Fraunhofer Diffraction

Start with Fresnel diffraction integral

$$E(x,y) = \frac{e^{jkz}}{j\lambda z} \iint E(x',y') \exp\left[\frac{jk}{2z}\left((x-x')^2 + (y-y')^2\right)\right] dx'\,dy'$$

$$= \frac{e^{jkz}}{j\lambda z} \iint E(x',y') \exp\left[\frac{jk}{2z}\left(x^2 + (x')^2 - 2xx' + y^2 + (y')^2 - 2yy'\right)\right] dx'\,dy'$$

$$= \frac{e^{jkz}}{j\lambda z} \iint E(x',y') \exp\left[\frac{jk}{2z}(x^2+y^2)\right] \exp\left[\frac{jk}{2z}\left((x')^2 + (y')^2 - 2xx' - 2yy'\right)\right] dx'\,dy'$$

$$= \frac{e^{jkz}}{j\lambda z} \exp\left[\frac{jk}{2z}(x^2+y^2)\right] \iint E(x',y') \exp\left[\frac{jk}{2z}\left(x'^2 + y'^2 - 2xx' - 2yy'\right)\right] dx'\,dy'$$

$$= \frac{e^{jkz}}{j\lambda z} \exp\left[\frac{jk}{2z}(x^2+y^2)\right] \iint E(x',y') \exp\left[\frac{jk}{2z}(x'^2+y'^2)\right] \exp\left[\frac{jk}{2z}(-2xx' - 2yy')\right] dx'\,dy'$$

In the Fraunhofer Approximation

$$\exp\left[\frac{jk}{2z}(x'^2+y'^2)\right] \approx 1$$

$$\frac{k}{2z}(x'^2+y'^2) << 2\pi$$

$$\left(\frac{2\pi}{\lambda}\right)\left(\frac{1}{2z}\right)\left((x')^2+(y')^2\right) << 2\pi$$

$$\boxed{z >> \frac{(x')^2 + (y')^2}{2\lambda}}$$

Fraunhofer Diffraction integral

$$U(x,y) = \frac{e^{jkz}}{j\lambda z}\, e^{j\frac{k}{2z}(x^2+y^2)} \iint E(x',y')\, e^{-j\frac{2\pi}{\lambda z}(xx'+yy')}\, dx'\,dy'$$

This integral is similar to the Fourier transform integral

$$\mathcal{F}(g) = \int_{-\infty}^{\infty} g(x')\, e^{-j2\pi f_x x'}\, dx'$$

$$-j\frac{2\pi}{\lambda z} xx' = -j2\pi f_x x' \Rightarrow f_x = \frac{x}{\lambda z} \quad \text{and } f_y = \frac{y}{\lambda z}$$

The Fourier transform operation can be regarded as a decomposition of a complicated function into a collection of more simple complex-exponential functions.

In standard signal processing the complex exponentials correspond to sinusoidal signals.

In optical analysis the complex exponentials correspond to plane waves

1D Fourier transform:
$$U(f) = \int_{-\infty}^{\infty} u(t)\, e^{-j 2\pi f t}\, dt$$
$$u(t) = \int_{-\infty}^{\infty} U(f)\, e^{+j 2\pi f t}\, df$$

In this case $U(f)$ is the frequency spectrum of the signal

Extend this into a 2D Fourier transform
$$U(f_x, f_y) = \iint_{-\infty}^{\infty} u(x,y)\, \exp[-j 2\pi (f_x x + f_y y)]\, dx\, dy$$
$$u(x,y) = \iint_{-\infty}^{\infty} U(f_x,f_y)\, \exp[+j 2\pi (f_x x + f_y y)]\, df_x\, df_y$$

$U(f_x, f_y)$ is the spectrum of functions of the form $\exp[j 2\pi (f_x x + f_y y)]$

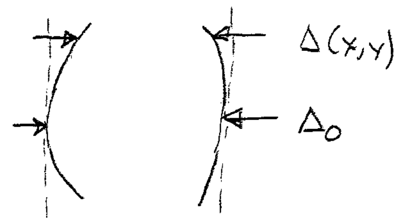
What are the physical meaning of these functions?

