{-1}\left(\sqrt{\frac{\epsilon_2}{\epsilon_1}}\right) = \sin^{-1}\left(\frac{n_2}{n_1}\right)$. This requires the propagation constant to be in the range

$$n_1 k_o \sin(\theta_c) < \beta < n_1 k_o \sin(90^o) \tag{2.32}$$

$$n_2 k_o < \beta < n_1 k_o. \tag{2.33}$$

We could use the concept of computing E_z or H_z and then applying Maxwell's equations to obtain the remaining field components just like we did in the metallic waveguides. However, the fact that we are now in two dimensions makes things a little easier. We will still use TE and TM to indicate that the field has no longitudinal electric or magnetic field, respectively. In the case of TE modes, the electric field will only have a $\hat{y}$ component (why?). We can then easily find the magnetic field using Faraday's law. A similar argument holds for TM modes, as we will see shortly.

Two other key concept concerning dielectric waveguides deserve attention. The first is that, due to the symmetry of the geometry, the fields will either be symmetric or anti-symmetric about the y-z plane. The second is that in order for the field to be guided by the high-permittivity dielectric slab, the fields outside the slab must be evanescent, i.e. they decay in the x direction. We will use these observations in the formulations that follow.

2.3.1 TE Modes

The electric field for the TE modes must satisfy our wave equation as given by

$$\frac{d^2 E_y}{dx^2} + \left(\omega^2 \mu\epsilon - \beta^2\right) E_y = 0 \tag{2.34}$$

This wave equation is valid in all regions. However, remember that the permittivity ϵ is different in the two regions.

Our experience with rectangular waveguides tells us what the solutions must be (before application of the boundary conditions). However, our argument about symmetry makes it so that within the slab, the variation will either be

$$E_y = A \cos k_{1x} x \tag{2.35}$$

or

$$E_y = B \sin k_{1x} x. \tag{2.36}$$

Note that we used the $\sin$ and $\cos$ form rather than the $\exp$ form. This is because we know that the fields form standing waves within the waveguide region. By substituting these into the wave equation (Eq. 2.34), it can be shown that

$$k_{1x}^2 = \omega^2 \mu \epsilon_1 - \beta^2 = k_1^2 - \beta^2. \tag{2.37}$$

The fields outside of the slab (in the cladding regions) are also of the same basic form.

$$\begin{array}{rcll} E_y &=& C e^{jk_{2x}x} & x \geq d \\ E_y &=& D e^{jk_{2x}x} & x \leq -d \end{array} \tag{2.38}$$

These field solutions use the exponential form because we know that they will not form standing waves. By substituting these in Eq. (2.34), it can be shown that

$$k_{2x}^2 = \omega^2 \mu \epsilon_2 - \beta^2 = k_2^2 - \beta^2. \tag{2.39}$$

In order to maintain guiding, the fields in the cladding must be evanescent or decay in amplitude with distance away from the slab. This requirement caused the propagation constant to be in the range of $n_2 k_o < \beta$. Therefore, the propagation constant in the cladding regions is complex as given by

$$k_{2x} = \pm j\alpha_2, \tag{2.40}$$

where

$$\alpha_2 = \pm\sqrt{\beta^2 - k_2^2}. \tag{2.41}$$

The sign of k_{2x} is chosen such that the fields decay with distance away from the waveguide. The resulting fields in the cladding regions are then given by

$$\begin{array}{rcll} E_y &=& Ce^{-\alpha_2 x} & x \geq d \\ E_y &=& De^{\alpha_2 x} & x \geq -d \end{array} \tag{2.42}$$

Therefore, the electric field in the various regions is given by

$$\overline{E} = \hat{y} \begin{cases} E_2 e^{-\alpha x - j\beta z} & x \geq d \\ E_1 \left\{ \begin{array}{c} \sin k_x x \\ \cos k_x x \end{array} \right\} e^{-j\beta z} & |x| \leq d \\ \left\{ \begin{array}{c} - \\ + \end{array} \right\} E_2 e^{+\alpha x - j\beta z} & x \leq -d \end{cases} \tag{2.43}$$

where the top and bottom lines in the braces refer to the antisymmetric and symmetric modes, respectively. The solution that uses the $\cos$ is called the symmetric solution and the solution that uses the $\sin$ is called the anti-symmetric solution.

Using Faraday's law, we can now compute the magnetic fields,

$$\overline{H} = \frac{-1}{j\omega\mu} \nabla \times \overline{E} = \begin{cases} \frac{E_2}{\omega\mu_0} \left(-\hat{x}\beta - \hat{z}j\alpha\right) e^{-\alpha x - j\beta z} & x \geq d \\ \frac{E_1}{\omega\mu} \left(-\hat{x}\beta \left\{ \begin{array}{c} \sin k_x x \\ \cos k_x x \end{array} \right\} - \hat{z} j k_x \left\{ \begin{array}{r} -\cos k_x x \\ \sin k_x x \end{array} \right\} \right) e^{-j\beta z} & |x| \leq d \\ \left\{ \begin{array}{c} - \\ + \end{array} \right\} \frac{E_2}{\omega\mu_0} \left(-\hat{x}\beta + \hat{z}j\alpha\right) e^{+\alpha x - j\beta z} & x \leq -d \end{cases} \tag{2.44}$$

Note that in these field expressions, we have used the fact that the z variation is of the form $e^{-j\beta z}$ both inside and outside the slab. Why do we know that this propagation constant is the same in both regions? Note also that we have 4 unknowns: E_2/E_1, k_x, α, and β.

Since these fields must obey the wave equation (with $\partial^2/\partial y^2 = 0$), we know that

$$k_x^2 + \beta^2 = k_1^2 = \omega^2 \mu \epsilon_1 \tag{2.45}$$

$$-\alpha^2 + \beta^2 = k_2^2 = \omega^2 \mu_0 \epsilon_2 \tag{2.46}$$

which gives us two constraints for determining our unknowns. We need two additional constraints in order to find all 4 unknowns.

Let's start by enforcing continuity of tangential electric fields at the core-cladding interface. We will first consider the symmetric modes. Therefore, at $x = d$

$$E_1 \cos(k_x d) e^{-j\beta z} = E_2 e^{-\alpha d} e^{-j\beta z} \qquad \rightarrow \qquad \cos(k_x d) E_1 - e^{-\alpha d} E_2 = 0 \tag{2.47}$$

Note that applying continuity at $x = -d$ results in an identical equation, so this does not help us. This stems from the symmetry of the problem, and in reality we have already used this symmetry to break the problem into symmetric and antisymmetric modes.

Since we need one more equation, we will apply continuity of tangential (i.e. $\hat{z}$ component) magnetic field at the boundary. At $x = d$ we have

$$-jk_x \frac{E_1}{\omega\mu} \sin(k_x d) e^{-j\beta z} = -j\alpha \frac{E_2}{\omega\mu} e^{-\alpha d} e^{-j\beta z} \qquad \rightarrow \qquad k_x \sin(k_x d) E_1 - \alpha e^{-\alpha d} E_2 = 0 \tag{2.48}$$

and again, we get the exact same equation at $x = -d$. The easiest thing to do is to divide these two equations to simplify them.

$$\frac{k_x \sin(k_x d) E_1}{\cos(k_x d) E_1} = \frac{\alpha e^{-\alpha d} E_2}{e^{-\alpha d} E_2} \tag{2.49}$$

$$\alpha = k_x \tan(k_x d) \tag{2.50}$$

which can be re-written as

$$\boxed{(\alpha d) = (k_x d)\tan(k_x d) \qquad \text{symmetric TE modes}} \tag{2.51}$$

Subtracting (2.45) and (2.46) yields

$$k_x^2 + \alpha^2 = \omega^2 \mu_0 \epsilon_0 (\epsilon_1 - \epsilon_2) \tag{2.52}$$

or as

$$(k_x d)^2 + (\alpha d)^2 = \omega^2 \mu_0 \epsilon_0 (\epsilon_1 - \epsilon_2) d^2 \tag{2.53}$$

We can actually solve this graphically, as will be shown later. Alternately, we can combine these two equations to obtain

$$\begin{aligned} \alpha^2 + k_x^2 &= \omega^2 \mu_o \epsilon_o \left(n_1^2 - n_2^2\right) \\ k_x^2 \tan^2(k_x d) + k_x^2 &= \omega^2 \mu_o \epsilon_o \left(n_1^2 - n_2^2\right) \\ \tan^2(k_x d) + 1 &= \frac{\omega^2 \mu_o \epsilon_o \left(n_1^2 - n_2^2\right)}{k_x^2} \\ \tan(k_x d) &= \sqrt{\frac{\omega^2 \mu_0 \epsilon_0 (n_1 - n_2)}{k_x^2} - 1} \end{aligned} \tag{2.54}$$

and solve this with a nonlinear solver on a calculator or computer.

1. Solutions in the range $(m-1)\pi/2 \le k_x d \le m\pi/2$ $m = 1, 3, 5, \ldots$ we will call TE$_m$ modes. These correspond to the symmetric TE modes.

2. Cutoff occurs when the mode is no longer guided, which occurs as soon as α becomes negative. So, we define cutoff as the frequency at which $\alpha = 0$. Using (2.51), this implies that $\tan(k_x d) = 0$, so that $k_x d = (m-1)\pi/2$, $m = 1, 3, 5, \ldots$. Using (2.53) with $\alpha = 0$ leads to

$$f_{c,m} = \frac{c(m-1)}{4d\sqrt{\mu_r \epsilon_r - 1}} \tag{2.55}$$

for the cutoff frequencies of the TE modes. Note that $f_{c,1} = 0$, so the lowest order mode propagates at any frequency. Furthermore, since at cutoff $\beta = k_2$ and $\beta^2 + k_x^2 = k_1^2$, the angle of incidence of the wave on the dielectric boundary can be expressed as

$$\theta_i = \sin^{-1} \frac{\beta}{\sqrt{\beta^2 + k_x^2}} = \sin^{-1} \frac{k_2}{k_1} = \sin^{-1} \sqrt{\frac{\epsilon_2}{\epsilon_1}} = \theta_c \tag{2.56}$$

which you may recognize as the critical angle. So, cutoff occurs when the angle of incidence on the boundary is smaller than the critical angle. Makes sense, doesn't it?

