The original diffraction integral

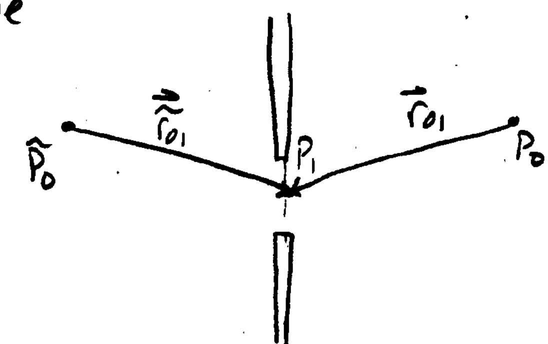
$$U(P_0) = \frac{1}{4\pi} \int_{\Sigma} \frac{e^{jkr_{01}}}{r_{01}} \left[\frac{\partial U}{\partial n} - jk\, U \cos(\vec{n}, \vec{r}_{01}) \right] ds$$

call Kirchoff Theory

Has some internal inconsistencies because both the field and its derivative must be specified on the boundary

To remove these inconsistencies we want to eliminate the need for either the field or its derivative on the boundary. We do this by choosing a different Greens function.

Suppose G is generated by a point source and its mirror image



$$G_-(P_1) = \frac{\exp(jkr_{01})}{r_{01}} - \frac{\exp(jk\tilde{r}_{01})}{\tilde{r}_{01}}$$

Along the aperture $G_-(P_1) = 0$
The integral equation becomes

$$U(P_0) = -\frac{1}{4\pi} \int_{\Sigma} -U \frac{\partial G_-}{\partial n}$$

$$\frac{\partial G_-}{\partial n}(P_1) = \cos(\vec{n}, \vec{r}_{01}) \left(jk - \frac{1}{r_{01}}\right) \frac{\exp(jkr_{01})}{r_{01}} - \cos(\vec{n}, \vec{\tilde{r}}_{01}) \left(jk - \frac{1}{\tilde{r}_{01}}\right) \frac{\exp(jk\tilde{r}_{01})}{\tilde{r}_{01}}$$

on P_1 $\quad r_{01} = \tilde{r}_{01}$
$\cos(\vec{n}, \vec{r}_{01}) = -\cos(\vec{n}, \vec{\tilde{r}}_{01})$

$$\frac{\partial G_-}{\partial n} = 2 \frac{\partial G}{\partial n} \quad \text{from before}$$

$$U_I(P_0) = -\frac{1}{2\pi} \int_{\Sigma} U \frac{\partial G}{\partial n}\, ds$$

The other solution would be to have $\frac{\partial G}{\partial n} = 0$ so choose

$$G_+(P_1) = \frac{\exp(jkr_{01})}{r_{01}} + \frac{\exp(jk\tilde{r}_{01})}{r_{01}}$$

$$U_{II}(P_0) = \frac{1}{4\pi}\int_\Sigma \frac{\partial U}{\partial n} G_+ \, ds$$

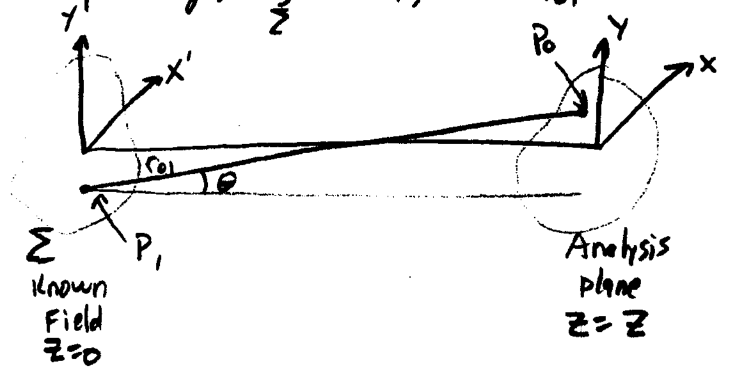
$$G_+ = 2G$$

$$U_{II}(P_0) = \frac{1}{2\pi}\int_\Sigma \frac{\partial U}{\partial n} G \, ds$$

$$U_I(P_0) = \frac{1}{2\pi}\int_\Sigma U \frac{\partial G}{\partial n} ds = \frac{1}{2\pi}\int_\Sigma U \cos(\vec{n}, \vec{r}_{01})\left(jk - \frac{1}{r_{01}}\right)\frac{e^{jkr_{01}}}{r_{01}}$$

