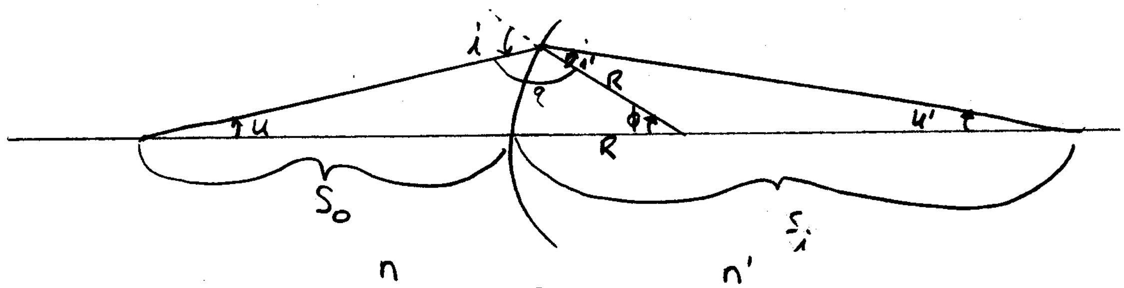
Let's start by looking at a single spherical surface.
We will use Snell's law.

Here is our sign convention:

Light travels from left to right
Distances up are positive
Distances to the right are positive
counterclockwise (ccw) angles are positive
For light traveling right to left use n as negative



Apply Snell's law at the interface
$$n \sin(i) = n' \sin(i')$$

use the paraxial approximation
$$n i = n' i'$$

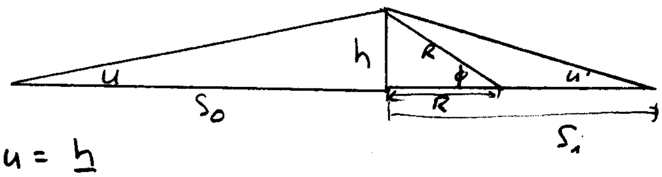
Need to relate u to i and u' to i'
$$u + \phi + q = 180$$
$$q = 180 - u - \phi$$
$$q + i = 180$$
$$i = 180 - 180 + u + \phi$$
$$|i| = |u| + |\phi|$$
from our sign convention
u positive
ϕ negative
i positive

$$i = u - \phi$$

Now relate u' to i'
$$|i'| + |u'| + 180 - |\phi| = 180$$
$$|i'| = -|u'| + |\phi|$$
i' positive
ϕ negative
u' negative

$$i' = u' - \phi$$
Plug into first equation
$$n(u - \phi) = n'(u' - \phi)$$

Now we need to relate the angle equation to position



$$\tan u \approx u = \frac{h}{s_o}$$

$$\sin\phi \approx \phi = \frac{h}{R}$$

$$\tan u' \approx u' = \frac{h}{s_i}$$

$$n(u-\phi) = n'(u'-\phi)$$

$$n\left(\frac{h}{s_o} - \frac{h}{R}\right) = n'\left(\frac{h}{s_i} - \frac{h}{R}\right)$$

$$\frac{n}{s_o} - \frac{n}{R} = \frac{n'}{s_i} - \frac{n'}{R}$$

$$\frac{n}{s_o} - \frac{n'}{s_i} = (n-n')\frac{1}{R}$$

OR

$$\boxed{\frac{n'}{s_i} - \frac{n}{s_o} = (n'-n)\left(\frac{1}{R}\right)} \qquad \frac{1}{R} \equiv C \quad \text{curvature}$$

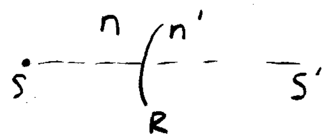
This is the conjugate equation

Thin Lens

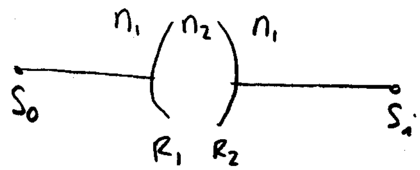
Now use the conjugate equation for a single surface to get the conjugate equation for a thin lens.