Observe also that the cutoff condition of $\beta = k_2$ means that the propagation constant becomes that of the surrounding medium.

3. Note that k_x is frequency dependent, unlike in the rectangular waveguide. From the two dispersion relations, we can see that $\omega\sqrt{\mu_0\epsilon_2} \leq \beta \leq \omega\sqrt{\mu\epsilon_1}$, which should be intuitive.
4. As the frequency gets larger, $\alpha \to \infty$ which means that the field decays very rapidly outside the dielectric. The behavior of the mode becomes like that of a parallel plate waveguide filled with a dielectric.

Note that we could repeat the entire procedure for the antisymmetric TE modes. The dispersion relation (2.53) remains the same. The guidance condition becomes

$$\boxed{(\alpha d) = -(k_x d)\cot(k_x d) \qquad \text{antisymmetric TE modes}} \tag{2.57}$$

Again, cutoff occurs for $k_x d = (m-1)\pi/2$, $m = 2, 4, 6, \ldots$. These are therefore the even order TE modes.

2.3.2 TM Modes

We can repeat the whole process for TM modes. In this case, we have

$$\overline{H} = \hat{y} \begin{cases} H_2 e^{-\alpha x - j\beta z} & x \geq d \\ H_1 \left\{ \begin{array}{c} \sin k_x x \\ \cos k_x x \end{array} \right\} e^{-j\beta z} & |x| \leq d \\ \left\{ \begin{array}{c} - \\ + \end{array} \right\} H_2 e^{+\alpha x - j\beta z} & x \leq -d \end{cases} \tag{2.58}$$

where the top and bottom lines in the braces refer to the antisymmetric and symmetric modes, respectively. Using Ampere's law, we can now compute the electric fields

$$\overline{E} = \frac{1}{j\omega\epsilon} \nabla \times \overline{H} = \begin{cases} \frac{H_2}{\omega\epsilon_0} \left(\hat{x}\beta + \hat{z}j\alpha\right) e^{-\alpha x - j\beta z} & x \geq d \\ \frac{H_1}{\omega\epsilon} \left(\hat{x}\beta \left\{ \begin{array}{c} \sin k_x x \\ \cos k_x x \end{array} \right\} + \hat{z}jk_x \left\{ \begin{array}{c} -\cos k_x x \\ \sin k_x x \end{array} \right\} \right) e^{-j\beta z} & |x| \leq d \\ \left\{ \begin{array}{c} - \\ + \end{array} \right\} \frac{H_2}{\omega\epsilon_0} \left(\hat{x}\beta - \hat{z}j\alpha\right) e^{+\alpha x - j\beta z} & x \leq -d \end{cases} \tag{2.59}$$

We go through the exact same sequence of steps for this case. The dispersion relations remain the same. The guidance conditions become

$$
\begin{aligned}
(\alpha d) &= \frac{\epsilon_0}{\epsilon}(k_x d)\tan(k_x d) && \text{symmetric TM modes} && (2.60)\\
(\alpha d) &= -\frac{\epsilon_0}{\epsilon}(k_x d)\cot(k_x d) && \text{antisymmetric TM modes} && (2.61)
\end{aligned}
$$

2.3.3 Graphical Mode Solution

Solving the simultaneous equations (2.53) and (2.51) (symmetric modes) or (2.57) (antisymmetric modes) for the values of k_x and α is difficult, because the equations are transcendental. But it is easy to find an approximate solution using a graphical technique. This also provides valuable physical insight.

Each solution involves a guidance equation and a dispersion equation. These equations are:

$$(\alpha d) = (k_x d)\tan(k_x d) \tag{2.62}$$

$$(\alpha d) = -(k_x d)\cot(k_x d) \tag{2.63}$$

$$(k_x d)^2 + (\alpha d)^2 = \omega^2 \mu_0 \epsilon_0 (\epsilon_1 - \epsilon_2) d^2 \tag{2.64}$$

Notice that we have multiplied each of the equations by d to make them dimensionless, which allows us to use one plot for any value of the frequency or slab thickness. The idea is to plot these equations on the same axes. Think of $k_x d$ as x and αd as y. These equations then become

$$(\alpha d) = (k_x d)\tan(k_x d) \quad \rightarrow \quad y = x\tan(x) \tag{2.65}$$

$$(\alpha d) = -(k_x d)\cot(k_x d) \quad \rightarrow \quad y = -x\cot(x) \tag{2.66}$$

$$(k_x d)^2 + (\alpha d)^2 = \omega^2 \mu_0 \epsilon_0 (\epsilon_1 - \epsilon_2) d^2 \quad \rightarrow \quad x^2 + y^2 = \omega^2 \mu_0 \epsilon_0 (\epsilon_1 - \epsilon_2) d^2 \tag{2.67}$$

- The first two equations are periodic functions. These functions are independent of both the premittivities and the the thickness of the waveguide. Figure 2.6 shows these plots.
- The second equation is a circle with a radius of $R = \omega\sqrt{\mu_0 \epsilon_0 (\epsilon_1 - \epsilon_2)}d$. Thus, the radius of this circle changes with waveguide parameters.

The second equation can be easily ploted as a quarter circle onto the lines shown in Fig. 2.6. Each intersection of the curves corresponds to a mode. The coordinates of the point give us the value of $k_x d$ and αd, which can be easily divided by d to give k_x and α.

Dielectric Waveguide Example

How many modes exist in a dielectric waveguide that has the following parameters? index of refraction of the core $n_1 = 1.6$, index of refraction of the cladding $n_2 = 1.5$, wavelength $\lambda = 1.0\mu m$, waveguide core thickness $2d = 4\mu m$.

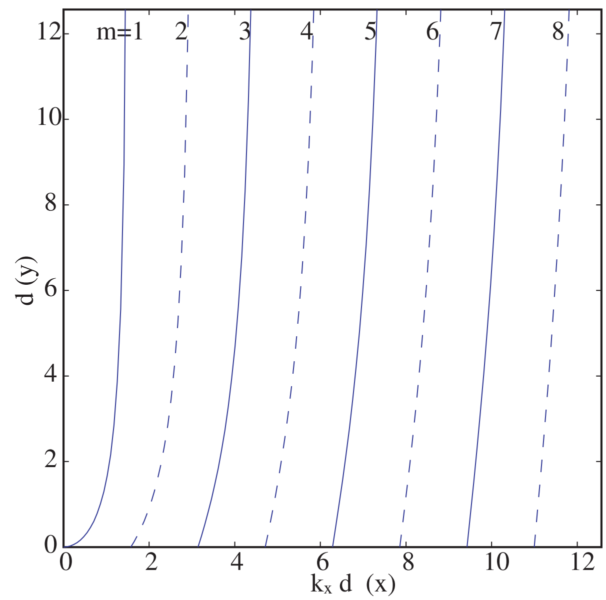

Figure 2.6: Plots of the guidance condition for a TE dielectric slab waveguide. The solid lines are for the symmetric modes and the dashed lines are for teh antisymmetric modes.

The equations are

$$\alpha d = k_y d \tan\left(k_y d\right) \tag{2.68}$$

$$\alpha d = -k_y d \cot\left(k_y d\right) \tag{2.69}$$

$$\left(k_y d\right)^2 + \left(\alpha d\right)^2 = \left(k_o d\right)\left(n_1^2 - n_2^2\right) \tag{2.70}$$

Using $k_y d = x$ and $\alpha d = y$ these equations become

$$y = x \tan x \tag{2.71}$$

$$y = -x \cot x \tag{2.72}$$

$$x^2 + y^2 = \left(k_o d\right)^2\left(n_1^2 - n_2^2\right) \tag{2.73}$$

For this example the radius of the circle is given by

$$r = \frac{2\pi}{1.0}\frac{4}{2}\sqrt{1.6^2 - 1.5^2} \tag{2.74}$$

$$r = 2.23\pi = 7.0 \tag{2.75}$$

The equation $x \tan x$ is equal to zero when $x = 0\pi, 2\pi, 3\pi, ...m\pi$ and is equal to ∞ when $x = \frac{\pi}{2}, \frac{3\pi}{2}, \frac{5\pi}{2}, ...\frac{\pi}{2} + m\pi$.

The equation $-x \cot x$ is equal to zero when $x = \frac{\pi}{2}, \frac{3\pi}{2}, \frac{5\pi}{2}, ...\frac{\pi}{2} + m\pi$ and is equal to ∞ when $x = \pi, 2\pi, 3\pi, ...m\pi$. And when $x = 0$ $-x \cot x = -1$.

The radius of the circle for this problem is $r = 7.0 = 2.23\pi$. There are 3 even modes (0, π, 2π) and 2 odd modes (0.5π, 1.5π).

