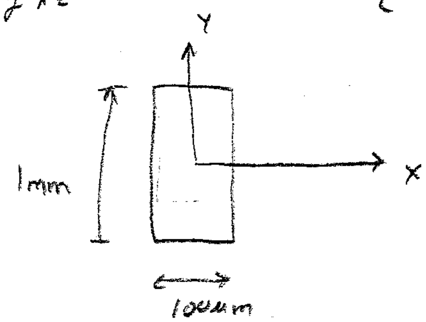
Example 1 — Square aperture

A $100\mu m \times 1mm$ aperture illuminated by a laser $\lambda = 500nm$.
(a) Find the diffraction pattern
(b) Find the necessary distance away from the aperture

Assume that the incident field is a plane wave

$$E(x,y) = \frac{e^{jkz}}{j\lambda z} e^{j\frac{k}{2z}(x^2+y^2)} \mathcal{F}\{P(x,y)\}\Big|_{f_x = \frac{x}{\lambda z},\ f_y = \frac{y}{\lambda z}}$$



$$p(x) = \begin{cases} 1 & |x| \le 50\mu m \\ 0 & \text{else} \end{cases}$$

this is similar to

$$\text{rect}(x) = \begin{cases} 1 & |x| \le \frac{1}{2} \\ 0 & \text{else} \end{cases}$$

$$\text{rect}(ax) = \begin{cases} 1 & |ax| \le \frac{1}{2} \\ 0 & \text{else} \end{cases} = \begin{cases} 1 & |x| \le \frac{1}{2a} \\ 0 & \text{else} \end{cases}$$

let $a = \frac{1}{100\mu m}$

$$p(x) = \text{rect}\left(\frac{x}{100\mu m}\right) = \begin{cases} 1 & |x| \le 50\mu m \\ 0 & \text{else} \end{cases}$$

Now take the Fourier transform

$$\mathcal{F}\{\text{rect}(ax)\} = \frac{1}{a}\,\text{Sinc}\left(\frac{F_x}{a}\right)$$

$$P(F_x) = (100\times10^{-6})\,\text{Sinc}(100\times10^{-6}F_x)$$

similar for the y-direction

$$P(F_y) = (10^{-3})\,\text{Sinc}(10^{-3}F_y)$$

$$F_x = \frac{x}{\lambda z} \qquad F_y = \frac{y}{\lambda z}$$

$$E(x,y) = \frac{e^{jkz}}{j\lambda z} e^{j\frac{k}{2z}(x^2+y^2)} (100\times10^{-6})(10^{-3})\,\text{Sinc}\left(\frac{100\times10^{-6}x}{\lambda z}\right)\text{Sinc}\left(\frac{10^{-3}y}{\lambda z}\right)$$

Zeros at $\frac{100\times10^{-6}x}{\lambda z} = \pm 1$

$$\Theta_x = \frac{x}{z} = \frac{\lambda}{100\times10^{-6}} = \frac{0.5\times10^{-6}}{100\times10^{-6}}$$

$$\Theta_x = 5 \text{ mrad} = 0.29^\circ$$

$$\frac{10^3 y}{\lambda z} = \pm 1$$

$$\Theta_y = \frac{y}{z} = \frac{\lambda}{10^{-3}} = \frac{0.5}{1000} = 0.5\text{mrad} = 0.029^\circ$$

Valid distance $z \gg \frac{(x)^2+(y')^2}{2\lambda}$

$$z \gg \frac{(50\times10^{-6})^2 + (500\times10^{-6})^2}{(2)(0.5\times10^{-6})}$$

$$z \gg 0.25 \text{ m}$$

A lens with a focal length of $f = 10mm$ has a limiting aperture of $D = 5mm$. The lens also has a square obscuration in the middle of the lens. The lens is illuminate with a uniform beam of illuminance $10 \frac{mW}{m^2}$. The obscaration has a size of $2mm \times 2mm$. Use $\lambda = 500nm$

Start with the circular aperture.

$$\text{circ}\left(\frac{r}{2.5mm}\right) = \begin{cases} 1 & r < 2.5mm \\ 0 & \text{else} \end{cases}$$

$$\mathcal{F}\left\{\text{circ}\left(\frac{r}{2.5mm}\right)\right\} = A_c\left[2\,\frac{J_1\left(2\pi(2.5\times10^{-3})f_x\right)}{(2\pi)(2.5\times10^{-3})f_x}\right]$$