Conjugate equation

$$\frac{n'}{s'} - \frac{n}{s} = \frac{(n'-n)}{R}$$

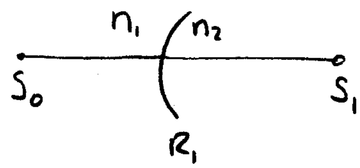


Thin Lens



<u>First Surface</u>

$S = S_o$
$S' = S_1$
$n = n_1$
$n' = n_2$
$R = R_1$

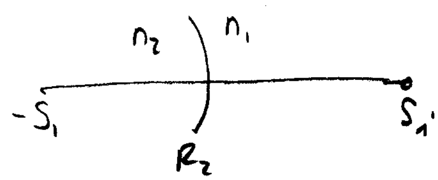


$$\frac{n_2}{S_1} - \frac{n_1}{S_o} = \frac{n_2 - n_1}{R_1}$$

$$\frac{n_2}{S_1} = \frac{n_1}{S_o} + \frac{n_2 - n_1}{R_1}$$

<u>Second surface</u>

$S = -S_1$
$S' = S_i$
$n = n_2$
$n' = n_1$
$R = R_2$



Assume zero separation between lens surfaces

$$\frac{n_1}{S_i} - \frac{n_2}{S_1} = \frac{n_1 - n_2}{R_2}$$

substitute from above

$$\frac{n_1}{S_i} - \frac{n_1}{S_o} - \frac{n_2 - n_1}{R_1} = \frac{n_1 - n_2}{R_2}$$

$$\frac{1}{S_i} - \frac{1}{S_o} = \left(\frac{1}{n_1}\right)\left(\frac{n_1 - n_2}{R_2} + \frac{n_2 - n_1}{R_1}\right)$$

$$\boxed{\frac{1}{S_i} - \frac{1}{S_o} = \left(\frac{n_2 - n_1}{n_1}\right)\left(\frac{1}{R_1} - \frac{1}{R_2}\right) = \frac{1}{f}}$$