What is the waveguide thickness for single mode operation? We need

$$r < 0.5\pi \tag{2.76}$$

$$\frac{2\pi}{1.0} d\sqrt{1.6^2 - 1.5^2} < \frac{\pi}{2} \tag{2.77}$$

$$d < 0.449 \tag{2.78}$$

or a slab thickness of $2d = 0.9\mu m$

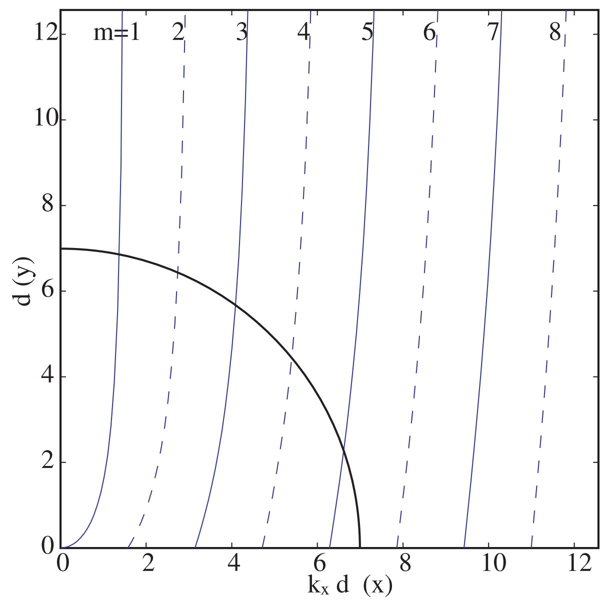

Figure 2.7: Graphical mode solution with $n_1 = 1.6$, $n_2 = 1.5$, $2d = 4\mu m$, and $\lambda = 1.0\mu m$.

2.3.4 Example - TM Modes

For the TM modes, the graphical solution method is complicated slightly by the fact that Eqs. (2.60) and (2.61) depend on the permittivity of the slab. This means that we need to plot curves for different values of ϵ_r. The required plot is shown below.

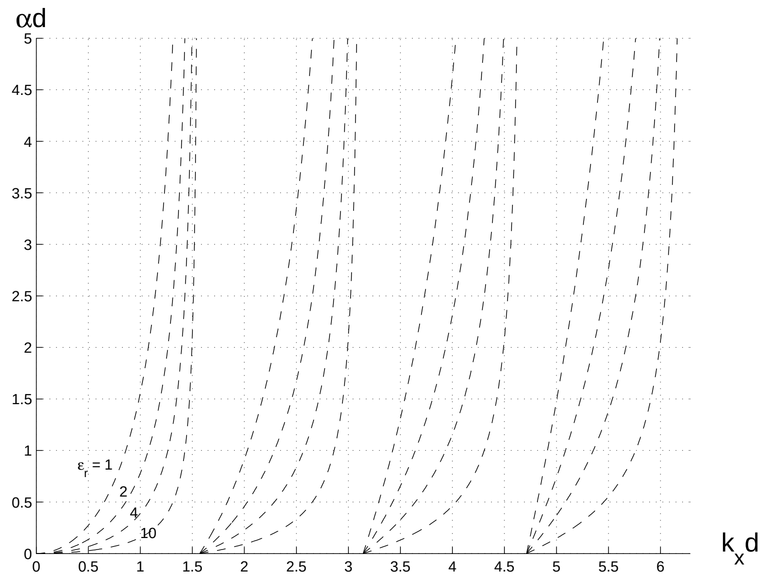

Figure 2.8: Graphical solution curves for TM modes of a dielectric slab waveguide for various values of ϵ_r. (Note that the $\epsilon_r = 1$ curve can also be used for TE modes.)

Consider an example with slab thickness 2 cm, relative permittivity 2, and operating frequency 20 GHz. The first few cutoff frequencies are

$$f_{c,1} = 0$$

$$f_{c,2} = 7.5\,\text{GHz}$$

$$f_{c,3} = 15\,\text{GHz}$$

$$f_{c,4} = 22.5\,\text{GHz}$$

From these values, we know that there are three propagating TM modes (and three TE modes as well). In order to find the values of k_x and α, we must find the intersections of a circle of radius $\omega d\sqrt{\epsilon_r - 1}/c \simeq 4.2$ with the curves in Fig. 2.8.

For the dominant mode, this intersection point lies at $\alpha d \simeq 4$, so that $\alpha = 4$ Np/cm. This tells us that after $1/\alpha \simeq .25$ cm, the fields outside the slab have decayed by a factor of $1/e$. The values of α as well as other constants can be found similarly for the higher order propagating modes. Will these modes be more or less tightly bound to the slab?

2.3.5 Excitation and Orthogonality of Modes

We have determined that within a waveguide there us a countable subset of allowable field solutions , or modes. The total EM field within is a sumation of all of these modes as given by

$$\begin{aligned} E_z &= \sum_{m,n} \left[A_{mn}^{TM,+} \sin\left(\frac{m\pi x}{a}\right) \sin\left(\frac{n\pi y}{b}\right) e^{-j\beta z} + A_{mn}^{TM,-} \sin\left(\frac{m\pi x}{a}\right) \sin\left(\frac{n\pi y}{b}\right) e^{j\beta z} \right] \\ H_z &= \sum_{m,n} \left[A_{mn}^{TE,+} \cos\left(\frac{m\pi x}{a}\right) \cos\left(\frac{n\pi y}{b}\right) e^{-j\beta z} + A_{mn}^{TE,-} \cos\left(\frac{m\pi x}{a}\right) \cos\left(\frac{n\pi y}{b}\right) e^{j\beta z} \right] \quad (2.79) \end{aligned}$$

What determines the values of the coefficients $A_{mn}^{TM,+}$, $A_{mn}^{TM,-}$, $A_{mn}^{TE,+}$, and $A_{mn}^{TE,-}$? They are fixed by the location, orientation, and strength of the driving source.

Mathematically, we can study this problem by using the principle of *orthogonality of modes*. Using Maxwell's equations, we can compute $\overline{J}$ in terms of the modes and amplitudes in the above equations. For any waveguide, these modes form a complete and orthogonal set of functions. This means that once we have found an expression for $\overline{J}$ in terms of the modal fields, we can expand the given source function $\overline{J}(\vec{r})$ into a sum over the same mode functions with known coefficients, and obtain the coefficients $A_{mn}^{TM,+}$, $A_{mn}^{TM,-}$, $A_{mn}^{TE,+}$, and $A_{mn}^{TE,-}$ by equating the two expansions term by term. This is very similar to finding the Fourier series coefficients of a function.

Let's start by getting some insight from simple physical considerations. The simplest waveguide excitation is a probe which extends through a small hole into the waveguide. Often, this is simply the center conductor of a coaxial cable. The current on the probe produces an electric field, which in turn drives one or more waveguide modes (Fig. 2.9). For strongest excitation of a given mode, the probe should be located at a maximum and oriented in the same direction as the electric field. If the probe is at a null of the electric field or is orthogonal to it, the mode is not excited (or weakly excited due to the finite size of the probe).

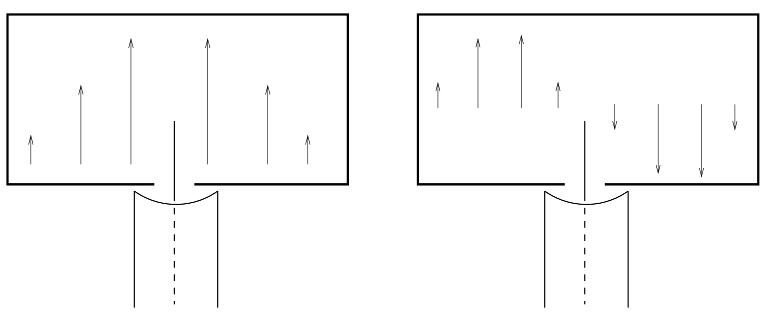

Figure 2.9: Excitation of waveguide modes. A probe in the center of the waveguide excites the TE_{10} mode (left) but not the TE_{20} mode (right).

We will now look at the problem of determining mode amplitudes from a given source from a mathematical point of view. To make the analysis a little simpler, we will consider the parallel plate waveguide instead of the rectangular or cylindrical waveguide, although these can be treated in a similar way.

The excitation source is typically modeled as a varying current sheet. We place in the waveguide a current density $\overline{J}_s = \hat{y} J_s(x)$. This current will excite modes in the waveguide. Because of the current direction, we already know that it will only excite TE modes. However, we will also show this mathematically. However, we don't know which of the TE modes will be excited. In general, we can write the total electric field in the waveguide for $z > 0$ as an infinite (Fourier) superpositions of TE modes as given by

$$\overline{E}(z>0) = \sum_{m=1}^{\infty} \hat{y} E_m^+ \sin(k_{xm} x) e^{-j\beta_m z} \tag{2.80}$$

where

$$k_{xm} = \frac{m\pi}{d} \tag{2.81}$$
$$\beta_m = \sqrt{k^2 - k_{xm}^2} \tag{2.82}$$

We can obtain the magnetic field from Faraday's law as

$$\overline{H}(z>0) = -\sum_{m=1}^{\infty} \frac{E_m^+}{\eta}\left[\hat{x}\frac{\beta_m}{k_0}\sin(k_{xm}x) - \hat{z}\frac{jk_{xm}}{k_0}\cos(k_{xm}x)\right] e^{-j\beta_m z} \tag{2.83}$$

Similarly for $z < 0$ we have

$$\overline{E}(z<0) = \sum_{m=1}^{\infty} \hat{y} E_m^- \sin(k_{xm} x) e^{j\beta_m z} \tag{2.84}$$
$$\overline{H}(z<0) = -\sum_{m=1}^{\infty} \frac{E_m^-}{\eta}\left[-\hat{x}\frac{\beta_m}{k_0}\sin(k_{xm}x) - \hat{z}\frac{jk_{xm}}{k_0}\cos(k_{xm}x)\right] e^{j\beta_m z} \tag{2.85}$$

Notice that on both sides of the source, the fields are traveling away from the source, as would be expected. The unknowns are E_m^+ and E_m^-.

