**Review**
Fundamental Concepts and Techniques of Calculus

## Detailed Solutions to Selected Exercises:

# Basic Techniques of Algebra

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1. Perform Arithmetic Operations with Fractions:

   (a) We apply the rules for adding, subtraction, multiplying and dividing franctions and obtain step by step that

$$\begin{aligned}\left(\tfrac{5}{2}-\tfrac{1}{2}\right)\div\tfrac{2}{7} &= \tfrac{5-1}{2}\div\tfrac{2}{7}=\tfrac{4}{2}\div\tfrac{2}{7}\\ &= \tfrac{2}{1}\div\tfrac{2}{7}=\tfrac{2}{1}\cdot\tfrac{7}{2}=\tfrac{2\cdot 7}{1\cdot 2}\\ &= \tfrac{14}{2}=7\end{aligned}$$

   (b)

   (c) Here, we observe that the integers $n$ is identified with the fraction $\frac{n}{1}$. Therefore, multiplying or dividing mixed terms can be easily reduced to working with fractions:

$$\begin{aligned}12\div\tfrac{1}{4} &= 12\cdot\tfrac{4}{1}=\tfrac{12}{1}\cdot\tfrac{4}{1}=\tfrac{12\cdot 4}{1\cdot 1}=\tfrac{48}{1}\\ &= 48\end{aligned}$$

   (d)

2. Simplify (and Rationalize) Expressions Involving Exponents and Radicals:

   (a)

   (b)

   (c) Here, we apply the laws of exponent and obtain step by step

$$\begin{aligned}\left(-\tfrac{125}{27}\right)^{-\frac{1}{3}} &= -\left(\tfrac{125}{27}\right)^{-\frac{1}{3}}\\ &= -\left(\tfrac{27}{125}\right)^{\frac{1}{3}}=-\frac{27^{\frac{1}{3}}}{125^{\frac{1}{3}}}=-\tfrac{3}{5}\end{aligned}$$

   (d)

   (e)

   (f)

   (g)

   (h)

   (i) We expand the fraction by $\sqrt{10}$:

$$\begin{aligned}\frac{5}{\sqrt{10}} &= \frac{5}{\sqrt{10}}\cdot 1=\frac{5}{\sqrt{10}}\cdot\frac{\sqrt{10}}{\sqrt{10}}=\frac{5\sqrt{10}}{10}\\ &= \frac{5\sqrt{10}}{10}=\tfrac{1}{2}\sqrt{10}\end{aligned}$$

   (j) We expand the fraction by $10^{2/3}$:

$$\begin{aligned}\frac{5}{\sqrt[3]{10}} &= \frac{5}{10^{1/3}}\cdot\frac{10^{2/3}}{10^{2/3}}=\frac{5\cdot 10^{2/3}}{10^{1/3}\cdot 10^{2/3}}\\ &= \frac{5\cdot 10^{2/3}}{10^{1/3+2/3}}=\frac{5\cdot 10^{2/3}}{10}=\tfrac{1}{2}\sqrt[3]{100}.\end{aligned}$$

   (k) We expand the fraction by $\sqrt{5}-\sqrt{6}$ and use the identity $(a+b)(a-b)=a^2-b^2$:

$$\begin{aligned}\frac{3}{\sqrt{5}+\sqrt{6}} &= \frac{3\big(\sqrt{5}-\sqrt{6}\big)}{\big(\sqrt{5}+\sqrt{6}\big)\big(\sqrt{5}-\sqrt{6}\big)}\\ &= \frac{3\big(\sqrt{5}-\sqrt{6}\big)}{\big(\sqrt{5}\big)^2-\big(\sqrt{6}\big)^2}\\ &= \frac{3\big(\sqrt{5}-\sqrt{6}\big)}{5-6}\\ &= \frac{3\big(\sqrt{5}-\sqrt{6}\big)}{-1}\\ &= 3\big(\sqrt{6}-\sqrt{5}\big)\end{aligned}$$

3. Simplify Expressions Involving Absolute Value:

   (a) Since $-5<0$, we have $|-5|=-(-5)=5$. Moreover, since $5>\sqrt{5}$, it follows that $\frac{1}{5}<\frac{1}{\sqrt{5}}$, since the function $x\mapsto\frac{1}{x}$ is order-reversing. Hence $\frac{1}{5}-\frac{1}{\sqrt{5}}<0$ and thus $\left|\frac{1}{5}-\frac{1}{\sqrt{5}}\right|=-\left(\frac{1}{5}-\frac{1}{\sqrt{5}}\right)=-\frac{1}{5}+\frac{1}{\sqrt{5}}$. Therefore,

$$\begin{aligned}\left|\frac{1}{5}-\frac{1}{\sqrt{5}}\right| &= -\frac{1}{5}+\frac{1}{\sqrt{5}}-5\\ &= \frac{1}{\sqrt{5}}-5\tfrac{1}{5}=\tfrac{1}{5}\sqrt{5}-5\tfrac{1}{5}.\end{aligned}$$

   (b)

   (c)

4. Simplify Expressions Involving Factorials:

   (a) (b) (c)

5. Manipulate Logarithmic Expressions:

   (a)

   (b)

   (c) Recall that the base-2 exponential function $\exp_2$ is the inverse function of $\log_2$. Applying $\exp_2$ on both sides of the equation, we see that $\log_2 4^3 = y \Leftrightarrow 2^y = 4^3 = \big(2^2\big)^3 = 2^6$. Since $\exp_2$ is one-to-one, it follows that $y = 6$.