Let's use U_I (First Rayleigh-Sommerfeld Solution)
with $r_{01} \gg \lambda$

$$U_I(P_0) = \frac{1}{j\lambda}\iint_\Sigma U(P_1)\frac{\exp(jkr_{01})}{r_{01}}\cos\theta \, ds \qquad \cos\theta = \cos(\vec{n}, \vec{r}_{01})$$



$$\cos\theta = \frac{Z}{r_{01}} \qquad r_{01} = \sqrt{(X-X')^2 + (Y-Y')^2 + Z^2}$$

$$U_I(P_0) = \frac{Z}{j\lambda}\iint_\Sigma U(P_1)\frac{\exp(jkr_{01})}{r_{01}^2}\, dx'\, dy'$$

We want to simplify r_{01}.
Assume $z \gg x-x'$ or $y-y'$

use the expansion $\sqrt{1+b} = 1 + \frac{1}{2}b - \frac{1}{8}b^2 + \cdots$

$$r_{01} = Z\left(1 + \frac{1}{2}\left(\frac{X-X'}{Z}\right)^2 + \frac{1}{2}\left(\frac{Y-Y'}{Z}\right)^2 + \cdots\right]$$

The terms needed are different for $\frac{1}{r_{01}^2}$ and $\exp[jkr_{01}]$

for $\frac{1}{r_{01}^2} \simeq \frac{1}{z^2}$ $\quad \exp[jkr_{01}] \simeq \exp\left[jkZ\left(1 + \frac{1}{2}\left(\frac{X-X'}{Z}\right)^2 + \frac{1}{2}\left(\frac{Y-Y'}{Z}\right)^2\right)\right]$

Let's use the first Rayleigh - Sommerfeld solution

$$U_I(P_0) = \frac{1}{2\pi} \int_{\Sigma} U \frac{\partial G}{\partial n} \, ds$$

$$= \frac{1}{2\pi} \int_{\Sigma} U \cos(\vec{n}, \vec{r}_{01}) \left(jK - \frac{1}{r_{01}} \right) \frac{e^{jKr_{01}}}{r_{01}} \, ds$$

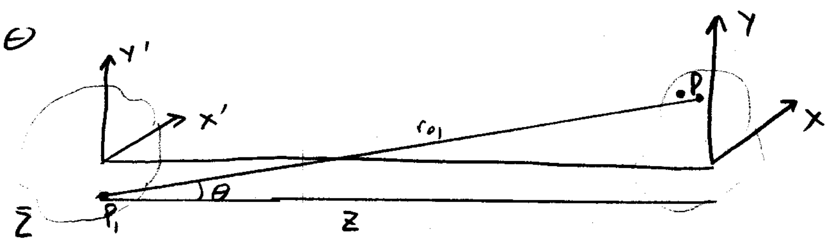
Assume $r_{01} \gg \lambda$ or $r_{01} \gg 10\,\mu m$

$$jK - \frac{1}{r_{01}} = j \frac{2\pi}{\lambda} - \frac{1}{r_{01}} \approx j \frac{2\pi}{\lambda}$$

$$U_I(P_0) = \frac{1}{2\pi} \int_{\Sigma} U \cos(\vec{n}, \vec{r}_{01}) \, j \frac{2\pi}{\lambda} \frac{e^{jKr_{01}}}{r_{01}} \, ds$$