Not keeping constant amplitude terms

$$E_1(x,y) = A_c\left[2\,\frac{J_1\left((2\pi)(2.5\times10^{-3})\frac{r}{\lambda f}\right)}{(2\pi)(2.5\times10^{-3})\left(\frac{r}{\lambda f}\right)}\right]$$

$$E_1(x,y) = 6.25\times10^{-6}\,\pi\left[2\,\frac{J_1\left(10^6\pi r\right)}{10^6\pi r}\right]$$

Now the square obscuration

$$\mathcal{F}\left\{\text{rect}\left(\frac{x}{2\times10^{-3}}\right)\text{rect}\left(\frac{y}{2\times10^{-3}}\right)\right\} = w^2\,\text{Sinc}\left(2\times10^{-3}f_x\right)\text{Sinc}\left(2\times10^{-3}f_y\right)$$

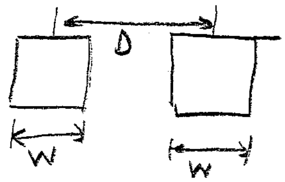
$$E_2(x,y) = w^2\,\text{Sinc}\left(\frac{2\times10^{-3}}{(0.5\times10^{-6})(10\times10^{-3})}\,x\right)\text{Sinc}\left(\frac{2\times10^{-3}}{(0.5\times10^{-6})(10\times10^{-3})}\,y\right)$$

$$(4\times10^{-6})\;\text{Sinc}\left(4\times10^{5}x\right)\;\text{Sinc}\left(4\times10^{5}y\right)$$

$$E_{tot} = E_1 - E_2$$

use $\lambda = 500\,nm$

$f = 10\,mm$



Aperture 1: $t_1(x,y) = rect\left(\frac{x - D/2}{W}\right) rect\left(\frac{y}{W}\right)$

$$T_1(f_x, f_y) = \mathcal{F}\{t\} = e^{-j2\pi D/2 f_x}\, \text{sinc}(W f_x)\, \text{sinc}(W f_y)$$

$$E_1(x,y) = W^2 e^{-j\pi D x/\lambda f}\, \text{sinc}\left(W \frac{x}{\lambda f}\right) \text{sinc}\left(W \frac{y}{\lambda f}\right)$$

$$t_2(x,y) = rect\left(\frac{x + D/2}{W}\right) rect\left(\frac{y}{W}\right)$$

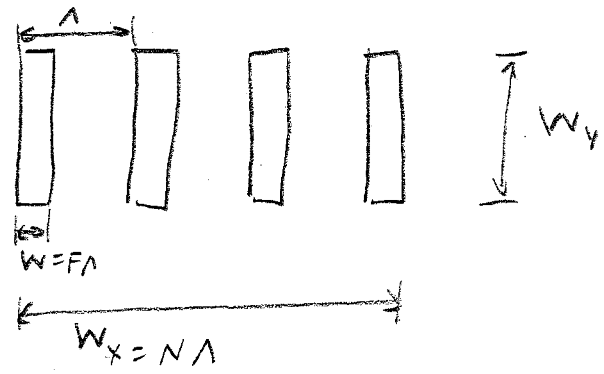
$$T_2(f_x, f_y) = W^2 e^{+j2\pi D/2 f_x}\, \text{sinc}(W f_x)\, \text{sinc}(W f_y)$$

$$E_2(x,y) = W^2\, \text{sinc}\left(W \frac{x}{\lambda f}\right) \text{sinc}\left(W \frac{y}{\lambda f}\right) e^{+j\pi D x/\lambda f}$$

$$E_{tot}(x,y) = W^2\, \text{sinc}\left(\frac{Wx}{\lambda f}\right) \text{sinc}\left(\frac{Wy}{\lambda f}\right)(2)\, \frac{\left(e^{j\pi D x/\lambda f} + e^{-j\pi D \frac{x}{\lambda f}}\right)}{2}$$

$$E_{tot} = E_0\, \text{sinc}\left(\frac{Wx}{\lambda f}\right) \text{sinc}\left(\frac{Wy}{\lambda f}\right) \cos\left(\pi \frac{Dx}{\lambda f}\right)$$