   (d)

   (e)

   (f) Using the logarithmic rules, we obtain

$$\begin{aligned}&\tfrac{1}{3}\big[\log_8 y + 2\log_8(y+4)\big] - \log_8(y-1)\\ &= \tfrac{1}{3}\big[\log_8 y + \log_8(y+4)^2\big] - \log_8(y-1)\\ &= \tfrac{1}{3}\log_8\big(y(y+4)^2\big) - \log_8(y-1)\\ &= \log_8\Big(y(y+4)^2\Big)^{1/3} - \log_8(y-1)\\ &= \log_8\left(\frac{y^{1/3}(y+4)^{2/3}}{y-1}\right).\end{aligned}$$

   (g) We apply the well-known logarithmic rules step by step and obtain

$$\begin{aligned}&4\ln 6 - 3\ln 2 + 2\ln 3 - \ln 8 =\\ &= 4\ln(2\cdot 3) - 3\ln 2 + 2\ln 3 - \ln 2^3\\ &= 4\big(\ln 2 + \ln 3\big) - 3\ln 2 + 2\ln 3 - 3\ln 2\\ &= 4\ln 2 + 4\ln 3 - 3\ln 2 + 2\ln 3 - 3\ln 2\\ &= (4-3-3)\ln 2 + (4+2)\ln 3\\ &= -2\ln 2 + \ln 3.\end{aligned}$$

   (h) We apply the logarithmic rules and obtain

$$\begin{aligned}&5\ln 6 - \ln 30 + \ln\tfrac{1}{5} =\\ &= 5\ln 6 - \ln(5\cdot 6) - \ln 5\\ &= 5\ln 6 - \big(\ln 5 + \ln 6\big) - \ln 5\\ &= 5\ln 6 - \ln 5 - \ln 6 - \ln 5\\ &= 4\ln 6 - 2\ln 5.\end{aligned}$$

6. Factor a Polynomial and Solve:

   (a) We us the formula $a^2 - b^2 = (a-b)(a+b)$ to factor the polynomial:

$$\begin{aligned}0 &= 9x^2 - 4 = (3x)^2 - 2^2\\ &= (3x-2)(3x+2).\end{aligned}$$

   Hence, $3x-2=0$ or $3x+2=0$ and thus $x = \frac{2}{3}$ or $x = -\frac{2}{3}$.

   (b)

   (c) We use the formulae $a^2 + 2ab + b^2 = (a+b)^2$ and $a^2 - b^2 = (a-b)(a+b)$ to factor the polynomial:

$$\begin{aligned}0 &= 4x^4 + 4x^2 + 1\\ &= \big(2x^2\big)^2 + 2(2x^2)\cdot 1 + 1^2\\ &= \big(2x^2+1\big)^2 = \big(2(x^2+\tfrac{1}{2})\big)^2\\ &= 4(x^2+\tfrac{1}{2})^2\\ &= 4\big(x + \tfrac{i}{2}\sqrt{2}\big)^2\big(x - \tfrac{i}{2}\sqrt{2}\big)^2.\end{aligned}$$

   Hence, $x = -\frac{i}{2}\sqrt{2}$ or $x = \frac{i}{2}\sqrt{2}$ are double roots of the given polynomial.

   (d)

   (e)

7. Complete the Square and find the Vertex:

   (a)

   (b)

$$\begin{aligned}y &= 3x^2 - 6x + 12 = 3(x^2 - 2x) + 12\\ &= 3\big((x-1)^2 - (-1)^2\big) + 12\\ &= 3(x-1)^2 - 3 + 12\\ &= 3(x-1)^2 + 9.\end{aligned}$$

   The vertex of a parabola is its local extremum, which is clearly attained when the term $(x-1)^2$ has its minimum, hence $V = (1,9)$.

   (c)

$$\begin{aligned}y &= 2x^2 + 20x + 3 = 2(x^2 + 10x) + 3\\ &= 2\big((x+5)^5 - 25\big) + 3\\ &= 2(x+5)^2 - 47,\end{aligned}$$

   hence the vertex is $V = (-5,-47)$.

8. Solving Quadratic Equations by Factoring or Using the Quadratic Formula:

   (a)

   (b) We use the formula $x^2 + (a+b)x + ab = (x+a)(x+b)$ to factor the polynomial:

$$0 = x^2 - x - 2 = (x-2)(x+1),$$

   hence, $x = 2$ or $x = -1$.

(c) We use the formula $\alpha_1\alpha_2 x^2 + (\alpha_1\beta_2 + \alpha_2\beta_1)x + \beta_1\beta_2 = (\alpha_1 x + \beta_1)(\alpha_2 x + \beta_2)$ to factor the polynomial:

$$0 = 2x^2 - 5x - 3 = (2x+1)(x-3),$$

hence, $2x+1=0$ or $x-3=0$ and thus $x = -\frac{1}{2}$ or $x = 3$.

(d)

(e) Using the Binomial formula, we can easily factor the given polynomial

$$0 = x^2 - 6x + 9 = (x-3)^2,$$

which implies that $x_1 = x_2 = 3$.