$$= -\frac{1}{j\lambda} \int_{\Sigma} U \cos\theta \frac{\exp(jKr_{01})}{r_{01}} \, ds$$

what is $\cos\theta$



$$\cos\theta = \frac{Z}{r_{01}}$$

$$U_I(P_0) = -\frac{1}{j\lambda} \int_{\Sigma} U \left(\frac{Z}{r_{01}} \right) \frac{\exp(jKr_{01})}{r_{01}} \, ds$$

$$= -\frac{Z}{j\lambda} \int_{\Sigma} U \frac{\exp(jKr_{01})}{r_{01}^2} \, ds$$

$$r_{01} = \sqrt{(X-X')^2 + (Y-Y')^2 + Z^2}$$

Approximation : $Z \gg |X-X'|$ or $|Y-Y'|$

$$r_{01} = Z \sqrt{\left(\frac{X-X'}{Z}\right)^2 + \left(\frac{Y-Y'}{Z}\right)^2 + 1}$$

Use a binomial expansion

$$(1+x)^K = 1 + Kx + \frac{K(K-1)}{2!} x^2 + \dots + \frac{K(K-1)\dots(K-n+1)}{n!} x^n$$

$$(1+\Delta)^{1/2} = 1 + \frac{\Delta}{2} + \frac{\left(\frac{1}{2}\right)\left(-\frac{1}{2}\right)}{2} \Delta^2$$

$$= 1 + \frac{\Delta}{2} - \frac{\Delta^2}{8}$$

$$r_{01} = z\left[1 + \frac{1}{2}\left(\frac{x-x'}{z}\right)^2 + \frac{1}{2}\left(\frac{y-y'}{z}\right)^2 - \frac{1}{8}\left(\frac{x-x'}{z}\right)^4 - \frac{1}{8}\left(\frac{y-y'}{z}\right)^4\right]$$

two terms of interest

$$\frac{1}{r_{01}} \approx \frac{1}{z^2} \qquad \exp[jkr_{01}] \approx \exp\left[jkz\left(1 + \frac{1}{2}\left(\frac{x-x'}{z}\right)^2 + \frac{1}{2}\left(\frac{y-y'}{z}\right)^2\right)\right]$$

This is the Fresnel approximation

When is it valid?
Look at the next term

$$\exp[jkr_{01}] = \exp\Big[\underbrace{jkz\left(1 + \frac{1}{2}\left(\frac{x-x'}{z}\right)^2 + \frac{1}{2}\left(\frac{y-y'}{z}\right)^2\right)}_{\text{Value}} + \underbrace{jkz\left(-\frac{1}{8}\left(\frac{x-x'}{z}\right)^4 - \frac{1}{8}\left(\frac{y-y'}{z}\right)^4\right)}_{\text{error}}\Big]$$

For negligible error

$$\left(\frac{2\pi}{\lambda}\right)(z)\left(\frac{1}{8}\right)\left[\left(\frac{x-x'}{z}\right)^4 + \left(\frac{y-y'}{z}\right)^4\right] \ll \pi$$

Let a = maximum distance between d and d'

$$\left(\frac{2\pi}{\lambda}\right)\left(\frac{z}{8}\right)\frac{a^4}{z^4} \ll \pi$$

$$\frac{a^4}{4\lambda z^3} \ll 1$$

convert to angle $\sin\theta = \frac{a}{z}$

$\theta \approx \frac{a}{z}$

$$\frac{a}{4\lambda}\theta^3 \ll 1$$

Example: $\lambda = 1\mu m$
1mm diameter hole

Let's look at a 10mm area plane
$a = 11mm$

$$z^3 \gg \frac{a^4}{4\lambda}$$

$$z \gg \frac{(11\times10^{-3})^{4/3}}{(4\times10^{-6})^{1/3}}$$

$$\boxed{z \gg 0.15m} \qquad \text{or} \qquad \theta \ll \left(\frac{4\times10^{-6}}{11\times10^{-3}}\right)^{1/3}$$

