Review
Fundamental Concepts and Techniques of Calculus

Detailed Solutions to Selected Exercises:
Trigonometry

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1. Apply the Basic Trigonometric Identities to Simplify Expressions:

   (a) We will use the Double-Angle Identity

   $$\sin(2x) = 2\sin x\cos x$$

   and the Power Identity

   $$\cos^2 x = \tfrac{1}{2}\big(1+\cos 2x\big)$$

   and obtain

   $$\begin{aligned}\sin x\cos x + \cos^2 x &= \\ &= \tfrac{1}{2}\sin(2x) + \tfrac{1}{2} + \tfrac{1}{2}\cos(2x)\\ &= \tfrac{1}{2}\big(1+\cos(2x)+\sin(2x)\big).\end{aligned}$$

   (b) We will factor out $2\sin x\cos x$ and apply the Double Angle Identities several times:

   $$\begin{aligned}2\sin x\cos^3 x - 2\sin^3 x\cos x &= \\ &= 2\sin x\cos x\big(\cos^2 x - \sin^2 x\big)\\ &= \sin(2x)\cos(2x) = \tfrac{1}{2}\sin(4x).\end{aligned}$$

   (c) We will first factor the numerator using $a^2 - b^2 = (a-b)(a+b)$ and then apply the Pythagorean Identity $1+\tan^2 x = \sec^2 x$:

   $$\begin{aligned}&\frac{\sec^4 x - \tan^4 x}{\sec^2 x + \tan^2 x} = \\ &= \frac{(\sec^2 x - \tan^2 x)(\sec^2 x + \tan^2 x)}{\sec^2 x + \tan^2 x}\\ &= \sec^2 x - \tan^2 x = 1.\end{aligned}$$

   (d) Simplifying the expression, we obtain

   $$\begin{aligned}\frac{2\sin^2 x}{\cos^3 x}\cdot\left(\frac{\cos x}{2\sin x}\right)^2 &= \frac{2\sin^2 x\cos^2}{\cos^3 x\, 4\ \sin^2 x}\\ &= \frac{1}{2\cos x} = \tfrac{1}{2}\sec x\end{aligned}$$

   (e) We can reduce the fraction by first factoring out $\sin x\cos x$ in the numerator:

   $$\begin{aligned}&\frac{3\sin x}{\cos^2 x}\cdot\frac{\cos^2 x + \cos x\sin x}{\sin^2 x - \cos^2 x} = \\ &= \frac{3(\sin x\cos^2 x + \sin^2 x\cos x)}{\cos^2 x(\sin^2 x - \cos^2 x)}\\ &= \frac{3\sin x\cos x(\cos x + \sin x)}{\cos^2 x(\sin x - \cos x)(\sin x + \cos x)}\\ &= \frac{3\sin x}{\cos x(\sin x - \cos x)}\\ &= \frac{3\tan(x)}{\sin x - \cos x}.\end{aligned}$$

   (f) We first compactify the denominator by applying the Pythagorean Identity $1+\tan^2\theta = \sec^2\theta$. Then we reduce the fraction after rewriting it in terms of $\sin$ and $\cos$:

   $$\begin{aligned}\frac{2\tan\theta}{1+\tan^2\theta} = \frac{2\tan\theta}{\sec^2\theta} &= \frac{2\sin\theta\cos^2\theta}{\cos\theta}\\ &= 2\sin\theta\cos\theta = \sin(2\theta).\end{aligned}$$

   (g) We separate the fraction and then rewrite $\tan$ in terms of $\sin$ and $\cos$:

   $$\begin{aligned}\frac{\tan x - \sin x}{2\tan x} &= \frac{\tan x}{2\tan x} - \frac{\sin x}{2\tan x}\\ &= \frac{1}{2} - \frac{\sin x\cos x}{2\sin x}\\ &= \tfrac{1}{2} - \tfrac{1}{2}\cos x = \tfrac{1}{2}(1-\cos x).\end{aligned}$$