Now, enforcing continuity of the tangential electric field across the boundary at $z = 0$ leads to

$$\hat{z} \times \left[\overline{E}(z>0) - \overline{E}(z<0)\right]\Big|_{z=0} = 0$$
$$\sum_{m=1}^{\infty}(E_m^+ - E_m^-)\sin\left(\frac{m\pi x}{d}\right) = 0 \tag{2.86}$$

Similarly, for the magnetic field,

$$\hat{z} \times \left[\overline{H}(z>0) - \overline{H}(z<0)\right]\Big|_{z=0} = \overline{J}_s$$
$$-\sum_{m=1}^{\infty}\frac{\beta_m}{\eta k_0}(E_m^+ + E_m^-)\sin\left(\frac{m\pi x}{d}\right) = J_s(x) \tag{2.87}$$

What we need to do next is invert Eqs. (2.86) and (2.87) somehow so that we can get the unknown mode amplitudes E_m^+ and E_m^- in terms of the given source J_s.

Now, if you have seen Fourier series, these equations might give you an idea how to proceed. Let's multiply both sides of (2.86) by $(2/d)\sin(n\pi x/d)$ and integrate from 0 to d in x.

$$\sum_{m=1}^{\infty}(E_m^+ - E_m^-)\frac{2}{d}\int_0^d \sin\left(\frac{m\pi x}{d}\right)\sin\left(\frac{n\pi x}{d}\right)\,dx = 0$$
$$\sum_{m=1}^{\infty}(E_m^+ - E_m^-)\delta_{mn} = 0$$
$$E_n^+ - E_n^- = 0$$
$$E_n^+ = E_n^- \tag{2.88}$$

This tells us that each mode propagates away from the source on either side with the same amplitude, which should be intuitive. Note that we have used the identity

$$\frac{2}{d}\int_0^d \sin\left(\frac{m\pi x}{d}\right)\sin\left(\frac{n\pi x}{d}\right)\,dx = \delta_{mn} \tag{2.89}$$

where δ_{mn} is the Kronecker delta function ($\delta_{nn} = 1$, $\delta_{mn} = 0$ if $m \neq n$). This is an *orthogonality* condition.

We now use this result in (2.87), and then multiply by $(2/d)\sin(n\pi x/d)$ and integrate from 0 to d in x. The result is

$$\begin{aligned}
\sum_{m=1}^{\infty} \frac{2E_m^+\beta_m}{\eta k_0}\frac{2}{d}\int_0^d \sin\left(\frac{m\pi x}{d}\right)\sin\left(\frac{n\pi x}{d}\right)\,dx &= -\frac{2}{d}\int_0^d J_s(x)\sin\left(\frac{n\pi x}{d}\right)\,dx \\
\sum_{m=1}^{\infty} \frac{2E_m^+\beta_m}{\eta k_0}\delta_{mn} &= -\frac{2}{d}\int_0^d J_s(x)\sin\left(\frac{n\pi x}{d}\right)\,dx \\
\frac{2E_n^+\beta_n}{\eta k_0} &= -\frac{2}{d}\int_0^d J_s(x)\sin\left(\frac{n\pi x}{d}\right)\,dx
\end{aligned} \tag{2.90}$$

or

$$E_n^+ = -\frac{\eta k_0}{\beta_n d}\int_0^d J_s(x)\sin\left(\frac{n\pi x}{d}\right)\,dx \tag{2.91}$$

So, if we are given a current distribution $J_s(x)$, we can determine the strength of each TE mode excited in the waveguide. This should be a very satisfying result—for each mode, the amplitude is given by the inner product of the current source with the function that gives the variation of the modal fields in the plane of the source. In this case, because the variation is sinusoidal, the amplitudes turn out to be the Fourier sine series expansion coefficients of the current source.

Now, what about TM modes? The general form of the electric and magnetic fields are given by

$$\overline{H}(z > 0) = \sum_{m=1}^{\infty} \hat{y} H_m^+ \cos(k_{xm}x) e^{-j\beta_m z} \tag{2.92}$$

$$\overline{E}(z > 0) = \sum_{m=1}^{\infty} \eta H_m^+ \left[\hat{x}\frac{\beta_m}{k_0}\cos(k_{xm}x) + \hat{z}\frac{jk_{xm}}{k_0}\sin(k_{xm}x)\right] e^{-j\beta_m z} \tag{2.93}$$

$$\overline{H}(z < 0) = \sum_{m=1}^{\infty} \hat{y} H_m^- \cos(k_{xm}x) e^{j\beta_m z} \tag{2.94}$$

$$\overline{E}(z < 0) = \sum_{m=1}^{\infty} \eta H_m^- \left[-\hat{x}\frac{\beta_m}{k_0}\cos(k_{xm}x) + \hat{z}\frac{jk_{xm}}{k_0}\sin(k_{xm}x)\right] e^{j\beta_m z} \tag{2.95}$$

Now, enforcing continuity of the tangential electric field across the boundary at $z = 0$ leads to

$$\begin{aligned}
\hat{z} \times \left[\overline{E}(z > 0) - \overline{E}(z < 0)\right]\Big|_{z=0} &= 0 \\
\sum_{m=1}^{\infty} \hat{x}\frac{\eta\beta_m}{k_0}\left(H_m^+ + H_m^-\right)\cos\left(\frac{m\pi x}{d}\right) &= 0
\end{aligned} \tag{2.96}$$

But since

$$\int_0^d \frac{2}{d}\cos\left(\frac{m\pi x}{d}\right)\cos\left(\frac{n\pi x}{d}\right) dx = \delta_{mn} \tag{2.97}$$

$$H_m^+ = -H_m^- \tag{2.98}$$

Similarly, for the magnetic field boundary condition

$$\hat{z} \times \left[\overline{H}(z>0) - \overline{H}(z<0)\right]\Big|_{z=0} = \hat{y} J_s(x) \tag{2.99}$$

The left hand side of is $\hat{x}$-directed so

$$H_m^+ = H_m^- \tag{2.100}$$

This can only be true if $H_m^+ = H_m^- = 0$.

If we only want to know whether a given mode is excited, we do not need to go through the trouble of finding the exact solutions for the electric and magnetic fields. But rather we use the expression

$$P = -\frac{1}{2}\int_v \overline{E} \cdot \overline{J}^* dv. \tag{2.101}$$

The total electric field can then be written as a superposition of the modes as given by

$$\begin{aligned}
P &= -\frac{1}{2}\int_v \sum_{m=0}^{\infty} \overline{E}_m \cdot \overline{J}^* dv && (2.102)\\
&= \sum_{m=0}^{\infty}\left(-\frac{1}{2}\int_v \overline{E}_m \cdot \overline{J}^* dv\right) && (2.103)\\
&= \sum_{m=0}^{\infty} P_m && (2.104)
\end{aligned}$$

So if a modal field is perpendicular to the source current $\left(\overline{E}_m \cdot \overline{J} = 0\right)$, then the mode will not be excited.

2.4 General Guided Wave Solutions

Let's assume a wave traveling along a cylindrical metal structure that is invariant in the z direction (*e.g.* a pipe), as shown in Fig. 2.10. Note that the structure may consist of more than one conductor, such as two parallel wires or two concentric cylinders. We also assume that the electromagnetic wave supported by this structure travels as $e^{-j\beta z}$, where β is a propagation constant much like k_z for a plane wave. Unlike free space, however, we expect the value of β to be dependent on the shape of the cross section of the structure.

Using these assumptions, we can write the fields in terms of *transverse* and *longitudinal* vector components as

$$\overline{E}(x,y,z) = \Bigg[\underbrace{\hat{x}E_x(x,y) + \hat{y}E_y(x,y)}_{\overline{E}_T(x,y)} + \hat{z}E_z(x,y)\Bigg] e^{-j\beta z} \tag{2.105}$$

$$\overline{H}(x,y,z) = \Bigg[\underbrace{\hat{x}H_x(x,y) + \hat{y}H_y(x,y)}_{\overline{H}_T(x,y)} + \hat{z}H_z(x,y)\Bigg] e^{-j\beta z} \tag{2.106}$$

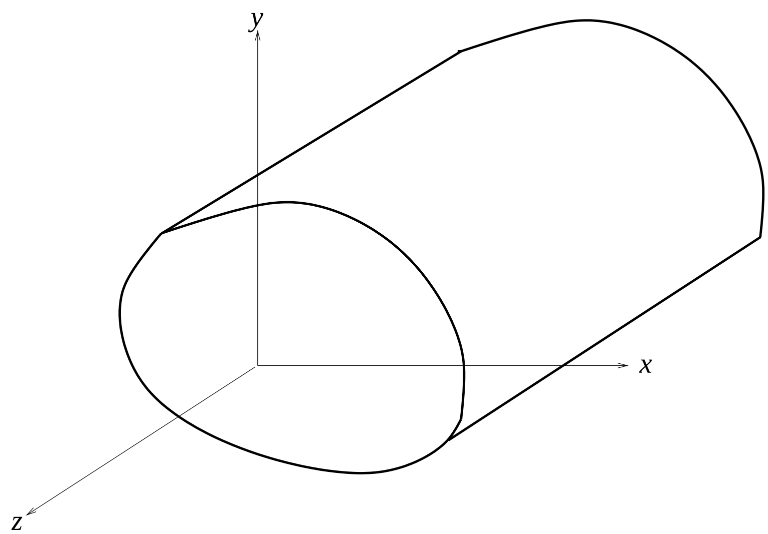

Figure 2.10: An arbitrary cylindrical waveguide, oriented so that its invariant axis is in the z direction.

where β is unknown. Note that the z dependence of the fields is contained entirely in a complex exponential term, which is factored out of the quantity in square brackets.