$$U = -\frac{z}{j\lambda}\iint_{\Sigma} U\left(\frac{1}{z^2}\right)\exp\left[jkz\left(1 + \frac{1}{2}\left(\frac{x-x'}{z}\right)^2 + \frac{1}{2}\left(\frac{y-y'}{z}\right)^2\right)\right] ds$$

$$U = -\frac{\exp(jkz)}{j\lambda z}\iint_{\Sigma} U \exp\left[\frac{jkz}{2}\left(\left(\frac{x-x'}{z}\right)^2 + \left(\frac{y-y'}{z}\right)^2\right)\right] ds$$

This can be cast in several ways:

(1) As a Fourier transform

$$U = \frac{e^{jkz}}{j\lambda z}\iint_{-\infty}^{\infty} U(x', y') \exp\left[\frac{jk}{2z}\left(x^2 - 2xx' + x'^2 + y^2 - 2yy' + y'^2\right)\right] dx'dy'$$

$$U = \frac{e^{jkz}}{j\lambda z} e^{\frac{jk}{2z}(x^2+y^2)}\iint_{-\infty}^{\infty} U(x',y')\, e^{\frac{jk}{2z}(x'^2+y'^2)}\, e^{-j\frac{k}{z}(xx'+yy')}\, dx'\, dy'$$

Fourier transform is: $V(\omega) \equiv \int_{-\infty}^{\infty} V(t)\, e^{j\omega t}\, dt$

let $t \equiv x'$ or y'

$\omega \equiv \frac{k}{z}x$ or $\frac{k}{z}y$

$$U = \frac{e^{jkz}}{j\lambda z} e^{\frac{jk}{2z}(x^2+y^2)}\, \mathcal{F}\left\{U(x,y)\, e^{j\frac{k}{2z}(x^2+y^2)}\right\}\Bigg|_{\substack{\omega_x = \frac{kx}{z} \\ \omega_y = \frac{ky}{z}}}$$

(2) As a convolution

$$U(x,y) = \iint_{-\infty}^{\infty} U(x',y')\, h(x-x', y-y')\, dx'\, dy'$$

$$h(x,y) = \frac{e^{jkz}}{j\lambda z}\exp\left[\frac{jk}{2z}(x^2+y^2)\right]$$

Fresnel Slit Diffraction

Calculate the diffraction of a slit

$$T(x) = \begin{cases} 1 & |x| < a \\ 0 & \text{else} \end{cases}$$

Start with the Fresnel diffraction integral

$$E(x,y) = \frac{e^{jkz}}{j\lambda z} \underbrace{\int_{-\infty}^{\infty} \exp\left[-\frac{jk}{2z}(y-y')^2\right] dy'}_{A(y)} \underbrace{\int_{-a}^{a} \exp\left[-\frac{jk}{2z}(x-x')^2\right] dx'}_{B(x)}$$

Start with y-integral

$$A(y) = \int_{-\infty}^{\infty} \exp\left[-j\frac{k}{2z}(y-y')^2\right] dy' \qquad u = y-y' \quad du = dy'$$

$$= \int_{-\infty}^{\infty} \exp\left[-j\frac{k}{2z}u^2\right] du \qquad v = \sqrt{\frac{jk}{2z}}\,u \quad dv = \sqrt{\frac{jk}{2z}}\,du$$

$$= \sqrt{\frac{2z}{jk}} \int_{-\infty}^{\infty} e^{-v^2} dv$$

use erfc(x)

$$\operatorname{erfc}(x) = \frac{2}{\sqrt{\pi}} \int_{x}^{\infty} e^{-t^2} dt$$

$$= \sqrt{\frac{2z}{jk}} \frac{\sqrt{\pi}}{2} \frac{2}{\sqrt{\pi}} \int_{-\infty}^{\infty} e^{-v^2} dv$$