   (h) With the Addition formula for $\sin$, follows

   $$\begin{aligned}&\sin(\alpha+\beta)\sin(\alpha-\beta) = \\ &= \big(\sin\alpha\cos\beta + \sin\beta\cos\alpha\big)\\ &\quad\cdot\big(\sin\alpha\cos\beta - \sin\beta\cos\alpha\big)\\ &= \sin^2\alpha\cos^2\beta - \sin^2\beta\cos^2\alpha\\ &= \tfrac{1}{2}\big(1-\cos 2\alpha\big)\tfrac{1}{2}\big(1+\cos 2\beta\big)\\ &\quad - \tfrac{1}{2}\big(1-\cos 2\beta\big)\tfrac{1}{2}\big(1+\cos 2\alpha\big)\end{aligned}$$

from which follows using the abbreviations $A := \cos 2\alpha$ and $B := \cos 2\beta$

$$
\begin{aligned}
&= \tfrac{1}{4}\Big((1-A)(1+B)-(1-B)(1+A)\Big)\\
&= \tfrac{1}{4}\Big(1-A+B-AB-(1+A-B-AB)\Big)\\
&= \tfrac{1}{4}\Big(1-A+B-AB-1-A+B+AB\Big)\\
&= \tfrac{1}{4}(-2A+2B) = -\tfrac{1}{2}A+\tfrac{1}{2}B\\
&= -\tfrac{1}{2}\cos 2\alpha + \tfrac{1}{2}\cos 2\beta.
\end{aligned}
$$

(i) We combine the fractions and reduce after applying the third Pythagorean Identity:

$$
\begin{aligned}
\left(\frac{\cot x}{\csc x}\right)^2 + \frac{1}{\csc^2 x} &=\\
&= \frac{\cot^2 x}{\csc^x} + \frac{1}{\csc^2 x}\\
&= \frac{\cot^2 x + 1}{\csc^2 x}\\
&= \frac{\csc^2 x}{\csc^2 x} = 1.
\end{aligned}
$$

(j) We separate the fraction and obtain

$$
\begin{aligned}
\frac{\cos\theta + \sin\theta}{\cos\theta} &= \frac{\cos\theta}{\cos\theta} + \frac{\sin\theta}{\cos\theta}\\
&= 1 + \tan\theta.
\end{aligned}
$$

(k) Applying the Pythagorean identities $1 + \tan^2 x = \sec^2 x$ and $1 = \cos^2 x + \sin^2 x$, we obtain

$$
\begin{aligned}
\left(\cos^2 x\right)\cdot\frac{1+\tan^2 x}{1-\cos^2 x} &=\\
&= \frac{\cos^2 x \sec^2 x}{\sin^2 x}\\
&= \frac{\cos^2 x}{\sin^2 x \cos^2 x}\\
&= \frac{1}{\sin^2 x} = \csc^2 s.
\end{aligned}
$$

2. Recall the Sum and Difference Formulas for Sine and Cosine:

   (a) $\sin(x+y) = \sin x\cos y + \sin y \cos x$

   (b) $\cos(x+y) = \cos x \cos y - \sin x \sin y$

   (c) $\sin(x-y) = \sin x \cos y - \sin y \cos x$

   (d) $\cos(x-y) = \cos x \cos y + \sin x \sin y$

3. The three Pythagorean identities:

$$
\begin{aligned}
\cos^2 x + \sin^2 &= 1\\
1 + \tan^2 x &= \sec^2 x\\
1 + \cot^2 x &= \csc^2 x
\end{aligned}
$$