The simplest waves that we have analyzed are plane waves as given by:
$$P(x,y,z) = \exp[j \vec{k} \cdot \vec{r}] = \exp[j (k_x x + k_y y + k_z z)]$$
which is very similar to $\exp[j 2\pi (f_x x + f_y y)]$

At the $z=0$ plane
$$P(x,y,0) = \exp[j (k_x x + k_y y)]$$

To make them equal let
$$k_x = 2\pi f_x \quad \text{and} \quad k_y = 2\pi f_y$$

A lens adds a phase transformation



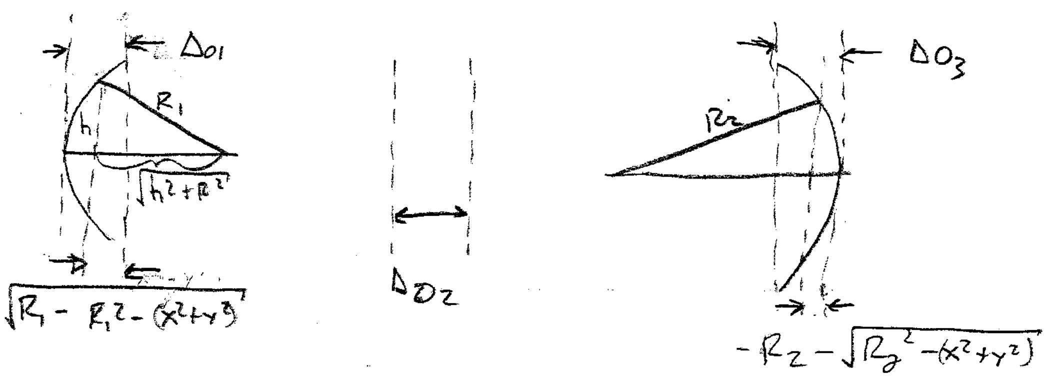
The total phase delay caused by the lens is

$$\phi(x,y) = kn\,\Delta(x,y) + k\left(\Delta_0 - \Delta(x,y)\right)$$

The lens transmission becomes

$$T_{lens} = \exp(jk\Delta_0)\,\exp\left[jk(n-1)\,\Delta(x,y)\right]$$

To find the thickness function $\Delta(x,y)$ separate the lens into 3 parts



$$\Delta_1(x,y) = \Delta_{01} - \left(R_1 - \sqrt{R_1^2 - (x^2+y^2)}\right)$$

$$= \Delta_{01} - R_1\left(1 - \sqrt{1 - \frac{x^2+y^2}{R_1^2}}\right)$$

$$\Delta_3(x,y) = \Delta_{03} - \left(-R_2 - \sqrt{R_2^2 - (x^2+y^2)}\right)$$

$$= \Delta_{03} + R_2\left(1 - \sqrt{1 - \frac{x^2+y^2}{R_2^2}}\right)$$

$$\Delta(x,y) = \Delta_{01} + \Delta_{02} + \Delta_{03} - R_1\left(1 - \sqrt{1 - \frac{x^2+y^2}{R_1^2}}\right) + R_2\left(1 - \sqrt{1 - \frac{x^2+y^2}{R_2^2}}\right)$$

In the paraxial approximation

$$\sqrt{1 - \frac{x^2+y^2}{R_1^2}} \approx 1 - \frac{x^2+y^2}{2R_1^2}$$

$$\sqrt{1 - \frac{x^2+y^2}{R_2^2}} \approx 1 - \frac{x^2+y^2}{2R_2^2}$$

$$\Delta(x,y) = \Delta_{01} + \Delta_{02} + \Delta_{03} - R_1\left(\cancel{1} - \cancel{1} + \frac{x^2+y^2}{2R_1^2}\right) + R_2\left(\cancel{1} - \cancel{1} + \frac{x^2+y^2}{2R_2^2}\right)$$

$$= \Delta_{01} + \Delta_{02} + \Delta_{03} - \left(\frac{x^2+y^2}{2}\right)\left(\frac{1}{R_1} - \frac{1}{R_2}\right)$$

$$T_{lens} = \exp(jK\Delta_0)\ \exp\left(-jK(n-1)\left(\frac{x^2+y^2}{2}\right)\left(\frac{1}{R_1} - \frac{1}{R_2}\right)\right)$$

$$\frac{1}{f} \equiv (n-1)\left(\frac{1}{R_1} - \frac{1}{R_2}\right)$$

$$T_{lens} = \exp(jK\Delta_0)\ \exp\left(-j\frac{K}{2f}(x^2+y^2)\right)$$

So the incident field is multiplied by the lens transmission function

$$E(x,y) = \frac{e^{jKz}}{j\lambda z} e^{jK\Delta_0} \iint E(x',y')\, e^{-j\frac{K}{2f}(x'^2+y'^2)}\, e^{j\frac{K}{2z}(x'^2+y'^2)}\, e^{-j\frac{K}{z}(xx'+yy')}\, dx'\,dy'$$

$$= \frac{e^{jKz}}{j\lambda z} e^{jK\Delta_0} \iint E(x',y')\, e^{-j\frac{K}{2}(x'^2+y'^2)\left(\frac{1}{f}-\frac{1}{z}\right)}\, e^{-j\frac{K}{z}(xx'+yy')}\, dx'\,dy'$$

if the analysis plane is at the lens focus $f = z$

$$E(x,y) = \frac{e^{jKz}}{j\lambda z} e^{jK\Delta_0} \iint E(x',y')\, e^{-j\frac{2\pi}{\lambda f}(xx'+yy')}\, dx'\,dy'$$

$$E(x,y) = \frac{e^{jKz}}{j\lambda z} e^{jK\Delta_0}\ \mathcal{F}\{E(x,y)\}\Big|_{f_x = \frac{x}{\lambda f},\ f_y = \frac{y}{\lambda f}}$$