Distances to left are negative to right positive

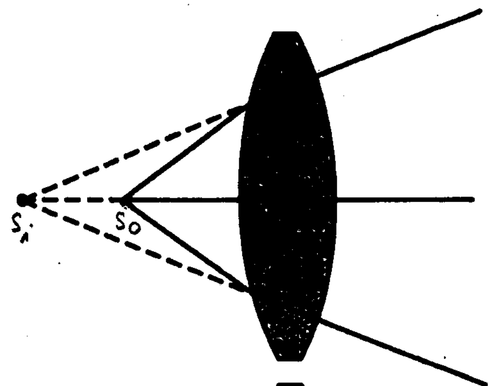
if So to left is positive

$$\frac{1}{S_i} + \frac{1}{S_o} = \frac{1}{f}$$

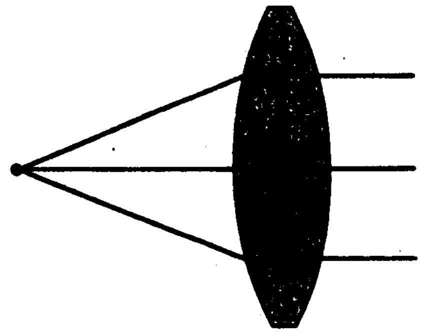
$f > 0$

$s_o < 0$ to left
$|s_o| < f$

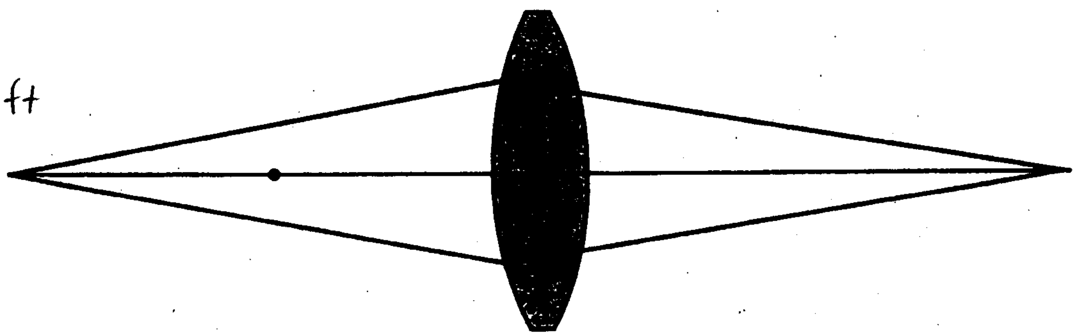
$s_i < 0$ to left



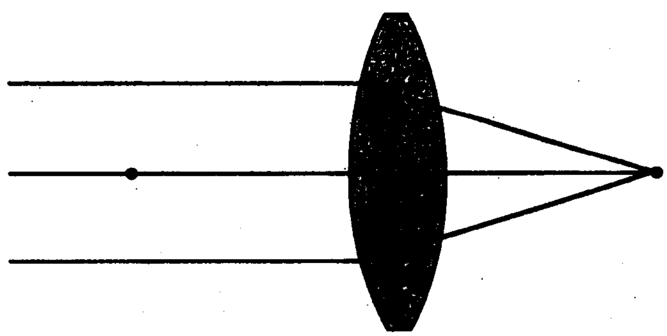
$s_o = -f$
$s_i = \infty$



$s_o < 0$ to left
$|s_o| > f$
$s_i > 0$

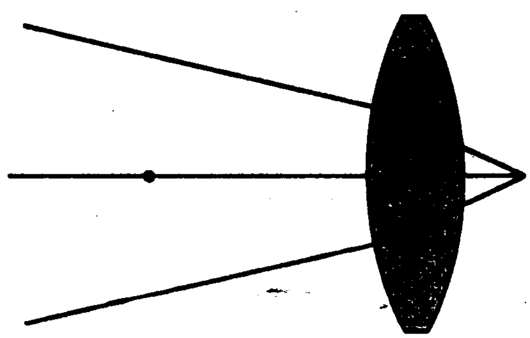


$s_o = -\infty$
$s_i = f$



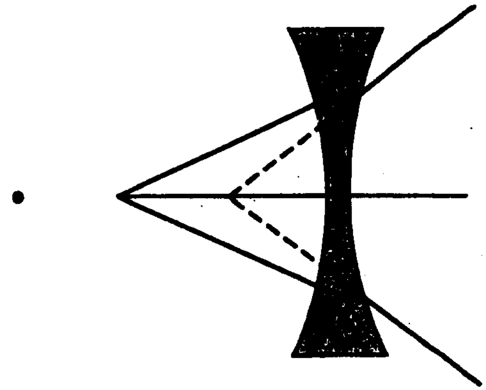
$s_o > 0$ to right

$s_i > 0$

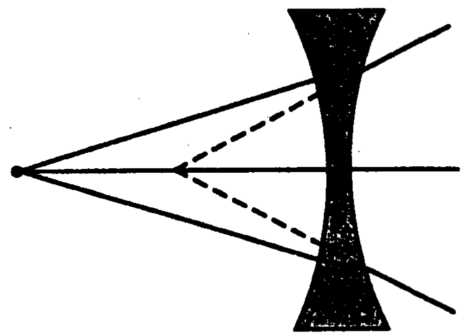


$f < 0$

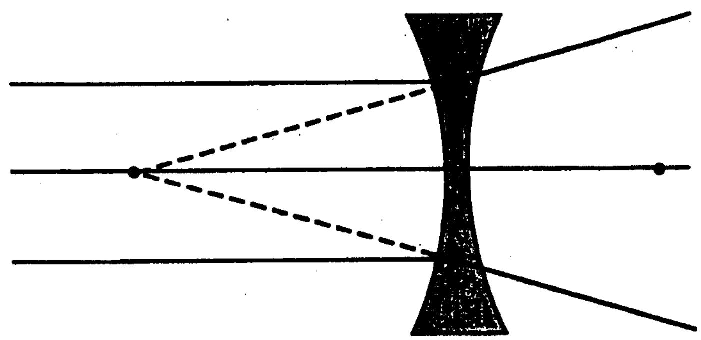
$s_o < 0$