(f) We use the "quadratic formula" to solve the quadratic equations, since it does not have a rational zero:

$$\begin{aligned} x &= \frac{-(-2) \pm \sqrt{(-2)^2 - 4 \cdot 1 \cdot 2}}{2 \cdot 1} \\ &= \frac{2 \pm \sqrt{-4}}{2} = \frac{2 \pm 2i}{2} = 1 \pm i. \end{aligned}$$

(g)

9. Solving Inequalities:

(a)

(b)

(c)

(d)

(e) We consider the left side of the inequality as a rational function and set

$$\begin{aligned} r(x) &= \frac{1}{x-1} + \frac{4}{x-6} = \frac{5x-10}{(x-1)(x-6)} \\ &= \frac{5(x-2)}{(x-1)(x-6)} \end{aligned}$$

Clearly, the domain of $f$ is $D_r := \mathbb{R} \setminus \{1, 6\}$. As a rational function $r$ is continuous on its domain $D_r$ and by the Intermediate-value Theorem of Calculus can only change its sign (change from positive values to negative or vice versa) at its only zero $x_0 = 2$ or at its singularities $x_1 = 1$ and $x_2 = 6$.

In order to determine the set $S$ of all such points $x \in D_r$ for which $r(x) > 0$, it suffices to determine the sign of $r$ at one "test point" of each of the subintervals $(-\infty, 1)$, $(1, 2)$, $(2, 6)$ and $(6, \infty)$ of $D_r$ obtained by removing the zeroes of the numberator and denominator of $r$ from $\mathbb{R}$. We choose the points $0, \frac{3}{2}, 3, 7$ as test points on those intervals and find (please verify!) that $r(0) < 0$, $r\left(\frac{3}{2}\right) > 0$, $r(3) < 0$ and $r(7) > 0$. Hence, we conclude that

$$r(x) \begin{cases} < 0, & x \in (-\infty, 1) \\ > 0, & x \in (1, 2) \\ < 0, & x \in (2, 6) \\ > 0, & x \in (6, \infty). \end{cases}$$

Thus, the solution set of the given inequality is $S := (1, 2) \cup (6, \infty)$.

(f)

10. Solving Inequalities Involving Absolute Values:

(a)

(b)

(c) We are asked to find all $x \in \mathbb{R}$ that satisfy the inequality $7 < |8x - 72|$. First note that

$$\begin{aligned} |8x - 72| &= \begin{cases} 8x - 72, & 8x - 72 \geq 0 \\ -(8x - 72), & 8x - 72 < 0 \end{cases} \\ &= \begin{cases} 8x - 72, & x \geq 9 \\ -8x + 72, & x < 9. \end{cases} \end{aligned}$$

We will therefore solve the inequality on the subdomains $D_1 := [9, \infty)$ and $D_2 := (-\infty, 9)$, where $|8x - 72|$ can be expressed without $|\cdot|$, separately:

(i) First, let $x \in D_1 = [9, \infty)$, then $|8x - 72| = 8x - 72$ and

$$7 < |8x - 72|$$

is equivalent to the following inequalities

$$\begin{aligned} 7 &< 8x - 72 \\ 79 &< 8x \\ \tfrac{79}{8} &< x, \end{aligned}$$

which implies that the solution set of the given inequality on $D_1$ is $S_1 := (\frac{79}{8}, \infty)$.

(ii) Next, let $x \in D_2 = (-\infty, 9)$, then $|8x - 72| = -8x + 72$ and similar as in the preceeding case

$$7 < |8x - 72|$$

is equivalent to the inequalities

$$\begin{aligned} 7 &< -8x + 72 \\ 8x &< 65 \\ x &< \tfrac{65}{8}, \end{aligned}$$

from which follows that the solution set of the given inequality on $D_2$ is $S_2 := (-\infty, \frac{65}{8})$.

The solution set of the inequality $7 < |8x - 72|$ on $\mathbb{R}$ is therefore the union of the solution sets on the subdomains, i.e.

$$S := S_1 \cup S_2 = (\tfrac{79}{8}, \infty) \cup (-\infty, \tfrac{65}{8}).$$

(d) We are to determine all $x \in \mathbb{R}$ satisfying the inequality $|x-3| - |x+1| \le x$. To this end, we first observe that

$$|x-3| = \begin{cases} x-3, & x-3 \ge 0 \\ -(x-3), & x-3 < 0 \end{cases}$$
$$= \begin{cases} x-3, & x \ge 3 \\ -x+3, & x < 3 \end{cases}$$

and

$$|x+1| = \begin{cases} x+1, & x+1 \ge 0 \\ -(x+1), & x+1 < 0 \end{cases}$$
$$= \begin{cases} x+1, & x \ge -1 \\ -x-1, & x < -1 \end{cases}$$

Therefore, we will subdivide the domain $\mathbb{R}$ into the subregions $D_1; = [3, \infty)$, $D_2 := [-1, 3)$ and $D_3 := (-\infty, -1)$ on which the terms $|x-3|$ and $|x+1|$ can be simultaneously replaced by absolute value free expressions. On these subdomains, the given inequality can be rewritten without absolute value expressions:

(i) Let $x \in D_1 = [3, \infty)$, then $|x-3| = x-3$ and $|x+1| = x+1$ and

$$|x-3| - |x+1| \le x$$

in turn is equivalent to

$$x - 3 - (x+1) \le x$$
$$x - 3 - x - 1 \le x$$

and thus to

$$-4 \le x,$$

which implies that the solution set of the given inequality on $D_1$ is $S_1 := D_1 = [3, \infty)$.