The region inside the structure is source-free. Therefore, Maxwell's equations are

$$\nabla \times \overline{E} = -j\omega\mu\overline{H} \qquad \nabla \times \overline{H} = j\omega\epsilon\overline{E} \tag{2.107}$$

Expanding in terms of partial derivatives gives

$$\hat{x}\left(\frac{\partial E_z}{\partial y} - \frac{\partial E_y}{\partial z}\right) + \hat{y}\left(\frac{\partial E_x}{\partial z} - \frac{\partial E_z}{\partial x}\right) + \hat{z}\left(\frac{\partial E_y}{\partial x} - \frac{\partial E_x}{\partial y}\right) = -j\omega\mu\left[\hat{x}H_x + \hat{y}H_y + \hat{z}H_z\right] \tag{2.108}$$

$$\hat{x}\left(\frac{\partial H_z}{\partial y} - \frac{\partial H_y}{\partial z}\right) + \hat{y}\left(\frac{\partial H_x}{\partial z} - \frac{\partial H_z}{\partial x}\right) + \hat{z}\left(\frac{\partial H_y}{\partial x} - \frac{\partial H_x}{\partial y}\right) = j\omega\epsilon\left[\hat{x}E_x + \hat{y}E_y + \hat{z}E_z\right] \tag{2.109}$$

Breaking this into vector components leads to

$$\frac{\partial E_z}{\partial y} + j\beta E_y = -j\omega\mu H_x \tag{2.110}$$

$$-j\beta E_x - \frac{\partial E_z}{\partial x} = -j\omega\mu H_y \tag{2.111}$$

$$\frac{\partial E_y}{\partial x} - \frac{\partial E_x}{\partial y} = -j\omega\mu H_z \tag{2.112}$$

$$\frac{\partial H_z}{\partial y} + j\beta H_y = j\omega\epsilon E_x \tag{2.113}$$

$$-j\beta H_x - \frac{\partial H_z}{\partial x} = j\omega\epsilon E_y \tag{2.114}$$

$$\frac{\partial H_y}{\partial x} - \frac{\partial H_x}{\partial y} = j\omega\epsilon E_z \tag{2.115}$$

By combining (2.110) and (2.114), we obtain

$$\frac{\partial E_z}{\partial y} + \frac{j\beta}{j\omega\epsilon}\left(-j\beta H_x - \frac{\partial H_z}{\partial x}\right) = -j\omega\mu H_x \tag{2.116}$$

$$j\omega\epsilon\frac{\partial E_z}{\partial y} + \beta^2 H_x - j\beta\frac{\partial H_z}{\partial x} = \omega^2\mu\epsilon H_x \tag{2.117}$$

which can be rearranged as

$$H_x = \frac{j}{k_c^2}\left(\omega\epsilon\frac{\partial E_z}{\partial y} - \beta\frac{\partial H_z}{\partial x}\right) \tag{2.118}$$

where $k_c^2 = k^2 - \beta^2$. What is k_c? We will look at this in a little bit more detail later. But it is effectively the transverse component of the wavevector. In the same way that β is the longitudinal component of the wavevector.

Note that had we eliminated H_x instead of E_y, we would have an expression for E_y in terms of H_z and E_z. Similarly, combining (2.111) and (2.113) leads to expressions for E_x and H_y in terms of H_z and E_z. Therefore, we can write

$$H_x = \frac{j}{k_c^2}\left(\omega\epsilon\frac{\partial E_z}{\partial y} - \beta\frac{\partial H_z}{\partial x}\right) \tag{2.119}$$

$$H_y = \frac{-j}{k_c^2}\left(\omega\epsilon\frac{\partial E_z}{\partial x} + \beta\frac{\partial H_z}{\partial y}\right) \tag{2.120}$$

$$k_c^2 = k^2 - \beta^2 \tag{2.121}$$

$$E_x = \frac{-j}{k_c^2}\left(\beta\frac{\partial E_z}{\partial x} + \omega\mu\frac{\partial H_z}{\partial y}\right) \tag{2.122}$$

$$E_y = \frac{j}{k_c^2}\left(-\beta\frac{\partial E_z}{\partial y} + \omega\mu\frac{\partial H_z}{\partial x}\right) \tag{2.123}$$

So, we can get all of the fields if we know the longitudinal components.

2.4.1 Transverse Electromagnetic (TEM) Waves

Consider a waveguide that consists of two parallel wires carrying a low frequency AC current. What are the electric and magnetic fields around the wires? The electric field between the wires is transverse to the direction of the wires, so $\overline{E}_T \neq 0$ and $E_z = 0$. By the right hand rule, the magnetic field is also transverse to the wires, so that $\overline{H}_T \neq 0$ and $H_z = 0$. But in the above expressions for the transverse fields, it would seem that if $E_z = H_z = 0$, then the transverse fields also must be zero. Are these equations wrong?

Let's look at this further. It is easy to see from Eqs. (2.119)-(2.123) that if $k_c \neq 0$, then $E_z = H_z = 0$ implies that $\overline{E} = \overline{H} = 0$. Therefore, k_c must vanish. If $k_c = 0$, then $\beta = k$, and we have a $0/0$ situation in our equations (which is indeterminate). We need therefore to go back to (2.110)-(2.115) under these conditions. For example, (2.110) and (2.114) simplify to

$$j\beta E_y = -j\omega\mu H_x \qquad -j\beta H_x = j\omega\epsilon E_y \tag{2.124}$$

or

$$\beta^2 E_y = \omega^2\mu\epsilon E_y = k^2 E_y \tag{2.125}$$

or $\beta = k$. The wave impedance is

$$Z_{\text{TEM}} = \frac{E_x}{H_y} = -\frac{E_y}{H_x} = \frac{\omega\mu}{\beta} = \frac{\omega\mu}{k} = \sqrt{\frac{\mu}{\epsilon}} = \eta \tag{2.126}$$

Parallel plate waveguides, coaxial cables, or any other structure with two separate conductors and a homogeneous dielectric between them can support TEM modes.

2.4.2 Transverse Electric (TE) Waves

Suppose that $E_z = 0$, $H_z \neq 0$. The wave equation for H_z in Cartesian coordinates assumes the form

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} + k^2\right) H_z = 0 \tag{2.127}$$

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \underbrace{k^2 - \beta^2}_{k_c^2}\right) H_z = 0 \tag{2.128}$$

This equation, together with the boundary condition that the electric field must vanish at the conductor surface, can be solved for E_z. Using Eqs. (2.119)-(2.123), all the other nonzero field components can be found from E_z.

For TE waves, the wave impedence is

$$Z_{\text{TE}} = \frac{E_x}{H_y} = -\frac{E_y}{H_x} = \frac{-\omega\mu\partial H_z/\partial y}{-\beta\partial H_z/\partial y} = \frac{\omega\mu}{\beta} = \frac{k\eta}{\beta} \tag{2.129}$$

Note that the wave impedance is frequency dependent.

2.4.3 Transverse Magnetic (TM) Waves

Suppose that $E_z \neq 0$, $H_z = 0$. The wave equation for E_z in Cartesian coordinates assumes the form

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + k_c^2\right) E_z = 0 \tag{2.130}$$

with

$$Z_{\text{TM}} = \frac{E_x}{H_y} = -\frac{E_y}{H_x} = \frac{-\beta\partial E_z/\partial x}{-\omega\epsilon\partial E_z/\partial x} = \frac{\beta}{\omega\epsilon} = \frac{\beta\eta}{k} \tag{2.131}$$

By the principle of superposition, we can construct any field solution for the waveguide by adding the TE and TM wave solutions. This is a very handy observation, since this allows us to reduce the full set of Maxwell's equations to two simple scalar equations, (2.128) and (2.130). Notice that we have done all of this without saying anything about the actual shape of the waveguide.

2.5 Rectangular Waveguides

We now assume that our "pipe" has a rectangular cross section of dimensions a and b in the x and y directions, respectively (Fig. 2.11). We have assumed in the above analysis that the wave propagates as $e^{-j\beta z}$. In the analysis to follow, we will actually prove this, which supports the use of the above analysis for these waveguides.

In a rectangular waveguide the wave is not simply bouncing up and down in the yz plane like the parallel plate waveguide. Certain waveguide modes can be thought of as bouncing off of all of the walls. This would look like a spiral. Therefore, the electric and magnetic fields can have components in all three coordinant

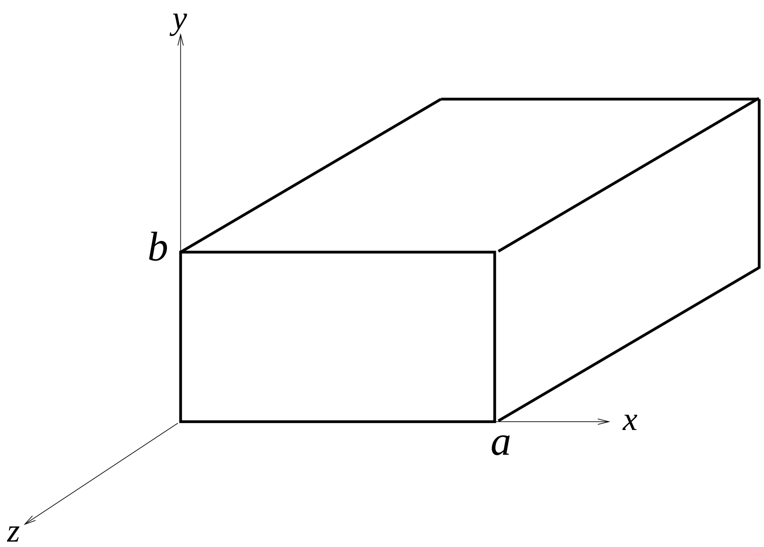

Figure 2.11: A rectangular waveguide of dimensions a by b.

directions. However based on Eqs. 2.119 - 2.123, the total vector fields can be calculated if we determine the E_z and H_z components. The various modes are divided into two different cases in which either E_z or H_z are equal to zero.

These two cases are:

Transverse Electric (TE): The electric field is perpendicular to the propagation direction $\overline{E} \perp \hat{z}$ ($E_z = 0$).

Transverse Magnetic (TM): The magnetic field is perpendicular to the propagation direction $\overline{H} \perp \hat{z}$ ($H_z = 0$).

