Review
Fundamental Concepts and Techniques of Calculus

Detailed Solutions to Selected Exercises:
Integral Calculus

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1. Integration By Parts:

   (a) We use integration by parts (R 4:6c) in the form

   $$\int uv' = uv - \int u'v$$

   and try to reduce the factor x in the integrand to 1 by choosing $u = x$:

   $$\begin{aligned}\int xe^x\,dx &= \int \underbrace{x}_{u}\underbrace{e^x}_{v'}\,dx \\ &= \underbrace{x}_{u}\underbrace{e^x}_{v} - \int \underbrace{1}_{u'}\underbrace{e^x}_{v}\,dx \\ &= xe^x - \int e^x\,dx \\ &= xe^x - e^x + C\end{aligned}$$

   (b)

   (c) We use integration by parts (R 4:6c) in the form

   $$\int uv' = uv - \int u'v$$

   and try to reduce the factor x in the integrand to 1 by choosing $u = x$:

   $$\begin{aligned}\int x\sin x\,dx &= \int \underbrace{x}_{u}\underbrace{\sin x}_{v'}\,dx \\ &= x(-\cos x) - \int 1\cdot(-\cos x)\,dx \\ &= -x\cos x + \int \cos x\,dx\end{aligned}$$

   which by R 4:5c) equals

   $$= -x\cos x + \sin x + C.$$

   (d) We will use integration (R 4:6c) by parts twice and will see that the remaining integral will be a multiple of the original integral:

   $$\begin{aligned}\int e^x\sin x\,dx &= \int \underbrace{e^x}_{u'}\underbrace{\sin x}_{v}\,dx \\ &= e^x\sin x - \int \underbrace{e^x}_{u'}\underbrace{\cos x}_{v}\,dx \\ &= e^x\sin x - \left(e^x\cos x + \int e^x\sin x\,dx\right) \\ &= e^x\big(\sin x - \cos x\big) - \int e^x\sin x\,dx\end{aligned}$$

   Collecting the term $\int e^x\sin x\,dx$ on the left side, we obtain

   $$2\int e^x\sin x\,dx = e^x\big(\sin x - \cos x\big)$$

   and thus

   $$\int e^x\sin x\,dx = \tfrac{e^x}{2}\big(\sin x - \cos x\big) + C.$$

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2. Trigonometric Integrals:

(a) Using the Second Pythagorean identity, we obtain the equation

$$\begin{aligned}\tan^3 x\sec^3 x &= \sec^2 x\tan^2 x(\sec x\tan x)\\ &= \sec^2(\sec^2 x-1)(\sec x\tan x).\end{aligned}$$

Thus

$$\begin{aligned}&\int \tan^3 x\sec^3 x\,dx =\\ &= \int \sec^2(\sec^2 x-1)(\sec x\tan x)\,dx\\ &= \int \sec^4 x(\sec x\tan x)\,dx\\ &\qquad - \int \sec^2 x(\sec x\tan x)\,dx\end{aligned}$$

which by the Generalized Powerrule (R 4:5b) equals

$$= \tfrac{1}{5}\sec^5 x - \tfrac{1}{3}\sec^3 x + C.$$

(b)

(c) Using the first Pythagorean identity, we obtain

$$\cos^3 x = \cos^2 x\cos x = (1-\sin^2 x)\cos x.$$

Hence

$$\begin{aligned}&\int \sin^4 x\cos^3 x\,dx =\\ &= \int \sin^4 x(1-\sin^2 x)\cos x\,dx\\ &= \int \sin^4 x\cos x\,dx - \int \sin^6 x\cos x\,dx\end{aligned}$$

which by the Generalized Powerrule (R 4:5b) equals

$$= \tfrac{1}{5}\sin^5 x - \tfrac{1}{7}\sin^7 x + C$$

(d)

(e) *First Method:* We demonstrate first, how the integral can be solved by reducing the power of cos: To this end, we first use (R 2:6b) and (R 2:7c) and reduce $\cos^3 A$:

$$\begin{aligned}\cos^3 A &= \cos A\cos^2 A\\ &= \tfrac{1}{2}\cos A(1+\cos 2A)\\ &= \tfrac{1}{2}\cos A + \tfrac{1}{2}\cos A\cos 2A\\ &\underset{(R\ 2:7c)}{=} \tfrac{1}{2}\cos A + \tfrac{1}{4}\cos A + \tfrac{1}{4}\cos 3A\\ &= \tfrac{3}{4}\cos A + \tfrac{1}{4}\cos 3A.\end{aligned}$$