$\lambda = 500nm$
$f = 10mm$



Model the aperture as 3 separate functions
(1) Individual infinite slit
(2) infinite sum of delta functions
(3) complete aperture size

(1) Individual slit

$$t_1(x,y) = \mathrm{rect}\left(\frac{x}{W}\right)$$

$$T_1(f_x, f_y) = \mathcal{F}\{t_1\} = W \,\mathrm{sinc}(Wf_x)\ \delta(f_y)$$

(2) Infinite sum of delta functions

First look at the perodicity in x-direction

$$\mathrm{comb}(x) = \sum_{n=-\infty}^{\infty} \delta(x-n) \qquad \text{we want} \quad \sum_{n=-\infty}^{\infty} \delta(x-n\Lambda)$$

$$\mathrm{comb}\left(\frac{x}{\Lambda}\right) = \sum_{n=-\infty}^{\infty} \delta\left(\frac{x}{\Lambda}-n\right) = \sum_{n=-\infty}^{\infty} \delta\left[\frac{1}{\Lambda}(x-n\Lambda)\right]$$

from the definition of the delta function

$$1 = \int_{-\infty}^{\infty} \delta(x)\, dx \qquad \int_{-\infty}^{\infty} \delta\left(\frac{x}{\Lambda}\right) dx = \Lambda \int_{-\infty}^{\infty} \delta(u)\, du$$

so $\delta\left(\frac{1}{\Lambda}(x-n\Lambda)\right) = \Lambda\, \delta(x-n\Lambda)$

$$\mathrm{comb}\left(\frac{x}{\Lambda}\right) = \Lambda \sum_{n=-\infty}^{\infty} \delta(x-n\Lambda)$$

$$t_2(x,y) = \frac{1}{\Lambda}\, \mathrm{comb}\left(\frac{x}{\Lambda}\right) \delta(y)$$

$$T_2(f_x, f_y) = \left(\frac{1}{\Lambda}\right)(\Lambda) \sum_{n=-\infty}^{\infty} \delta(\Lambda f_x - n)$$

(3) Complete aperture size

$$t_3(x,y) = \text{rect}\left(\frac{x}{N\Lambda}\right) \text{rect}\left(\frac{y}{W_y}\right)$$

$$T_3(f_x, f_y) = N\Lambda W_y \ \text{Sinc}(N\Lambda f_x) \ \text{Sinc}(W_y f_y)$$

Now put them together

$$t(x,y) = \left[t_3(x,y)\right] \left[t_1(x,y) \otimes t_2(x,y)\right]$$

$$T(f_x, f_y) = \left[T_3(f_x, f_y)\right] \otimes \left[T_1(f_x, f_y) \ T_2(f_x, f_y)\right]$$

$$T = \iint_{-\infty}^{\infty} \text{Sinc}[N\Lambda(f_x - \alpha)] \ \text{Sinc}[W_y(f_y - \beta)] \ \text{sinc}(W\alpha) \ \delta(\beta) \sum_{n=-\infty}^{\infty} \delta(\Lambda\alpha - n) \ d\alpha \, d\beta$$

$$= \sum_{n=-\infty}^{\infty} \int_{-\infty}^{\infty} \text{Sinc}[N\Lambda(f_x - \alpha)] \ \text{Sinc}(W\alpha) \ \delta(\Lambda\alpha - n) \ d\alpha \int_{-\infty}^{\infty} \text{Sinc}[W_y(f_y - \beta)] \ \delta(\beta) \ d\beta$$

$$= \sum_{n=-\infty}^{\infty} \text{Sinc}\left[N\Lambda\left(f_x - \frac{n}{\Lambda}\right)\right] \text{Sinc}\left(W\frac{n}{\Lambda}\right) \ \text{Sinc}(W_y f_y)$$