2.5.1 Transverse Magnetic (TM)

The magnetic field is perpendicular to the propagation direction: $(E_z \neq 0, H_z = 0)$

So we only have to solve for E_z. The wave equation for this component is

$$\left(\nabla^2 + k^2\right) E_z = 0 \tag{2.132}$$

We solved this when we first studied plane waves. Assume that E_z is of a form in which you can separate the variables resulting in

$$E_z = f(x)\; g(y)\; h(z), \tag{2.133}$$

where f, g, and h are unknown functions. This is a realistic assumption and is a common assumption used to solve a partial differential equation. Plug this form of the field into the wave equation to get

$$\begin{aligned} \left(\nabla^2 + k^2\right) E_z &= \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right) f(x)g(y)h(z) = 0 && (2.134)\\ &= f''gh + fg''h + fgh'' = 0 && (2.135) \end{aligned}$$

Divide by fgh to get

$$\frac{f''}{f} + \frac{g''}{g} + \frac{h''}{h} + k^2 = 0 \tag{2.136}$$

For the partial of f with respect to x $\left(\frac{f''}{f}\right)$, g, h, and k are all just constants. Equation 2.134 can be broken up into three different equations as given by

$$\frac{f''}{f} = -k_x^2 \tag{2.137}$$

$$\frac{g''}{g} = -k_y^2 \tag{2.138}$$

$$\frac{h''}{h} = -k_z^2. \tag{2.139}$$

Each of these equations is now just a second order differential equation with the following solutions

$$f(x) = Ae^{jk_x x} + Be^{-jk_x x} \tag{2.140}$$

$$g(x) = Ce^{jk_y y} + De^{-jk_y y} \tag{2.141}$$

$$h(x) = Ee^{jk_z z} + Fe^{-jk_z z} \tag{2.142}$$

And the total electric field is

$$E_z(x,y,z) = fgh \tag{2.143}$$

$$= \left(Ae^{jk_x x} + Be^{-jk_x x}\right)\left(Ce^{jk_y y} + De^{-jk_y y}\right)\left(Ee^{jk_z z} + Fe^{-jk_z z}\right) \tag{2.144}$$

Now we need to determine all of the unknowns. The waveguide is infinite in z resulting in no backwards propagating wave so ($E = 0$).

The waves in the x and y direction are standing waves rather than traveling waves so we will convert the exponential form into sinusoidal form as given by

$$E_z(x,y,z) = \left[Asin\left(k_x x\right) + Bcos\left(k_x x\right)\right]\left[Csin\left(k_y y\right) + Dcos\left(k_y y\right)\right]e^{-jk_z z}. \tag{2.145}$$

We need to find the following unknowns A, B, C, D, k_x, k_y, and k_z.

The electric field component is in the z-direction. Therefore, it is tangential at each waveguide edge resulting in boundary conditions given by

$$E_z\left(x=0\right) = E_z\left(x=a\right) = E_z\left(y=0\right) = E_z\left(y=b\right) = 0 \tag{2.146}$$

First $x = 0$ results in

$$B\left[Csin\left(k_y y\right) + Dcos\left(k_y y\right)\right]e^{-jk_z z} = 0 \tag{2.147}$$

For all values of y and z, this requires $\mathbf{B} = \mathbf{0}$.

For $y = 0$ results in

$$\left[Asin\left(k_x x\right) + Bcos\left(k_x x\right)\right]De^{-jk_z z} = 0 \tag{2.148}$$

For all values of x and z, this requires $\mathbf{D} = \mathbf{0}$.

The electric field is now given by

$$E_z(x, y, z) = Asin\,(k_x x)\,Csin\,(k_y y)\,e^{-jk_z z} = 0. \tag{2.149}$$

The constants can be combined resulting in

$$E_z(x, y, z) = E_o sin\,(k_x x)\,sin\,(k_y y)\,e^{-jk_z z} = 0, \tag{2.150}$$

where E_o is just related to the power of the waveguide mode.

Now use the other two boundaries to determine the propagation constants. For $x = a$

$$E_z(a, y, z) = E_o sin\,(k_x a)\,sin\,(k_y y)\,e^{-jk_z z} = 0. \tag{2.151}$$

In order for this to be equal to 0 for all y and z, then

$$sin\,(k_x a) = 0, \tag{2.152}$$

which results in

$$k_x a = m\pi \tag{2.153}$$

$$k_x = \frac{m\pi}{a}. \tag{2.154}$$

At the $y = b$ boundary

$$E_z(x, b, z) = E_o sin\,(k_x x)\,sin\,(k_y b)\,e^{-jk_z z} = 0, \tag{2.155}$$

resulting in

$$k_y = \frac{n\pi}{b}. \tag{2.156}$$

The remaining unknown k_z is determined by using the magnitude of the wavevector as given by

$$k_z = \sqrt{k^2 - \left(\frac{m\pi}{a}\right)^2 - \left(\frac{n\pi}{b}\right)^2} \tag{2.157}$$

The resulting solution for the TM_{mn} is

$$E_z(x, y, z) = E_o sin\left(\frac{m\pi}{a}x\right) sin\left(\frac{n\pi}{b}x\right) e^{-jk_z z}. \tag{2.158}$$

If we need to find the total electric and magnetic fields, we use the equations derived earlier (Eqs. 2.119 - 2.123) as given by

$$E_x = E_{xo} cos\,(k_x x)\,sin\,(k_y y)\,e^{-jk_z z} \tag{2.159}$$

$$E_y = E_{yo} sin\,(k_x x)\,cos\,(k_y y)\,e^{-jk_z z} \tag{2.160}$$

$$H_x = H_{xo} sin\,(k_x x)\,cos\,(k_y y)\,e^{-jk_z z} \tag{2.161}$$

$$H_y = H_{yo} cos\,(k_x x)\,sin\,(k_y y)\,e^{-jk_z z} \tag{2.162}$$

What is the lowest order mode? If either $m = 0$ or $n = 0$, then $E_z = 0$. If both $E_z = 0$ and $H_z = 0$, then the total power is 0, therefore $m \geq 1$ and $n \geq 1$. So the lowest order mode is TM_{11}.

2.5.2 Transverse Electric (TE)

Now solve the case for which the electric field is perpendicular to the propagation direction ($E_z = 0$ and $H_z \neq 0$). $\overline{H}$ satisfies the same wave equations as $\overline{E}$, so H_z has the form given by

$$H_z(x,y,z) = \left[Asin\left(k_x x\right) + Bcos\left(k_x x\right)\right]\left[Csin\left(k_y y\right) + Dcos\left(k_y y\right)\right]e^{-jk_z z}. \tag{2.163}$$

When we apply the boundary conditions H_z is tangential at the boundary. Since $H_t = J_s$, this boundary condition does not help specify the mode. Therefore, we need to calculate the electric field from the magnetic field.

$$\begin{aligned} E_x &= \frac{-j\omega\mu}{k^2-k_z^2}\frac{\partial H_z}{\partial y} && (2.164)\\ &= \frac{-j\omega\mu}{k^2-k_z^2}\left[Asin\left(k_x x\right) + Bcos\left(k_x x\right)\right]\left[Ck_y cos\left(k_y y\right) - Dk_y sin\left(k_y y\right)\right]e^{-jk_z z}. && (2.165)\end{aligned}$$

$$E_x(y=0) = 0 = \frac{-j\omega\mu}{k^2-k_z^2}\left[Asin\left(k_x x\right) + Bcos\left(k_x x\right)\right]\left[Ck_y\right]e^{-jk_z z}. \tag{2.166}$$

$$\mathbf{C} = \mathbf{0} \tag{2.167}$$

Plug in $\mathbf{C} = \mathbf{0}$ and apply the boundary condition at $y = b$ to get

$$E_x(y=b) = 0 = \frac{-j\omega\mu}{k^2-k_z^2}\left[Asin\left(k_x x\right) + Bcos\left(k_x x\right)\right]\left[-Dk_y sin\left(k_y b\right)\right]e^{-jk_z z}. \tag{2.168}$$

$$k_y = \frac{n\pi}{b} \tag{2.169}$$

Now do a similar process with E_y.

$$\begin{aligned} E_y &= \frac{j\omega\mu}{k^2-k_z^2}\frac{\partial H_z}{\partial x} && (2.170)\\ &= \frac{j\omega\mu}{k^2-k_z^2}\left[Ak_x cos\left(k_x x\right) - Bk_x sin\left(k_x x\right)\right]\left[Ck_y cos\left(k_y y\right) - Dk_y sin\left(k_y y\right)\right]e^{-jk_z z}. && (2.171)\end{aligned}$$

$$E_y(x=0) = 0 = \frac{j\omega\mu}{k^2-k_z^2}\left[Ak_x\right]\left[Ck_y cos\left(k_y y\right) - Dk_y sin\left(k_y y\right)\right]e^{-jk_z z}. \tag{2.172}$$

$$\mathbf{A} = \mathbf{0} \tag{2.173}$$

Plug in $\mathbf{A} = \mathbf{0}$ and apply the boundary condition at $x = a$ to get

$$E_y(x=a) = 0 = \frac{j\omega\mu}{k^2-k_z^2}\left[-Bk_x sin\left(k_x a\right)\right]\left[Ck_y cos\left(k_y y\right) - Dk_y sin\left(k_y y\right)\right]e^{-jk_z z}. \tag{2.174}$$

$$k_x = \frac{m\pi}{a} \tag{2.175}$$

The resulting solution for the TE_{mn} mode is given by

$$H_z = H_{zo}cos\left(\frac{m\pi}{a}x\right)cos\left(\frac{n\pi}{b}y\right)e^{-jk_z z} \tag{2.176}$$

$$k_z = \sqrt{k^2 - \left(\frac{m\pi}{a}\right)^2 - \left(\frac{n\pi}{b}\right)^2} \tag{2.177}$$

The other fields have the form:

$$E_x = E_{xo} cosk_x x sink_y y e^{-jk_z z} \tag{2.178}$$
$$E_y = E_{yo} sink_x x cosk_y y e^{-jk_z z} \tag{2.179}$$
$$H_x = H_{xo} sink_x x cosk_y y e^{-jk_z z} \tag{2.180}$$
$$H_y = H_{yo} cosk_x x sink_y y e^{-jk_z z} \tag{2.181}$$

What is the lowest order mode? H_z can have a non-zero amplitude even if $m = 0$ and $n = 0$. However, if both $m = 0$ and $n = 0$ then $E_x = E_y = 0$. In combination with the fact that $E_z = 0$, the total E is zero resulting in no power. Thus, the lowest order mode is either TE_{10} or TE_{01}.