4. Recall the Double and Half Angle Formulas for Sine and Cosine:

   (a) $\sin 2x = 2\cos x \sin x$

   (b) $\cos 2x = \cos^2 x - \sin^2 x = 2\cos^2 x - 1 = 1 - 2\sin^2 x$

   (c) $\tan 2x = \dfrac{2\tan x}{1-\tan^2 x}$

   (d) $\cos^2 x = \frac{1}{2}\big(1+\cos 2x\big)$

   (e) $\sin^2 x = \frac{1}{2}\big(1-\cos 2x\big)$

   (f) $\sin\frac{x}{2} = \pm\sqrt{\dfrac{1-\cos x}{2}}$

   (g) $\cos\frac{x}{2} = \pm\sqrt{\dfrac{1+\cos x}{2}}$

   (h) $\tan\frac{x}{2} = \dfrac{\sin x}{1+\cos x} = \dfrac{1-\cos x}{\sin x}$

5. Recall the Law of Sines and Law of Cosines; Determine which Law applies; Solve Triangle:

   (a) We need to compute a, β and γ. Clearly, $\gamma = \pi - \alpha - \beta = \pi - \frac{2}{3}\pi - \beta = \frac{1}{3}\pi = \beta$. a can be computed using the law of consine:

$$
\begin{aligned}
a^2 &= b^2 + c^2 - 2bc\cos\alpha\\
&= 9 + 25 - 2\cdot 3\cdot 5\cdot\cos\big(\tfrac{2}{3}\pi\big)\\
&= 34 - 30\big(\cos^2\tfrac{\pi}{3} - \sin^2\tfrac{\pi}{3}\big)\\
&= 34 - 30\big(\tfrac{1}{4} - \tfrac{3}{4}\big) = 49
\end{aligned}
$$

   hence

$$a = \sqrt{49} = 7.$$

   To compute β, we will use the law of sine:

$$\frac{\sin\beta}{\sin\alpha} = \frac{3}{7},$$

   hence

$$\sin\beta = \tfrac{3}{7}\sin\alpha = \tfrac{3}{7}\cdot\tfrac{1}{2}\sqrt{3} = \tfrac{3}{14}\sqrt{3}$$

   and therefore

$$\beta = \arcsin\big(\tfrac{3}{14}\sqrt{3}\big).$$

(b) We need to compute b, α and γ. Clearly, $\gamma = 180° - 62° - \alpha = 118° - \alpha$. b can be computed using the law of cosine:

$$\begin{aligned} b^2 &= a^2 + c^2 - 2ac\cos\beta \\ &= 225 + 49 - 2 \cdot 15 \cdot 7 \cdot \cos 62° \\ &= 274 - 210\cos 62° \end{aligned}$$

hence

$$b = \sqrt{274 - 210\cos 62°}.$$

Since, by the law of sine,

$$\frac{\sin\alpha}{\sin\beta} = \frac{a}{b}$$

it follows that

$$\sin\alpha = \frac{15}{b}\sin 62°,$$

hence

$$\alpha = \arcsin\left(\frac{15}{b}\sin 62°\right).$$

(c)
(d)
(e)

6. Recall the Trigonometric Values of Angles which are Multiples of $\frac{\pi}{4}$ And $\frac{\pi}{6}$, and find the exact Value:

(a) (b) (c) (d) (e) (f) (g)

7. Simplify the Trigonometric Expression and find the exact Value: (Hint: Use Right-Triangle Construction)

(a)
(b)
(c)
(d)
(e)
(f)
(g)

8. Graph the Trigonometric Equations:

(a) (b) (c) (d) (e) (f) (g) (h) (i) (j)

9. Solve for x in the Trigonometric Equations:

(a)
(b)
(c)
(d)
(e)
(f)
(g)
(h)
(i)

10. Convert Cartesian Coordinates to Polar Coordinates:

(a) (b) (c) (d) (e) (f)

11. Convert Polar Coordinates to Cartesian Coordinates:

(a) (b) (c) (d) (e) (f) (g) (h)

12. Define the Hyperbolic Trigonometric Functions in terms of the Exponential Functions:

(a)
(b)
(c)
(d)
(e)
(f)

13. Apply the Basic Hyperbolic Trigonometric Identities to simplify Expressions:

(a)
(b)
(c)
(d)
(e)
(f)