$|s_o| < f$



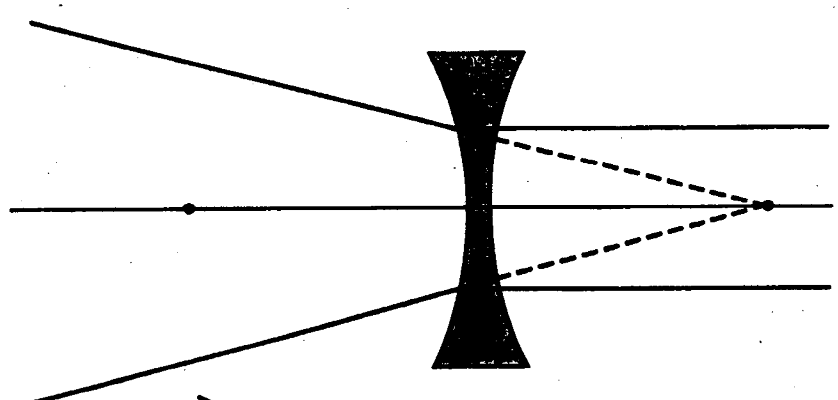
$s_o < 0$



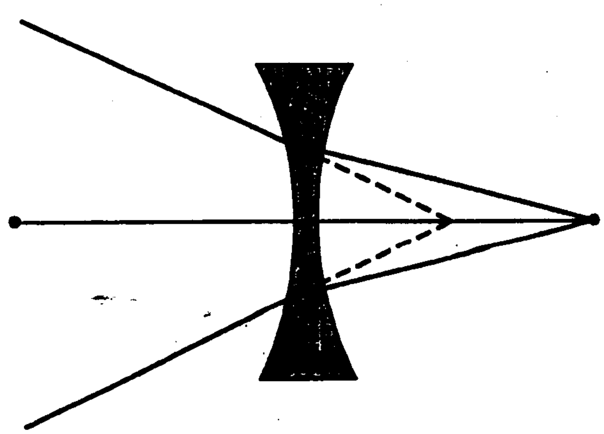
$s_o = \infty$
$s_i = -f$



$s_o = f$
$s_i = \infty$



$s_o > 0$



Now let's look producing an image from an object using a single thin lens.

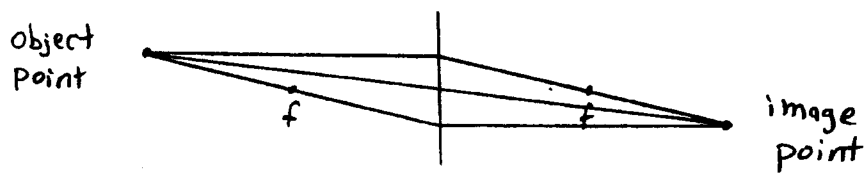
We will analyze the lens and image using the following properties:

(1) A ray passing through the focal point is bent parallel to the axis of symmetry

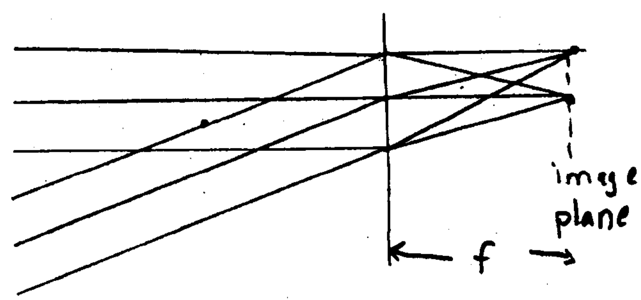
(2) An incident ray parallel to the lens axis is bent to pass through the back focal point

(3) A ray passing through the center of the lens is unchanged in direction.