Cutoff. Notice that β depends on the time frequency of operation mode numbers, m and n. For a given mode, if the frequency is such that $\beta = 0$, the mode does not propagate:

$$\beta^2 = k^2 - k_x^2 - k_y^2 = k^2 - k_c^2 = 0 \tag{2.182}$$

At this frequency,

$$k = k_c = \sqrt{\left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2} \tag{2.183}$$

We call this condition cutoff. The value k_c is therefore the cutoff wavenumber. Using $k = \omega/c$, the cutoff frequency at which propagation stops for the TE_{mn} mode is

$$f_{c,mn} = \frac{c}{2\pi}\sqrt{\left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2} \tag{2.184}$$

If the frequency of operation is below the cutoff frequency of the mode, so that $f < f_{c,mn}$, then $k < k_c$ and $\beta = -j\alpha$ becomes imaginary. The z dependence of the mode then becomes $e^{-j\beta z} = e^{-\alpha z}$ so that the field decays in z. This is called an **evanescent** wave.

Dominant mode. The dominant waveguide mode has the lowest value of f_c. Assuming that $a > b$, this corresponds to the TE_{10} mode, with cutoff frequency

$$f_{c,10} = \frac{c}{2\pi}\frac{\pi}{a} = \frac{c}{2a} \tag{2.185}$$

The fields for the dominant mode are

$$H_z = A_{10} \cos\frac{\pi x}{a} e^{-j\beta z} \tag{2.186}$$

$$E_y = -\frac{j\omega\mu a}{\pi} A_{10} \sin\frac{\pi x}{a} e^{-j\beta z} \tag{2.187}$$

$$H_x = \frac{j\beta a}{\pi} A_{10} \sin\frac{\pi x}{a} e^{-j\beta z} \tag{2.188}$$

Wavelength. This can be defined as the distance between equal phase planes along the waveguide in the z direction.

$$\lambda_g = \frac{2\pi}{\beta} = \frac{2\pi}{\sqrt{k^2 - k_c^2}} = \frac{2\pi}{k\sqrt{1 - (f_{c,mn}/f)^2}} = \frac{\lambda}{\sqrt{1 - (f_{c,mn}/f)^2}} \tag{2.189}$$

Note that $\lambda_g > \lambda$.

Phase velocity In the time domain, the wave travels as $\cos(\omega t - \beta z)$. To remain on a point of constant phase, we write $\omega t - \beta z = \phi =$ constant. Then

$$z = \frac{\omega t - \phi}{\beta} \tag{2.190}$$

$$v_p = \frac{dz}{dt} = \frac{\omega}{\beta} = \frac{\omega}{\sqrt{k^2 - k_c^2}} = \frac{c}{\sqrt{1 - (f_{c,mn}/f)^2}} \tag{2.191}$$

Note that $v_p > c$.

Group velocity In the time domain, the envelope or pulse travels as

$$v_g = \frac{d\omega}{d\beta} = \frac{c^2}{\omega}\sqrt{\left(\frac{\omega}{c}\right)^2 - \left(\frac{m\pi}{a}\right)^2 - \left(\frac{n\pi}{b}\right)^2} \tag{2.192}$$

$$= c^2\sqrt{\frac{1}{c^2} - \left(\frac{m\pi}{\omega a}\right)^2 - \left(\frac{n\pi}{\omega b}\right)^2} \tag{2.193}$$

(2.194)

Note that $v_g < c$.

2.6 Circular Waveguide

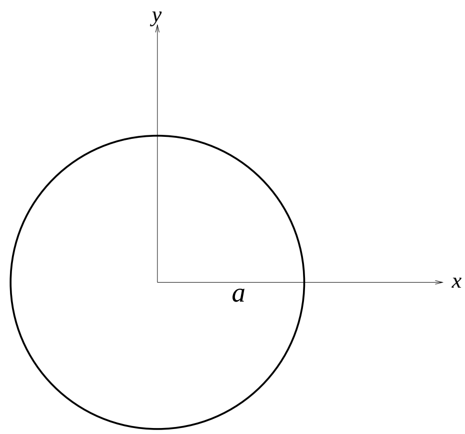

Figure 2.12: A circular waveguide of radius a.

For a circular waveguide of radius a (Fig. 2.12), we can perform the same sequence of steps in cylindrical coordinates as we did in rectangular coordinates to find the transverse field components in terms of the longitudinal (i.e. E_z, H_z) components. In cylindrical coordinates, the transverse field is

$$\overline{E}_T = \hat{\rho}E_\rho + \hat{\phi}E_\phi \qquad \overline{H}_T = \hat{\rho}H_\rho + \hat{\phi}H_\phi \tag{2.195}$$

Using this in Maxwell's equations (where the curl is applied in cylindrical coordinates) leads to

$$H_\rho = \frac{j}{k_c^2}\left(\frac{\omega\epsilon}{\rho}\frac{\partial E_z}{\partial \phi} - \beta\frac{\partial H_z}{\partial \rho}\right) \tag{2.196}$$

$$H_\phi = \frac{-j}{k_c^2}\left(\omega\epsilon\frac{\partial E_z}{\partial \rho} - \frac{\beta}{\rho}\frac{\partial H_z}{\partial \phi}\right) \tag{2.197}$$

$$E_\rho = \frac{-j}{k_c^2}\left(\beta\frac{\partial E_z}{\partial \rho} - \frac{\omega\mu}{\rho}\frac{\partial H_z}{\partial \phi}\right) \tag{2.198}$$

$$E_\phi = \frac{-j}{k_c^2}\left(\frac{\beta}{\rho}\frac{\partial E_z}{\partial \phi} - \omega\mu\frac{\partial H_z}{\partial \rho}\right) \tag{2.199}$$

where $k_c^2 = k^2 - \beta^2$ as before. Please note that here (as well as in rectangular waveguide derivation), we have assumed $e^{-j\beta z}$ propagation. For $e^{+j\beta z}$ propagation, we replace β with $-\beta$.

2.6.1 TE Modes

We don't need to prove that the wave travels as $e^{\pm j\beta z}$ again since the differentiation in z for the Laplacian is the same in cylindrical coordinates as it is in rectangular coordinates ($\partial^2/\partial z^2$). However, the ρ and ϕ derivatives of the Laplacian are different than the x and y derivatives. The wave equation for H_z is

$$(\nabla^2 + k^2)H_z = 0 \tag{2.200}$$

$$\left(\frac{\partial^2}{\partial \rho^2} + \frac{1}{\rho}\frac{\partial}{\partial \rho} + \frac{1}{\rho^2}\frac{\partial^2}{\partial \phi^2} + \frac{\partial^2}{\partial z^2} + k^2\right) H_z(\rho, \phi, z) = 0 \tag{2.201}$$

Using the separation of variables approach, we let $H_z(\rho,\phi,z) = R(\rho)P(\phi)e^{-j\beta z}$, and obtain

$$\left[R''P + \frac{1}{\rho}R'P + \frac{1}{\rho^2}RP'' + \underbrace{(k^2-\beta^2)}_{k_c^2}RP\right]e^{-j\beta z} = 0 \tag{2.202}$$

Multiplying by a common factor $\left(\frac{\rho^2}{RP}\right)$ leads to

$$\underbrace{\rho^2\frac{R''}{R} + \rho\frac{R'}{R} + \rho^2 k_c^2}_{\text{function of }\rho} + \underbrace{\frac{P''}{P}}_{\text{function of }\phi} = 0 \tag{2.203}$$

Because the terms in this equation sum to a constant, yet each depends only on a single coordinate, each term must be constant:

$$\frac{P''}{P} = -k_\phi^2 \qquad \rightarrow \qquad P'' + k_\phi^2 P = 0 \tag{2.204}$$

so that

$$P(\phi) = A_0\sin(k_\phi\phi) + B_0\cos(k_\phi\phi) \tag{2.205}$$

Using this result in (2.203) leads to

$$\rho^2\frac{R''}{R} + \rho\frac{R'}{R} + (\rho^2k_c^2 - k_\phi^2) = 0 \tag{2.206}$$

or

$$\boxed{\rho^2R'' + \rho R' + (\rho^2k_c^2 - k_\phi^2)R = 0} \tag{2.207}$$

This is known as *Bessel's Differential Equation*.

Now, we could use the Method of Frobenius to solve this equation, but we would just be repeating a well-known solution. The series you obtain from such a solution has very special properties (a lot like sine and cosine: you may recall that $\sin(x)$ and $\cos(x)$ are really just shorthand for power series that have special properties).

The solution is

$$R(\rho) = C_0J_{k_\phi}(k_c\rho) + D_0N_{k_\phi}(k_c\rho) \tag{2.208}$$

where $J_\nu(x)$ is the Bessel function of the first kind of order ν and $N_\nu(x)$ is the Bessel function of the second kind of order ν.