(ii) Next, let $x \in D_2 = [-1, 3)$, then $|x-3| = -x+3$ and $|x+1| = x+1$ and

$$|x-3| - |x+1| \le x$$

is equivalent to the following inequalities

$$-x + 3 - (x+1) \le x$$
$$-2x + 2 \le x$$
$$x \le 3x - \tfrac{2}{3} \qquad \le x,$$

from which follows that the solution set of the given inequality on $D_2$ is $S_2 := [\frac{2}{3}, 3)$.

(iii) Finally, let $x \in D_3 = (-\infty, -1)$, then $|x-3| = -x+3$ and $|x+1| = -x-1$. Then

$$|x-3| - |x+1| \le x$$

is again equivalent to the following inequalities

$$-x + 3 - (-x-1) \le x$$
$$-x + 3 + x + 1 \le x$$
$$5 \le x,$$

from which follows that the solution set of the given inequality on $D_3$ is empty, i.e. $S_3 := \emptyset$.

The solution set of $|x-3| - |x+1| \le x$ on $\mathbb{R}$ is therefore, the union

$$S = S_1 \cup S_2 \cup S_3$$
$$= [3, \infty) \cup [\tfrac{2}{3}, 3) \cup \emptyset$$
$$= [\tfrac{2}{3}, \infty)$$

11. Solve Exponential and Logarithmic Equations:

(a) Since the (base-2) exponential function is one-to-one, the identity $2^{x^2+3x} = 1024 = 2^{10}$ implies

$$10 = x^2 + 3x$$

and therefore

$$0 = x^2 + 3x - 10 = (x+5)(x-2),$$

hence $x = -5$ or $x = 2$.

(b) Clearly, the exponential equation

$$6\big(2^{3x-1}\big) - 7 = 9$$

is equivalent to

$$2^{3x-1} = \frac{16}{2 \cdot 3} = \frac{2^3}{3}$$

and thus to

$$2^{3x-4} = \frac{1}{3}.$$

Applying the logarithmic function $\log_2$ to this equations first yields

$$3x - 4 = -\log_2 3$$

and then

$$x = \tfrac{1}{3}\big(4 - \log_2 3\big).$$

(c) The exponential equation

$$0 = e^{2x} - 4e^x - 5$$

can be reduced to a quadratic equation by the substitution $y := e^x$:

$$0 = y^2 - 4y - 5 = (y-5)(y+1).$$

The solutions of the quadratic equation are $y = 5$ and $y = -1$. We have to resubstitute the permissible values of $y$ back into the equation $y = e^x$. The first solution $y = 5$, produces the equation $5 = e^x$ which implies that $x = \ln 5$, while the second solution $y = -1$ yields the equation $-1 = e^x$ which has the complex solution $x = i\pi$.

(d)

(e)

(f)

(g)

(h) First, we observe that the given equation has domain $(0, \infty)$. Thus, if $x \in (0, \infty)$, we can conclude using the logarithmic rules that

$$\begin{aligned} 1 &= 2\ln(x+2) - \frac{1}{2}\ln x^4 \\ &= 2\ln(x+2) - 2\ln x \\ &= 2\big(\ln(x+2) - \ln x\big) \\ &= 2\ln\left(\frac{x+2}{x}\right). \end{aligned}$$

Dividing the equation by 2, yields

$$\frac{1}{2} = \ln\left(\frac{x+2}{x}\right)$$

Applying the natural exponential function exp (which is the inverse function of ln) on both sides of the equation yields

$$e^{\frac{1}{2}} = \exp\left(\ln\left(\frac{x+2}{x}\right)\right) = \frac{x+2}{x}$$

from which by multiplying the equation by $x$ first follows that

$$x\sqrt{e} = x + 2$$

and then

$$x\big(\sqrt{e} - 1\big) = 2,$$

hence,

$$x = \frac{2}{\sqrt{e} - 1}.$$

(i)

(j)

12. Find the Equation of a Line:

(a)

(b)

(c)

(d)

(e)

13. Find the Distance and Midpoint between Two Points:

(a)

(b)

(c)

14. Graph the Quadratic Equation (Conic):

(a) We transform the equation

$$x^2 - 6x + 2y + 9 = 0$$

into normal form by "completing the square"

$$\begin{aligned} (x-3)^2 - 9 + 2y + 9 &= 0 \\ (x-3)^2 + 2y &= 0, \end{aligned}$$

hence, $y = -\frac{1}{2}(x-3)^2$, which is the equation of a parabola opened downwardly with vertex at $(3, 0)$.