Then

$$\begin{aligned}\cos^6 x = (\cos^2 x)^3 &\underset{(R\ 2:6b)}{=} \left[\tfrac{1}{2}(1+\cos 2x)\right]^3\\ &= \tfrac{1}{8}\left[1+3\cos 2x+3\cos^2 2x+\cos^3 2x\right]\end{aligned}$$

which by (R 2:6b) and the computation above equals

$$\begin{aligned}&= \tfrac{1}{8}\left[1+3\cos 2x+\tfrac{3}{2}+\tfrac{3}{2}\cos 4x\right.\\ &\qquad\left.+\tfrac{3}{4}\cos 2x+\tfrac{1}{4}\cos 6x\right]\\ &= \tfrac{1}{8}\left[\tfrac{5}{2}+\tfrac{15}{4}\cos 2x+\tfrac{3}{2}\cos 4x+\tfrac{1}{4}\cos 6x\right].\end{aligned}$$

Therefore,

$$\begin{aligned}\int \cos^6 x\,dx &= \tfrac{1}{8}\left[\tfrac{5}{2}x+\tfrac{15}{8}\sin 2x\right.\\ &\qquad\left.+\tfrac{3}{8}\sin 4x+\tfrac{1}{24}\sin 6x\right]+C\\ &= \tfrac{5}{16}x+\tfrac{15}{64}\sin 2x\\ &\qquad+\tfrac{3}{64}\sin 4x+\tfrac{1}{192}\sin 6x+C.\end{aligned}$$

Second Method: We will now show how the integral can be solved using the recurrence formula (R 4:5o)

$$\int \cos^n x\,dx = \tfrac{1}{n}\cos^{n-1}x\sin x + \tfrac{n-1}{n}\int \cos^{n-2}x\,dx.$$

Note that the recurrence allows us to compute $\int \cos^n x\,dx$ provided we have already computed

$\int \cos^{n-2} x\, dx$.

$$\int \cos^0 x\, dx = x$$
$$\int \cos^1 x\, dx = \sin x$$
$$\begin{aligned}\int \cos^2 x\, dx &= \tfrac{1}{2}\cos x \sin x + \tfrac{1}{2}\int \cos^0 x\, dx \\ &= \tfrac{1}{2}\cos x \sin x + \tfrac{1}{2}x\end{aligned}$$
$$\begin{aligned}\int \cos^3 x\, dx &= \tfrac{1}{3}\cos^2 x \sin x + \tfrac{2}{3}\int \cos^1 x\, dx \\ &= \tfrac{1}{3}\cos^2 x \sin x + \tfrac{2}{3}\sin x\end{aligned}$$
$$\begin{aligned}\int \cos^4 x\, dx &= \tfrac{1}{4}\cos^3 x \sin x + \tfrac{3}{4}\int \cos^2 x\, dx \\ &= \tfrac{1}{4}\cos^3 x \sin x \\ &\qquad + \tfrac{3}{4}\big(\tfrac{1}{2}\cos x \sin x + \tfrac{1}{2}x\big) \\ &= \tfrac{1}{4}\cos^3 x \sin x + \tfrac{3}{8}\cos x \sin x + \tfrac{3}{8}x\end{aligned}$$
$$\begin{aligned}\int \cos^5 x\, dx &= \tfrac{1}{5}\cos^4 x \sin x \\ &\qquad + \tfrac{4}{5}\big(\tfrac{1}{3}\cos^3 x \sin x + \tfrac{2}{3}\sin x\big) \\ &= \tfrac{1}{5}\cos^4 x \sin x \\ &\qquad + \tfrac{4}{15}\cos^2 x \sin x + \tfrac{8}{15}\sin x\end{aligned}$$
$$\begin{aligned}\int \cos^6 x\, dx &= \tfrac{1}{6}\cos^5 x \sin x \\ &+ \tfrac{5}{6}\big(\tfrac{1}{4}\cos^3 x \sin x + \tfrac{3}{8}\cos x \sin x + \tfrac{3}{8}\big) \\ &= \tfrac{1}{6}\cos^5 x \sin x \\ &+ \tfrac{5}{24}\cos^3 x \sin x + \tfrac{5}{16}\cos x \sin x + \tfrac{5}{16}x.\end{aligned}$$

(f) We will use the trigonometric identities

$$\sin x \cos x = \tfrac{1}{2}\sin(2x) \qquad (1)$$
$$\cos^2 x = \tfrac{1}{2}\big(1 + \cos(2x)\big) \qquad (2)$$
$$\sin^2 x = \tfrac{1}{2}\big(1 - \cos(2x)\big) \qquad (3)$$

to reduce the number of factors of the integrand (at the expense of creating higher "frequencies").