1. First, let's examine k_ϕ.

$$H_z(\rho,\phi,z) = \left[C_0J_{k_\phi}(k_c\rho) + D_0N_{k_\phi}(k_c\rho)\right]\left[A_0\sin(k_\phi\phi) + B_0\cos(k_\phi\phi)\right]e^{-j\beta z} \tag{2.209}$$

Clearly, $H_z(\rho,\phi,z) = H_z(\rho,\phi+2\pi\ell,z)$ where ℓ is an integer. This can only be true if $\boxed{k_\phi = \nu}$, where $\nu =$ integer.

$$H_z(\rho,\phi,z) = \left[C_0J_\nu(k_c\rho) + D_0N_\nu(k_c\rho)\right]\left[A_0\sin(\nu\phi) + B_0\cos(\nu\phi)\right]e^{-j\beta z} \tag{2.210}$$

2. It turns out that $N_\nu(k_c\rho) \to -\infty$ as $\rho \to 0$. Clearly, $\rho = 0$ is in the domain of the waveguide. Physically, however, we can't have infinite field intensity at this point. This leads us to conclude that $\boxed{D_0 = 0}$. We now have

$$H_z(\rho, \phi, z) = [A\sin(\nu\phi) + B\cos(\nu\phi)]\, J_\nu(k_c\rho)e^{-j\beta z} \tag{2.211}$$

3. The relative values of A and B have to do with the absolute coordinate frame we use to define the waveguide. For example, let $A = F\cos(\nu\phi_0)$ and $B = -F\sin(\nu\phi_0)$ (you can find a value of F and ϕ_0 to make this work). Then

$$A\sin(\nu\phi) + B\cos(\nu\phi) = F\sin[\nu(\phi - \phi_0)] \tag{2.212}$$

The value of ϕ_0 that makes this work can be thought of as the *coordinate reference* for measuring ϕ. So, we really are left with finding F, which is simply the mode amplitude and is therefore determined by the excitation.

4. We still need to determine k_c. The boundary condition that we can apply is the tangential component of the electric field at the metallic boundary ($\rho = a$) is zero. The tangential conponents are $p\hat{h}i$ and $\hat{z}$. Since $E_z = 0$, the boundary condition is $E_\phi(a, \phi, z) = 0$. Since

$$\begin{aligned} E_\phi(\rho, \phi, z) &= \frac{j\omega\mu}{k_c^2}\frac{\partial H_z}{\partial \rho} && (2.213) \\ &= \frac{j\omega\mu}{k_c^2} H_{zo}\sin(\nu\phi)k_c J_\nu'(k_c\rho)e^{-j\beta z} && (2.214) \end{aligned}$$

where

$$J_\nu'(x) = \frac{d}{dx}J_\nu(x), \tag{2.215}$$

our boundary condition indicates that $J_\nu'(k_c a) = 0$. So

$$k_c a = p_{\nu n}' \qquad \to \qquad k_c = \frac{p_{\nu n}'}{a} \tag{2.216}$$

where $p_{\nu n}'$ is the nth zero of $J_\nu'(x)$. Below is a table of a few of the zeros of $J_\nu'(x)$:

$J_\nu'(k_c a) = 0$	$n = 1$	$n = 2$	$n = 3$
$\nu = 0$	0.0000	3.8317	7.0156
$\nu = 1$	1.8412	5.3314	8.5363
$\nu = 2$	3.0542	6.7061	9.9695

5. We have already defined $k_c^2 = k^2 - \beta^2$, so

$$\beta^2 = k^2 - \left(\frac{p_{\nu n}'}{a}\right)^2 \tag{2.217}$$

Note that there is no "ϕ" term here. However, the ϕ variation of the fields in the waveguide does influence β. (How?)

6. Cutoff frequency ($\beta = 0$): Since $k = k_c = 2\pi f_{c,\nu n}/c$ at the mode cutoff frequency,

$$f_{c,\nu n} = \frac{c}{2\pi}\frac{p_{\nu n}'}{a} \tag{2.218}$$

7. Dominant Mode: We don't count the $\nu = 0$, $n = 1$ mode (TE_{01}) since $p'_{01} = 0$ resulting in zero fields. The dominant TE mode is therefore the mode with the smallest non-zero value of $p'_{\nu n}$, which is the TE_{11} mode.

8. The expressions for wavelength and phase velocity derived for the rectangular waveguide apply here as well. However, you must use the proper value for the cutoff frequency in these expressions.

2.6.2 TM Modes

The derivation is the same except that we are solving for E_z. We can therefore write

$$E_z(\rho, \phi, z) = E_{zo} \sin(\nu\phi) J_\nu(k_c\rho) e^{-j\beta z} \tag{2.219}$$

In this case the tangential electric field components are both E_z and E_ϕ, where

$$E_\phi = \frac{-j\beta}{k_c^2 \rho} \frac{\partial E_z}{\partial \phi} \tag{2.220}$$

Both $E_z(a, \phi, z) = 0$ and $E_\phi(a, \phi, z) = 0$ if $J_\nu\,(k_c a) = 0$ This leads to

$$k_c a = p_{\nu n} \qquad \rightarrow \qquad k_c = \frac{p_{\nu n}}{a} \tag{2.221}$$

where $p_{\nu n}$ is the nth zero of $J_\nu(x)$.

$J_\nu(k_c a) = 0$	$n = 1$	$n = 2$	$n = 3$
$\nu = 0$	2.4048	5.5201	8.6537
$\nu = 1$	3.8317	7.0156	10.1735
$\nu = 2$	5.1356	8.4172	11.6198

In this case, we have

$$\beta^2 = k^2 - \left(\frac{p_{\nu n}}{a}\right)^2 \tag{2.222}$$

$$f_{c,\nu n} = \frac{c}{2\pi} \frac{p_{\nu n}}{a} \tag{2.223}$$

It becomes clear the the TE_{11} mode is the dominant overall mode of the waveguide.

2.6.3 Bessel Functions

Here are some of the basic properties of Bessel functions:

$$J_\nu(x) = \sum_{n=0}^{\infty} \frac{(-1)^n (x/2)^{2n+\nu}}{n!(n+\nu)!} \tag{2.224}$$

$$N_\nu(x) = \lim_{p\to\nu} \frac{J_p(x)\cos(p\pi) - J_{-p}(x)}{\sin(p\pi)} \tag{2.225}$$

$$J_\nu(-x) = (-1)^\nu J_\nu(x), \quad \nu = \text{integer} \tag{2.226}$$

$$J_\nu(x) \simeq \sqrt{\frac{2}{\pi x}}\cos(x - \pi/4 - \nu\pi/2), \quad x \to \infty \tag{2.227}$$

$$N_\nu(x) \simeq \sqrt{\frac{2}{\pi x}}\sin(x - \pi/4 - \nu\pi/2), \quad x \to \infty \tag{2.228}$$

$$\frac{d}{dx}Z_\nu(x) = Z_{\nu-1}(x) - \nu Z_\nu(x)/x \tag{2.229}$$

where Z is any Bessel function. Figures 2.13 and 2.14 show Bessel functions of the first and second kinds of orders 0, 1, 2, 3.

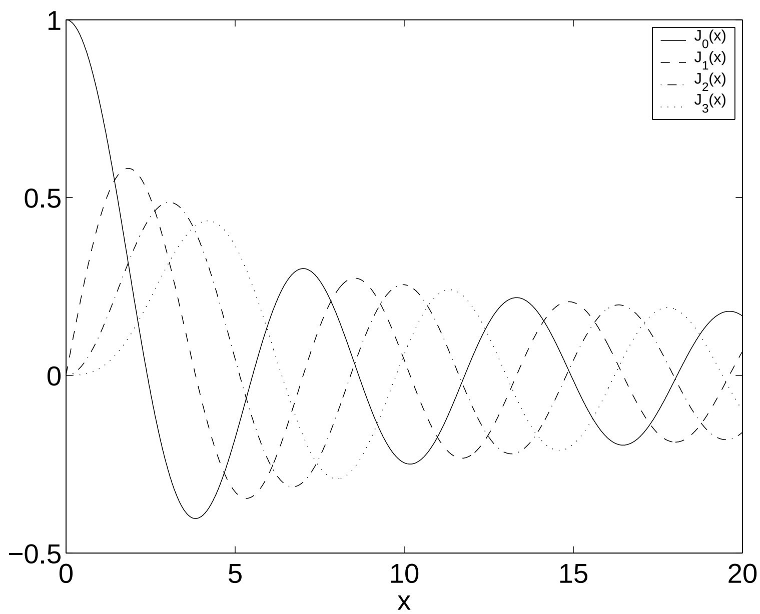

Figure 2.13: Bessel functions of the first kind.

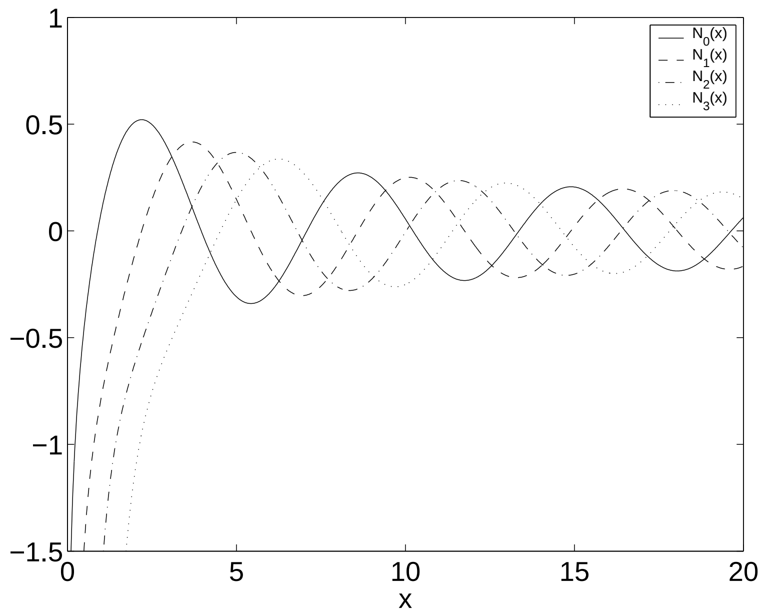

Figure 2.14: Bessel functions of the second kind.