(b) We transform the equation

$$16x^2 + 16y^2 - 16x + 24y - 3 = 0$$

into normal form by "completing the square"

$$\begin{aligned} 16(x^2 - x) + 16(y^2 + \frac{3}{2}) &= 3 \\ 16\big((x - \tfrac{1}{2})^2 - \tfrac{1}{4}\big) + 16\big((y + \tfrac{3}{4})^2 - \tfrac{9}{16}\big) &= 3 \\ 16\big(x - \tfrac{1}{2}\big)^2 + 16\big(y + \tfrac{3}{4}\big)^2 &= 16. \end{aligned}$$

Division by 16 yields

$$\big(x - \tfrac{1}{2}\big)^2 + \big(y + \tfrac{3}{4}\big)^2 = 1,$$

which is the equation of a circle of radius 1 and center $\big(\frac{1}{2}, -\frac{3}{4}\big)$.

(c) We transform the equation

$$x^2 + y^2 - 2x - 31 = 0$$

into normal form by "completing the square"

$$\begin{aligned}(x^2 - 2x) + y^2 - 31 &= 0\\ (x-1)^2 - 1 + y^2 - 31 &= 0\\ (x-1)^2 + y^2 &= 32,\end{aligned}$$

which is the equation of a circle of radius $4\sqrt{2}$ and center $(1, 0)$.

(d) We first transform the equation

$$-9x^2 - 24x + 4y^2 - 8y + 16 = 0$$

into normal form by "completing the square"

$$\begin{aligned}-9(x^2 + \tfrac{8}{3}x) + 4(y^2 - 2) &= -16\\ -9\big((x + \tfrac{4}{3})^2 - \tfrac{16}{9}\big) + 4\big((y-1)^2 - 1\big)\big) &= -16\\ -9\big(x + \tfrac{4}{3}\big)^2 + 16 + 4(y-1)^2 - 4 &= -16\\ -9\big(x + \tfrac{4}{3}\big)^2 + 4(y-1)^2 &= -28\\ \frac{\big(x + \frac{4}{3}\big)^2}{\frac{28}{9}} - \frac{(y-1)^2}{7} &= 1.\end{aligned}$$

Clearly,

$$\frac{\big(x + \frac{4}{3}\big)^2}{\frac{28}{9}} - \frac{(y-1)^2}{7} = 1. \qquad (1)$$

is the equation of a hyperbola obtained from the hyperbola

$$\frac{x^2}{\frac{28}{9}} - \frac{y^2}{7} = 1, \qquad (2)$$

which is in standard position, by a shift by the vector $\big(-\frac{4}{3}, 1\big)$.

We will first determine the asymptotes, vertices, center and foci of the hyperbola (2). Recall that the asymptotes of (2) can be easily determined from the modified equation

$$\frac{x^2}{\frac{28}{9}} - \frac{y^2}{7} = 0$$

by solving for $y$. Clearly, $y^2 = \frac{7 \cdot 9}{28}x^2 = \frac{9}{4}x^2$. Therefore, the asymptotes of the hyperbola (2) are given by the equations

$$y = \pm\tfrac{3}{2}x.$$

Moreover, the foci $(\pm c, 0)$ can be determined from the equation $c^2 = a^2 + b^2$, where $a^2 = \frac{28}{9}$ and $b^2 = 7$. Since $c = \sqrt{\frac{28}{9} + 7} = \frac{1}{3}\sqrt{28 + 63} = \frac{1}{3}\sqrt{91}$, the foci of (2) are given by

$$F_1 = \big(-\tfrac{1}{3}\sqrt{91}, 0\big), \quad F_2 = \big(\tfrac{1}{3}\sqrt{91}, 0\big).$$

The vertices $(\pm a, 0)$ are obtained from $a = \sqrt{\frac{28}{9}} = \frac{2}{3}\sqrt{7}$ and are

$$V_1 = \big(-\tfrac{2}{3}\sqrt{7}, 0\big), \quad V_2 = \big(\tfrac{2}{3}\sqrt{7}, 0\big).$$

Clearly the center $C$ of the hyperbola (2) is at $(0, 0)$ and constitutes the midpoint of the transverse axis determined by the foci $F_1$ and $F_2$.

In order to find the asymptotes, foci, vertices and center of the hyperbola (1), we only have to shift the corresponding equations or points, respectively.

Thus the asymptotes of (1) are

$$y - 1 = \pm\frac{3}{2}\big(x + \frac{4}{3}\big),$$

the foci

$$\begin{aligned}\tilde{F}_1 &= \big(-\tfrac{1}{3}\sqrt{91} - \tfrac{4}{3}, 0 + 1\big)\\ &= \big(-\tfrac{4}{3} - \tfrac{1}{3}\sqrt{91}, 1\big)\\ \tilde{F}_2 &= \big(\tfrac{1}{3}\sqrt{91} - \tfrac{4}{3}, 0 + 1\big)\\ &= \big(-\tfrac{4}{3} + \tfrac{1}{3}\sqrt{91}, 1\big)\end{aligned}$$

the vertices

$$\begin{aligned}\tilde{V}_1 &= \big(-\tfrac{2}{3}\sqrt{7} - \tfrac{4}{3}, 0 + 1\big)\\ &= \big(-\tfrac{4}{3} - \tfrac{2}{3}\sqrt{7}, 1\big)\\ \tilde{V}_2 &= \big(\tfrac{2}{3}\sqrt{7} - \tfrac{4}{3}, 0 + 1\big)\\ &= \big(-\tfrac{4}{3} + \tfrac{2}{3}\sqrt{7}, 1\big)\end{aligned}$$

and the center

$$\tilde{C} = \big(0 - \tfrac{4}{3}, 0 + 1\big) = \big(-\tfrac{4}{3}, 1\big).$$

15. Find the Composition of Two Functions:

    (a)