$$\begin{aligned}&\int \sin^4 x \cos^4 x\ dx = \\ &\underset{(1)}{=} \int \Big(\tfrac{1}{2}\sin 2x\Big)^4 = \tfrac{1}{16}\int \Big(\sin^2 2x\Big)^2 dx \\ &\underset{(3)}{=} \tfrac{1}{16}\int \Big(\tfrac{1}{2}\big(1 - \cos 4x\big)\Big)^2 dx \\ &= \tfrac{1}{64}\int \big(1 - 2\cos 4x + \cos^2 4x\big)dx \\ &\underset{(2)}{=} \tfrac{1}{64}\int \big(1 - 2\cos 4x + \tfrac{1}{2} + \tfrac{1}{2}\cos 8x\big)dx \\ &= \tfrac{1}{64}\int \big(\tfrac{3}{2} - 2\cos 4x + \tfrac{1}{2}\cos 8x\big)dx \\ &= \tfrac{1}{64}\big(\tfrac{3}{2}x - \tfrac{1}{2}\sin 4x + \tfrac{1}{16}\sin 8x\big) + C \\ &= \tfrac{3}{128}x - \tfrac{1}{128}\sin 4x + \tfrac{1}{1024}\sin 8x + C.\end{aligned}$$

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3. Substitution:

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(f)

(g) Using the substitution (R 4:6b)

$$y = \sin^{-1}(2x)$$
$$dy = \frac{2dx}{\sqrt{1-4x^2}}$$

and thus

$$\frac{dy}{2} = \frac{dx}{\sqrt{1-4x^2}}$$

follows immediately that

$$\begin{aligned}\int \frac{\sin^{-1} 2x}{\sqrt{1-4x^2}}\,dx &= \tfrac{1}{2}\int y\,dy \\ &= \tfrac{1}{4}y^2 + C \\ &= \tfrac{1}{4}\Big(\sin^{-1}(2x)\Big)^2 + C \\ &= \tfrac{1}{4}\sin^{-2}(2x) + C.\end{aligned}$$

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(s) The substitution (R 4:6b) $y = e^x$, $dy = e^x dx$ leads to the reduction

$$\int \frac{e^x}{\sqrt{9-e^{2x}}}\,dx = \int \frac{dy}{\sqrt{9-y^2}} =$$

This integral can be solved using trigonometric substitution (R 4:6(b)iii). With the substitution $y = 3\sin\theta$, $d\theta = 3\cos\theta$ we obtain

$$\begin{aligned} &= \int \frac{3\cos\theta\,d\theta}{\sqrt{9-9\sin^2\theta}} = 3\int \frac{\cos\theta\,d\theta}{\sqrt{9(1-\sin^2\theta)}} \\ &= 3\int \frac{\cos\theta\,d\theta}{\sqrt{9\cos^2\theta}} = \int \frac{\cos\theta\,d\theta}{\cos\theta} \\ &= \int d\theta = \theta + C \end{aligned}$$

provided that $\cos\theta > 0$. Thus, solving the equation $e^x = y = 3\sin\theta$ for theta, we find that $\theta = \sin^{-1}\left(\frac{1}{3}e^x\right)$ and conclude

$$\int \frac{e^x}{\sqrt{9-e^{2x}}}\,dx = \sin^{-1}\left(\tfrac{1}{3}e^x\right) + C.$$

(t) We first "complete the square" in the radicant: $x^2+4x+13 = (x+2)^2+9$. Then the integral

$$\begin{aligned} &\int \frac{x+3}{\sqrt{x^2+4x+13}}\,dx = \\ &\quad= \int \frac{x+3}{\sqrt{(x+2)^2+9}}\,dx \end{aligned}$$

and with the substitution (R 4:6b) $y = x+2$, $dy = dx$

$$= \int \frac{y+1}{\sqrt{y^2+9}}\,dy$$

With the trigonometric substitution (R 4:6(b)iii) $y = 3\tan\theta$ which implies $dy = 3\sec^2\theta\,d\theta$ the last integral changes to

$$\begin{aligned} &= \int \frac{3\tan\theta+1}{\sqrt{9\tan^2\theta+9}}(3\sec^2\theta)\,d\theta \\ &= 3\int \frac{3\tan\theta+1}{\sqrt{9(\tan^2\theta+1)}}(\sec^2\theta)\,d\theta \\ &= 3\int \frac{3\tan\theta+1}{\sqrt{9\sec^2\theta)}}(\sec^2\theta)\,d\theta \\ &= \int \frac{3\tan\theta+1}{\sec\theta}(\sec^2\theta)\,d\theta \end{aligned}$$