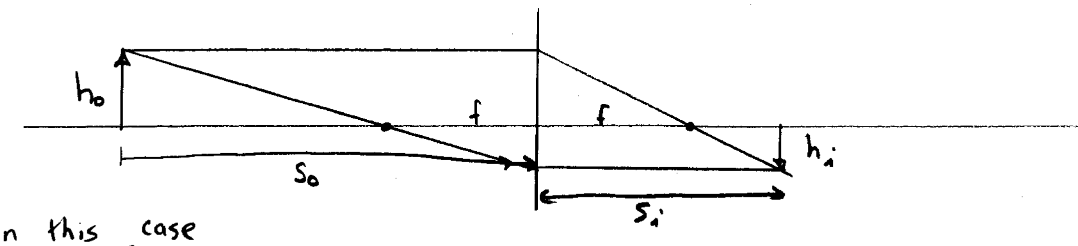
(4) A point in the object space is imaged to a point in the image space.



(5) Parallel rays image to a point
(A lens converts from incident angle to a point)



Let's look at a finite object. We are interested in
(1) Location of the image
(2) Size of the image

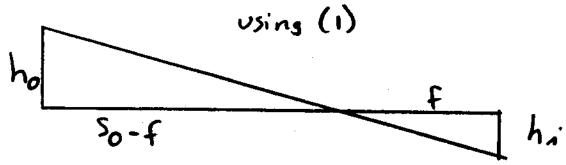


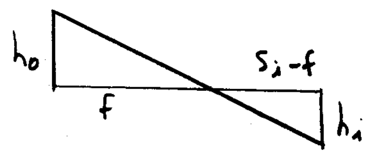
In this case

$S_o < 0$
$S_i > 0$
$h_o > 0$
$h_i < 0$
$f > 0$

USE SIMILAR TRIANGLES

using (1)





$$\frac{h_o}{|S_o| - f} = \frac{|h_i|}{f}$$

$$\frac{h_o}{-S_o - f} = \frac{-h_i}{f}$$

$$\frac{h_i}{h_o} = \frac{f}{f + S_o}$$

$$\frac{h_o}{f} = -\frac{h_i}{S_i - f}$$

$$\frac{h_i}{h_o} = \frac{f - S_i}{f}$$

Equate these 2 Equations

$$\frac{f}{f + S_o} = \frac{f - S_i}{f}$$

$$f^2 = (f + S_o)(f - S_i)$$

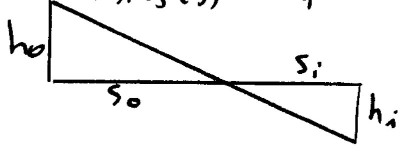
$$\cancel{f^2} = \cancel{f^2} + S_o f - S_i f - S_o S_i$$

$$\left(S_o S_i = S_o f - S_i f \right) \frac{1}{f S_o S_i}$$

$$\frac{1}{f} = \frac{1}{S_i} - \frac{1}{S_o}$$

Same conjugate equation

using (3)



$$\frac{h_o}{|S_o|} = \frac{|h_i|}{S_i}$$

$$M \equiv \frac{h_i}{h_o} = \frac{S_i}{S_o}$$

$M > 0$ erect image
$M < 0$ inverted image

$$\boxed{M = \frac{S_i}{S_o} = \frac{f}{f + S_o} = \frac{f - S_i}{f}}$$

The transverse magnification is defined as

$$M \equiv \frac{h_i}{h_o}$$

$$\boxed{M = +\frac{S_i}{S_o}}$$

A positive M is an erect image

A negative M is an inverted image

Define $x_o \equiv S_o + f$

$$\boxed{M = +\frac{x_o}{f}}$$