    (b)

16. Find the Inverse of a Function:

    (a)

    (b)

    (c)

(d) Clearly, the function $f:\mathbb{R}\to\mathbb{R}$ defined by

$$f(x)=x^2+9x+5=\left(x+\tfrac{9}{2}\right)^2-\tfrac{29}{4}$$

(note that its graph is a parabola!) ist not injective (one-to-one), since, for example,

$$\begin{aligned}f\left(\tfrac{1}{2}\right)&=\left(\tfrac{1}{2}+\tfrac{9}{2}\right)^2-\tfrac{29}{4}\\&=5^2-\tfrac{29}{4}=(-5)^2-\tfrac{29}{4}\\&=\left(-\tfrac{19}{2}+\tfrac{9}{2}\right)^2-\tfrac{29}{4}=f\left(-\tfrac{19}{2}\right).\end{aligned}$$

Therefore, $f$ is not invertible on $\mathbb{R}$. However, we can, for example, restrict it to the subinterval $[-\frac{9}{2},\infty)$. This restriction, which we will also denote by $f$, constitutes a bijection $f:[-\frac{9}{2},\infty)\to[-\frac{29}{4},\infty)$ as can be easily verified. We will determine its inverse: To this end, first set $y=\left(x+\frac{9}{2}\right)^2-\frac{29}{4}$. Next interchange $x$ and $y$, which leads to the equation $x=\left(y+\frac{9}{2}\right)^2-\frac{29}{4}$. Then solve for $y$: Taking the square root of

$$\left(y+\tfrac{9}{2}\right)^2=x+\tfrac{29}{4}$$

yields

$$\left|y+\tfrac{9}{2}\right|=\sqrt{x+\tfrac{29}{4}}$$

and

$$y+\tfrac{9}{2}=\sqrt{x+\tfrac{29}{4}}$$

since $y\in[-\frac{9}{2},\infty)$, implies $\left|y+\frac{9}{2}\right|=y+\frac{9}{2}$. Thus

$$y=-\tfrac{9}{2}+\sqrt{x+\tfrac{29}{4}}.$$

Hence, the inverse function

$$f^{-1}:[-\tfrac{29}{4},\infty)\to[-\tfrac{9}{2},\infty)$$

is given by

$$f^{-1}(x)=-\tfrac{9}{2}+\sqrt{x+\tfrac{29}{4}}$$

for $x\in[-\frac{29}{4},\infty)$.
We conclude by showing that $f^{-1}$ is indeed inverse to $f$ on $[-\frac{9}{2},\infty)$: Let $x\in[-\frac{9}{2},\infty)$ then

$$\begin{aligned}f^{-1}\big(f(x)\big)&=f^{-1}\left(\left(x+\tfrac{9}{2}\right)^2-\tfrac{29}{4}\right)\\&=-\tfrac{9}{2}+\sqrt{\left(x+\tfrac{9}{2}\right)^2-\tfrac{29}{4}+\tfrac{29}{4}}\\&=-\tfrac{9}{2}+\left|x+\tfrac{9}{2}\right|\\&=-\tfrac{9}{2}+x+\tfrac{9}{2}\\&=x.\end{aligned}$$

Conversely, let $x\in[-\frac{29}{4},\infty)$ then

$$\begin{aligned}f\big(f^{-1}(x)\big)&=f\left(-\tfrac{9}{2}+\sqrt{x+\tfrac{29}{4}}\right)\\&=\left(-\tfrac{9}{2}+\sqrt{x+\tfrac{29}{4}}+\tfrac{9}{2}\right)^2-\tfrac{29}{4}\\&=x+\tfrac{29}{4}-\tfrac{29}{4}\\&=x.\end{aligned}$$

(e)

17. Find the Domain and Range of a Function:

(a)

(b)

(c)

(d)

(e) (i) Domain of $f$: The domain of $f$ is the largest subset of $\mathbb{R}$ for which

$$\begin{aligned}f(x)&=\frac{\sqrt{9-x^2}}{x^2-3x+2}=\frac{\sqrt{9-x^2}}{(x-2)(x-1)}\\&=\frac{\sqrt{9-x^2}}{\left(x-\frac{3}{2}\right)^2-\frac{1}{4}}\end{aligned}$$

is well-defined. Therefore, we have to eliminate those $x\in\mathbb{R}$ for which $0=x^2-3x+2=(x-1)(x-2)$, i.e. $x=1$ and $x=2$, and confine ourselves to those $x\in\mathbb{R}$ for which $0\le 9-x^2=(3-x)(3+x)$, i.e. $x\in[-3,3]$. Thus

$$D_f=[-3,3]\setminus\{1,2\}.$$

(ii) Range of $f$: To determine the range of $f$, we will first compute the range of its restrictions to each of the subdomains given by the intervals $[-3,1)$, $(1,2)$ and $(2,3]$. Note that the *only* zeroes of $f$ are the boundary points $-3$, and 3 and also note that $f$ is continuous on its domain. We can therefore apply the Intermediate-value Theorem to each closed subinterval of its domain:

We begin with the subinterval $[-3,1)$. Since $f(-3)=0$, $f(x)\ge 0$ for all $x\in[-3,1)$ and $\lim_{x\to 1^-}f(x)=\infty$, as can be easily verified, we conclude using the Intermediate-value Theorem that the range of the restriction of $f$ to the interval $[-3,1)$ is $[0,\infty)$.