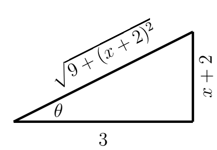
provided $\sec\theta > 0$, recall that $\sqrt{A^2} = |A|$, and thus

$$\begin{aligned} &= \int (3\tan\theta+1)\sec\theta\,d\theta \\ &= 3\int \sec\theta\tan\theta\,d\theta + \int \sec\theta\,d\theta \\ &= 3\sec\theta + \ln|\sec\theta+\tan\theta| + C \\ &= \sqrt{x^2+4x+13} + \ln|x+2+\sqrt{x^2+4x+13} \end{aligned}$$

since $3\tan\theta = y = x+2$, implies that

$$\begin{aligned} \tan\theta &= \tfrac{1}{3}(x+2) \\ \sec\theta &= \tfrac{1}{3}\sqrt{9+(x+2)^2} \\ &= \tfrac{1}{3}\sqrt{x^2+4x+13} \end{aligned}$$

as can be easily verified consulting the following right triangle:



(u)

(v)

(w)

(x)

(y) With the substitution $x = a\tan\theta$ and $dx = a\sec^2\theta\,d\theta$, cf. (R 4:6(b)iii), follows

$$\int \frac{dx}{x^2\sqrt{a^2+x^2}}\,dx =$$
$$= \int \frac{a\sec^2\theta\,d\theta}{a^2\tan^2\theta\sqrt{a^2+a^2\tan^2\theta}}$$
$$= \int \frac{a\sec^2\theta\,d\theta}{a^2\tan^2\theta\sqrt{a^2(1+\tan^2\theta)}}$$
$$= \int \frac{a\sec^2\theta\,d\theta}{a^2\tan^2\theta\sqrt{a^2\sec^2\theta)}}$$
$$= \int \frac{a\sec^2\theta\,d\theta}{a^3\tan^2\theta\sec\theta} = \frac{1}{a^2}\int\frac{\sec\theta\,d\theta}{\tan^2\theta}$$

provided $a > 0$ and $\sec\theta > 0$ and thus

$$= \frac{1}{a^2}\int\frac{\frac{1}{\cos\theta}\,d\theta}{\frac{\sin^2\theta}{\cos^2\theta}} = \frac{1}{a^2}\int\frac{\cos\theta}{\sin^2\theta}\,d\theta$$
$$= \frac{1}{a^2}\int\cot\theta\csc\theta\,d\theta$$
$$= -\frac{1}{a^2}\csc\theta + C$$

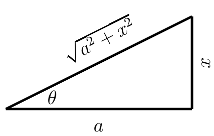
by (R 4:5g) and equals

$$= -\frac{\sqrt{a^2+x^2}}{a^2x} + C$$

since $x = a\tan\theta$ implies $\tan\theta = \frac{x}{a}$ and thus

$$\csc\theta = \frac{\sqrt{a^2+x^2}}{x}$$

as can be easily verified consulting the triangle.



(z)

4. Partial Fractions:

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(x)
(y)

5. Improper Integrals:

(a)
(b)
(c)
(d)
(e)
(f)

6. Applications of Integration:

(a) *Area Between Two Curves:*
   i.
   ii.
   iii.

(b) *Volume By Slicing:*
   i.
   ii.
   iii.
   iv.
   v.

(c) *Volume By Shells:*
   i.
   ii.
   iii.
   iv.
   v.

(d) *Arc Length:*

i.
ii.

(e) *Surface Area:*

i.
ii.
iii.
iv.

(f) *Mass From Density:*

i.
ii.
iii.
iv.

(g) *Center of Mass and Centroid:*

i.
ii.
iii.
iv.
v.
vi.
vii.
viii.
ix.

(h) *Fluid Pressure:*

i.
ii.

(i) *Work:*

i.
ii.
iii.
iv.

7. Parametric Curves:

(a) Express the Parametric Curve by an Equation in x and y:

i.
ii.
iii.
iv.
v.

(b) Find a Parametrization $x = (t)$, $y = y(t)$, $t \in [0, 1]$ for

i.
ii.
iii.
iv.

(c)

(d) Find the slope of the given curve at the given point and give an equation of the tangent line:

i.
ii.

(e) Find the Length of the Parametric Arc:

i.
ii.

8. Polar Coordinates:

(a) Write the Equation in Polar Coordinates:

i.
ii.
iii.

(b) Write the Equation in Cartesian Coordinates:

i.
ii.
iii.

(c) Sketch the Polar Curves:

i.
ii.
iii.
iv.
v.
vi.

(d) Calculate the Area enclosed by the Polar Curve:

i.
ii.
iii.
iv.
v.

(e) Find the Slope of the Polar Curve:

i.
ii.
iii.

(f) Find the Length of the Polar Curve:

i.
ii.