Circular aperture

$$G(f_x, f_y) = \iint_{-\infty}^{\infty} g(x,y) \exp[-j2\pi(f_x x + f_y y)]\, dx\, dy$$

transform into polar coordinates

$$r = \sqrt{x^2+y^2}$$
$$\theta = \tan^{-1}\left(\frac{y}{x}\right)$$
$$\rho = \sqrt{f_x^2 + f_y^2}$$
$$\phi = \tan^{-1}\left(\frac{f_y}{f_x}\right)$$

For a rotationally symmetric incident field the Fourier transform becomes

$$G(\rho) = \int_0^{2\pi}\int_0^{\infty} g(r) \exp[-j2\pi r\rho(\cos\theta\cos\phi + \sin\theta\sin\phi)]\,\rho\, d\theta\, dr$$

$$= \int_0^{2\pi}\int_0^{\infty} g(r) \exp[-j2\pi r\rho\cos(\theta-\phi)]\,\rho\, d\theta\, dr$$

The definition of the Bessel function is

$$J_0(a) = \frac{1}{2\pi}\int_0^{2\pi} \exp[-ja\cos(\theta-\phi)]\, d\theta$$

let $a = 2\pi r\rho$

$$G(\rho) = 2\pi\int_0^{\infty} r\, g(r)\, J_0(2\pi r\rho)\, dr$$

Since the Bessel function is even

$$g(r) = 2\pi\int_0^{\infty} \rho\, G(\rho)\, J_0(2\pi r\rho)\, d\rho$$

So the Fourier transform and inverse Fourier transform are the same

Now take the Fourier transform of a circular aperture

circle function $\quad \mathrm{circ}(r) = \mathrm{circ}(\sqrt{x^2+y^2}) = \begin{cases} 1 & \sqrt{x^2+y^2} < 1 \\ \frac{1}{2} & \sqrt{x^2+y^2} = 1/2 \\ 0 & \text{else} \end{cases}$

$$\mathcal{F}\{\mathrm{circ}(r)\} = 2\pi\int_0^1 r\, J_0(2\pi r\rho)\, dr$$

let $u = 2\pi r\rho \quad du = 2\pi\rho\, dr \quad r = \frac{u}{2\pi\rho}$

$r = 1 \rightarrow u = 2\pi\rho$

$$\mathcal{F}\{\mathrm{circ}(r)\} = 2\pi\int_0^{2\pi\rho} \frac{u}{2\pi\rho}\, J_0(u)\, \frac{du}{2\pi\rho}$$

$$= \frac{1}{2\pi\rho^2}\int_0^{2\pi\rho} u\, J_0(u)$$

Now we use the Bessel identity

$$\int_0^x \tau J_0(\tau)\, d\tau = x J_1(x)$$

$$\mathcal{F}\{\mathrm{circ}(r)\} = \frac{1}{2\pi\rho^2}(2\pi\rho)\, J_1(2\pi\rho)$$

$$= \frac{J_1(2\pi\rho)}{\rho}$$

For a circular aperture with radius R

$$\mathcal{F}\{\mathrm{circ}(\tfrac{r}{R})\} = 2\pi R^2 (2) \frac{J_1(2\pi R f_R)}{2\pi R f_R}$$

with a lens $$E(r) = \frac{e^{jkz} e^{j\frac{kr^2}{2}}}{j\lambda f} 2\pi R^2 \left[2 \frac{J_1(2\pi R \frac{r}{\lambda f})}{2\pi R \frac{r}{\lambda f}} \right]$$

$$I(r) = |E(r)|^2 = \left(\frac{2\pi R^2}{\lambda f}\right)^2 \left[2 \frac{J_1(2\pi R \frac{r}{\lambda f})}{2\pi R \frac{r}{\lambda f}} \right]^2$$

The location of the 1st null is $\left[2 \frac{J_1(\pi x)}{\pi x}\right]^2 = 0$ at $x = 1.22$

$$\frac{2R}{\lambda f} r = 1.22$$

$$r = \frac{1.22}{2} \lambda \frac{f}{R}$$

$$r = 1.22 \lambda \frac{f}{D} \quad \text{or} \quad d = 2.44 \lambda f\#$$

This is called the Airy Disk