We will now consider the behavior of $f$ on the subinterval $(1,2)$. Clearly, $\lim_{x\to 1^+}=\lim_{x\to 2^-}=-\infty$ and $f(x)<0$ for all $x\in(1,2)$. It can be easily verified that $f$ attains its global maximum on the subinterval $(1,2)$ where the denominator of $f$ which is given by the parabola $\left(x-\frac{3}{2}\right)^2-\frac{1}{4}$ has its smallest value which is precisely at $\frac{3}{2}$ where the parabola has its vertex. The corresponding function value of $f$ is

$$f\left(\tfrac{3}{2}\right)=\frac{\sqrt{9-\left(\frac{3}{2}\right)^2}}{-\frac{1}{4}}=-6\sqrt{3}.$$

Hence, again by the Intermediate-value Theorem, the range of the restriction of $f$ to the interval $(1,2)$ is given by the interval $[-6\sqrt{3},-\infty)$.
Finally, consider the interval $(2,3]$. Clearly, $f(x)\geq 0$ for all $x\in(2,3]$, $f(3)=0$ and $\lim_{x\to 2^+}=\infty$, as can be easily verified. Hence, again by the Intermediate-value Theorem, the range of the restriction of $f$ to the interval $(2,3]$ is $[0,\infty)$.
The range of $f$ on its total domain $D_f=[-3,1)\cup(1,2)\cup(2,3]$ is therefore the union of the ranges of the restrictions of $f$. Hence

$$\begin{aligned}\operatorname{range}(f)&=[0,\infty)\cup[-6\sqrt{3},-\infty)\cup[0,\infty)\\&=[-6\sqrt{3},-\infty)\cup[0,\infty).\end{aligned}$$

(f)

18. Determine Whether a Function is Even, Odd or Neither:

(a)

(b) We compute $f(-x)$ and check whether it equals $f(x)$ or $-f(x)$ or neither:

$$\begin{aligned}f(-x)&=(-x)\sqrt{1-(-x)^3}\\&=\left(-x\sqrt{1-x^2}\right)\\&=-f(x)\end{aligned}$$

Since, $f(-x)=-f(x)$ for all $x\in\mathbb{R}$, $f$ is odd on $\mathbb{R}$.

(c)

(d)

(e) We compute $f(-x)$ and check whether it equals $f(x)$ or $-f(x)$ or neither:

$$\begin{aligned}f(-x)&=\frac{3(-x)^3-7(-x)}{(-x)^4-2(-x)^2-7}\\&=\frac{-3x^3+7}{x^4-2x^2-7}\\&=-\frac{3x^3-7}{x^4-2x^2-7}\\&=-f(x).\end{aligned}$$

Since $f(-x)=-f(x)$ for all $x\in\mathbb{R}$, $f$ is odd on $\mathbb{R}$.

(f)

(g) We compute $f(-x)$ and check whether it equals $f(x)$ or $-f(x)$ or neither:

$$f(-x)=(-x)^3-5=-x^3-5$$

which is clearly not equal to either $f(x)$ or $-f(x)$ for all $x\in\mathbb{R}$: Choose for example $x=1$. Then, $f(-1)=-6$ while $f(1)=-4$. Hence, $f$ is neither odd nor even on $\mathbb{R}$.

(h) We compute $f(-x)$ and check whether it equals $f(x)$ or $-f(x)$ or neither:

$$\begin{aligned}f(-x)&=1+(-x)^2\cos\big(2(-x)\big)+(-x)\\&=1+x^2\cos(2x)-x\end{aligned}$$

Again, we see that $f(-x)$ is not equal to either $f(x)$ or $f(-x)$ for all $x\in\mathbb{R}$: Choose, for instance, $x=\frac{\pi}{4}$, then $f\left(-\frac{\pi}{4}\right)=1-\frac{\pi}{4}$, while $f\left(\frac{\pi}{4}\right)=1+\frac{\pi}{4}$. Hence $f$ is neither odd nor even on $\mathbb{R}$.

19. Graph Exponential and Logarithmic Functions:

(a)

(b)

(c)

20. Manipulate Complex Numbers:

(a)

(b)

(c) We expand the fraction by the conjugate of the doniminator $1-2i$:

$$\begin{aligned}\frac{6-7i}{1-2i}&=\frac{6-7i}{1-2i}\cdot\frac{1+2i}{1+2i}\\&=\frac{6\cdot 1+(-7i)(2i)+6\cdot 2i+(-7i)1}{1-(2i)^2}\\&=\frac{6+14+i(12-7)}{1+4}\\&=\frac{20+5i}{5}=4+i.\end{aligned}$$

(d) We first simplify the fraction and then expand by the conjugate of the denominator:

$$\begin{aligned}\frac{-3-i}{-1-2i}&=\frac{-(3+i)}{-(1+2i)}=\frac{3+i}{1+2i}\\&=\frac{3+i}{1+2i}\cdot\frac{1-2i}{1-2i}\\&=\frac{3+2+i(-6+1)}{1^2-(2i)^2}\\&=\frac{5-5i}{5}=1-i.\end{aligned}$$