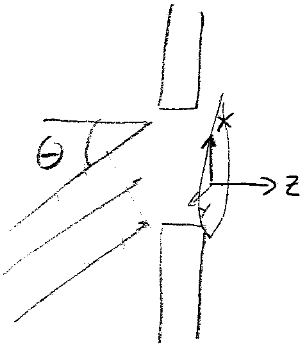
$$T(f_x, f_y) = \left[\text{Sinc}(W_y f_y)\right] \left[\sum_{n=-\infty}^{\infty} \text{Sinc}\left[N\Lambda\left(f_x - \frac{n}{\Lambda}\right)\right] \text{Sinc}\left(W\frac{n}{\Lambda}\right)\right]$$

$$T(x,y) = \left[\text{Sinc}\left(W_y \frac{y}{\lambda f}\right)\right] \left[\sum_{n=-\infty}^{\infty} \text{Sinc}\left(W\frac{n}{\Lambda}\right) \ \text{Sinc}\left[\frac{N\Lambda}{\lambda f}\left(x - \frac{n\lambda f}{\Lambda}\right)\right]\right]$$

Sinc functions centered at $x = \frac{n\lambda f}{\Lambda}$ and $y = 0$

width of orders defined by whole aperture size W_y and $N\Lambda$

Amplitude of orders defined by width of slit $\text{Sinc}\left(W\frac{n}{\Lambda}\right)$



incident field is a plane wave

$$E_i = E_0\, e^{-j\vec{k}\cdot\vec{r}} \qquad \vec{k} = k_0 \sin\theta\, \hat{x} + k_0 \cos\theta\, \hat{z}$$

$$E_i = E_0 \exp\left(-jk_0(\sin\theta\, x + \cos\theta\, z)\right)$$

at $z=0$ plane

$$E_i = E_0\, e^{-jk\sin\theta x}$$

Multiply by the aperture

$$E_i = E_0\, e^{-jk\sin\theta x}\, \mathrm{rect}\left(\frac{x}{W_x}\right) \mathrm{rect}\left(\frac{y}{W_y}\right)$$

$$\mathcal{F}\{E_i\}\Big|_{f_x = \frac{x}{\lambda f},\; f_y = \frac{y}{\lambda f}}$$

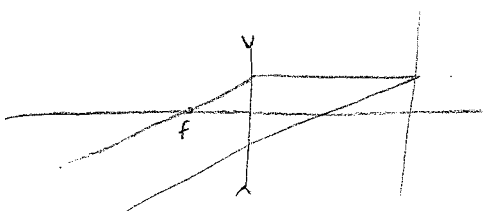
$$E_{out}(x,y) = W_y\, \mathrm{sinc}(W_y f_y)\; \mathcal{F}\left\{\mathrm{rect}\left(\frac{x}{W_x}\right) e^{-jk\sin\theta x}\right.$$

$$= W_y\, \mathrm{sinc}(W_y f_y)\; \mathcal{F}\left\{\mathrm{rect}\left(\frac{x}{W_x}\right) e^{-j2\pi \frac{\sin\theta}{\lambda} x}\right.$$

$$= W_y W_x \mathrm{sinc}(W_y f_y)\; \mathrm{sinc}\left(W_x\left(f_x - \frac{\sin\theta}{\lambda}\right)\right]$$

sinc is centered at $\frac{x}{\lambda f} = \frac{\sin\theta}{\lambda}$

$x = \sin\theta\, f$ in paraxial approximation $x = \theta f$



$\frac{x}{f} = \theta \qquad x = f\theta$

$$E_i(x,y) = E_0 \exp\left(-\frac{x^2+y^2}{W^2}\right)$$

Pass through a lens

$$E_{out}(x,y) = \mathcal{F}\{E_i\}\Big|_{f_X=\frac{x}{\lambda f},\ f_Y=\frac{y}{\lambda f}}$$

From the table:

$$f(x)=\exp(-\pi x^2) \Rightarrow \exp(-\pi f_x^2)$$

$$f(ax) = e^{-\pi(ax)^2} = e^{-\left(\frac{x}{W}\right)^2}$$

$$\pi(ax)^2 = \left(\frac{x}{W}\right)^2$$

$$a^2 = \frac{1}{\pi W^2} \qquad a = \frac{1}{\sqrt{\pi}W}$$

$$F(f_x) = \sqrt{\pi}W \quad \exp\left(-\pi\left(\sqrt{\pi}Wf_x\right)^2\right)$$

$$= \sqrt{\pi}W \quad \exp\left(-\left(\pi W f_x\right)^2\right)$$

$$E_{out}(x,y) = \exp\left(-\left(\frac{\pi W x}{\lambda f}\right)^2\right)$$

Gaussian with new waist radius of $\boxed{\frac{\lambda f}{\pi W} = W_0}$