$$= \sqrt{\frac{2z}{jk}} \frac{\sqrt{\pi}}{2} \underbrace{\operatorname{erfc}(-\infty)}_{2}$$

$$A(y) = \sqrt{\frac{2z}{jk}\pi}$$

Now the x-integral

$$B(x) = \int_{-a}^{a} \exp\left[-j\frac{k}{2z}(x-x')^2\right] dx' \qquad u = x'-x \quad du = dx'$$

$$= \int_{-a-x}^{a-x} \exp\left[-j\frac{k}{2z}u^2\right] du$$

if we try to use erfc the argument will be complex which is invalid for erfc so you can't use this function

Use Fresnel integrals

$$C(x) = \int_0^x \cos\left(\frac{\pi}{2}t^2\right) dt$$

$$S(x) = \int_0^x \sin\left(\frac{\pi}{2}t^2\right) dt$$

$$B(x) = \int_{-a-x}^{a-x} \exp\left[-j \frac{\pi}{\lambda z} u^2\right] du$$

$$\frac{\pi}{2} v^2 = \frac{\pi}{\lambda z} u^2$$

$$v = \sqrt{\frac{2}{\lambda z}}\, u \qquad dv = \sqrt{\frac{2}{\lambda z}}\, du$$

$u = a - x \qquad v = \sqrt{\frac{2}{\lambda z}}(a-x)$

$u = -a - x \qquad v = \sqrt{\frac{2}{\lambda z}}(-a-x)$

$$B(x) = \sqrt{\frac{\lambda z}{2}} \int_{\sqrt{\frac{2}{\lambda z}}(-a-x)}^{\sqrt{\frac{2}{\lambda z}}(a-x)} e^{-j \pi/2\, v^2}\, dv$$

$$e^{-j\theta} = \cos\theta - j \sin\theta$$

$$B(x) = \sqrt{\frac{\lambda z}{2}} \int_0^{\sqrt{\frac{2}{\lambda z}}(a-x)} e^{-j \pi/2\, v^2} dv + \int_0^{\sqrt{\frac{2}{\lambda z}}(a+x)} e^{-j \pi/2\, v^2} dv$$

$$= \sqrt{\frac{\lambda z}{2}} \Bigg(\int_0^{\sqrt{\frac{2}{\lambda z}}(a-x)} \cos\left(\frac{\pi}{2} v^2\right) dv - j \int_0^{\sqrt{\frac{2}{\lambda z}}(a-x)} \sin\left(\frac{\pi}{2} v^2\right) dv$$
$$+ \int_0^{\sqrt{\frac{2}{\lambda z}}(a+x)} \cos\left(\frac{\pi}{2} v^2\right) dv - j \int_0^{\sqrt{\frac{2}{\lambda z}}(a+x)} \sin\left(\frac{\pi}{2} v^2\right) dv \Bigg)$$

$$= \sqrt{\frac{\lambda z}{2}} \left[C\left(\sqrt{\frac{2}{\lambda z}}(a-x)\right) - j\, S\left(\sqrt{\frac{2}{\lambda z}}(a-x)\right) + C\left(\sqrt{\frac{2}{\lambda z}}(a+x)\right) - j\, S\left(\sqrt{\frac{2}{\lambda z}}(a+x)\right) \right]$$

Final answer

$$E(x,y) = \frac{e^{jkz}}{j \lambda z} \sqrt{\frac{2 z \pi}{j k}} \sqrt{\frac{\lambda z}{2}} \left[C\left(\sqrt{\frac{2}{\lambda z}}(a-x)\right) - j\, S\left(\sqrt{\frac{2}{\lambda z}}(a-x)\right) + C\left(\sqrt{\frac{2}{\lambda z}}(a+x)\right) - j\, S\left(\sqrt{\frac{2}{\lambda z}}(a+x)\right) \right]$$

$NF = \frac{a^2}{\lambda d}$