(e) We first add the two fractions and then transform the sum into standard form by expanding by the conjugate of its denominator:

$$\begin{aligned}\frac{i}{3-2i}+\frac{2i}{3+8i} &= \frac{i(3+8i)+2i(3-2i)}{(3-2i)(3+8i)}\\ &= \frac{3i-8+6i+4}{9+16+i(24-6)}\\ &= \frac{-4+9i}{25+18i}\\ &= \frac{-4+9i}{25+18i}\cdot\frac{25-18i}{25-18i}\\ &= \frac{-100+162+i(72+225)}{625+324}\\ &= \frac{62+297i}{949}\\ &= \frac{62}{949}+\frac{297}{949}\,i\end{aligned}$$

21. Finding Powers and Roots of Complex Numbers Using D'Moivres Theorem:

(a)

(b) We will first transform $-1-\sqrt{3}\,i$ into polar form. Since the point represented by the complex number $-1-\sqrt{3}\,i$ is located in the 4th quadrant, we can compute its argument by

$$\begin{aligned}\arg(-1-\sqrt{3}\,i) &= \arctan\left(\frac{-\sqrt{3}}{-1}\right)\\ &= \arctan\sqrt{3} = \tfrac{\pi}{3}.\end{aligned}$$

Its modulus or absolute value is

$$|-1-\sqrt{3}\,i| = \sqrt{(-1)^2+\left(-\sqrt{3}\right)^2} = 2.$$

Therefore, $-1-\sqrt{3}\,i$ has the polar representation

$$-1-\sqrt{3}\,i = 2e^{\frac{\pi}{3}i}$$

and by d'Moivres Theorem

$$\begin{aligned}\left(-1-\sqrt{3}\,i\right)^3 &= \left(2e^{\frac{\pi}{3}i}\right)^3 = 2^3e^{3\cdot\frac{\pi}{3}i}\\ &= 8e^{\pi i} = 8\big(\cos\pi + i\sin\pi\big)\\ &= 8\big((-1)+0i\big) = -8.\end{aligned}$$

(c) Recall that if $n=4k+r$ with $0\le r<4$ then

$$\begin{aligned}i^n &= i^{4k+r} = i^{4k}i^r = \left(i^4\right)^k i^r = 1^k i^r\\ &= i^r,\end{aligned}$$

and thus

$$i^n = \begin{cases}1 & n\equiv 0 \pmod 4\\ i & n\equiv 1 \pmod 4\\ -1 & n\equiv 2 \pmod 4\\ -i & n\equiv 3 \pmod 4).\end{cases}$$

In particular, since $39 = 4\cdot 9+3$, we have

$$i^{39} = i^3 = -i.$$

(d) We first transform $16+12i$ into polar form. Since the point corresponding to the complex number $16+12i$ is located in the 1st quadrant, its argument is given by

$$\begin{aligned}\varphi = \arg(16+12i) &= \arctan\left(\tfrac{12}{16}\right)\\ &= \arctan\left(\tfrac{3}{4}\right).\end{aligned}$$

Its modulus or absolute value is

$$\begin{aligned}|16+12i| &= 4|4+3i| = 4\sqrt{4^2+3^2}\\ &= 4\sqrt{25} = 20.\end{aligned}$$

Thus

$$16+12i = 20e^{\varphi i}.$$

By d'Moivres Theorem, the two square roots of $16+12i$ are given by

$$z_0 = \sqrt{20}e^{\frac{\varphi}{2}i} = 2\sqrt{5}e^{\frac{\varphi}{2}i}$$

and

$$z_0 = \sqrt{20}e^{\frac{\varphi}{2}i+\frac{2\pi}{2}} = 2\sqrt{5}e^{\frac{\varphi}{2}i+\pi}.$$

(e) Since $64 = 2^6$ is a real number, its argument $\arg(64)=0$. Thus its polar representation is

$$64 = 2^6e^{0i}.$$

By d'Moivres Theorem, the six roots of $64$ are given by the equations

$$z_k = 2e^{\frac{0i}{6}+\frac{2\pi ki}{6}} = 2\left(\cos\tfrac{\pi k}{3} + i\sin\tfrac{\pi k}{3}\right)$$

for $k=0,1,\ldots,5$

Explicitely

$$\begin{aligned}z_0 &= 2\\ z_1 &= 2\left(\cos\tfrac{\pi}{3}+i\sin\tfrac{\pi}{3}\right)\\ &= 2\left(\tfrac{1}{2}+\tfrac{i}{2}\sqrt{3}\right) = 1+i\sqrt{3}\\ z_3 &= 2\left(\cos\tfrac{2\pi}{3}+i\sin\tfrac{2\pi}{3}\right)\\ &= 2\left(-\tfrac{1}{2}+\tfrac{i}{2}\sqrt{3}\right) = -1+i\sqrt{3}\\ z_3 &= -2\\ z_4 &= -1-i\sqrt{3}\\ z_5 &= 1-i\sqrt{3}.\end{aligned}$$

Note that the six roots of 64 are equally spaced around the circle of radius 2 starting with the angle $0$ with counterclockwise increments of $\pi/3$.