

Review
Fundamental Concepts and Techniques of Calculus

Exercises:* Differential Calculus

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1. Two-Sided Limits: Find the Following Limits if they Exist:

(a) $\lim_{x \rightarrow 3} \frac{2x - 6}{x - 3}$ HSD

(b) $\lim_{x \rightarrow 3} \frac{x - 3}{x^2 - 6x - 9}$ HSD

(c) $\lim_{x \rightarrow 1} \frac{x - 1}{x^2 - 5x + 4}$ HSD

(d) $\lim_{x \rightarrow 0} \frac{1 - \frac{1}{x}}{1 + \frac{1}{x}}$ HSD

(e) $\lim_{x \rightarrow 1} \frac{3x - 3}{\sqrt{x} - 1}$ HSD

(f) $\lim_{x \rightarrow 2} \frac{\frac{1}{\sqrt{2}} - \frac{1}{\sqrt{x}}}{x - 2}$ HSD

(g) $\lim_{x \rightarrow 1} \frac{x^{2/3} - 1}{x - 1}$ HSD

(h) $\lim_{x \rightarrow 0} \frac{\tan^2 x}{x}$ HSD

(i) $\lim_{x \rightarrow \pi} \frac{\sin x}{x - \pi}$ HSD

(j) $\lim_{x \rightarrow 0} \frac{\sin 2x}{\sin 3x}$ HSD

(k) $\lim_{x \rightarrow \pi/4} \frac{\tan(\pi - 4x)}{x - \pi/4}$ S

(l) $\lim_{x \rightarrow 0} |x| \cos x$ HSD

(m) $\lim_{x \rightarrow \pi/4} \frac{\sin(x + \frac{1}{4}\pi) - 1}{x - \frac{1}{4}\pi}$ S

(e) $\lim_{x \rightarrow \infty} \frac{5x^3 + 4(\sqrt{x})^5 + 2x + \tanh x}{4x^3 + 2x + 4}$ S

(f) $\lim_{x \rightarrow \infty} (\sqrt{x^2 + 4x} - \sqrt{x})$ S

(g) $\lim_{x \rightarrow \infty} (\sqrt{x^2 + 4x} - x)$ S

(h) $\lim_{x \rightarrow \infty} \frac{1}{1 + e^{-x}}$ S

(i) $\lim_{x \rightarrow -\infty} \frac{1}{1 + e^{-x}}$ S

4. Squeeze Play, Pinching Theorem: Find the Following Limits:

(a) $\lim_{x \rightarrow \infty} \frac{\sin x}{x}$ S

(b) $\lim_{x \rightarrow 0} x \sin \frac{1}{x}$ S

(c) $\lim_{n \rightarrow \infty} \frac{e^n}{n!}$ HSD

(d) $\lim_{x \rightarrow -3} (x + 3)^2 \sin \frac{1}{x + 3}$ S

(e) $\lim_{x \rightarrow \pi} (x - \pi) \cos^2 \frac{1}{x - \pi}$ S

(f) $\lim_{x \rightarrow 1} |x - 1| \sin x$ S

2. One-Sided Limits: Find the Following Limits if they Exist:

(a) $\lim_{x \rightarrow 8^+} \frac{|8 - x|}{8 - x}$ S

(b) $\lim_{x \rightarrow 0^+} \frac{\sin x}{\sqrt{x}}$ HSD

(c) $\lim_{x \rightarrow 4^+} \frac{1}{x - 4}$ S

(d) $\lim_{x \rightarrow 4^-} \frac{1}{x - 4}$ S

(e) $\lim_{x \rightarrow 0^+} x \sin \frac{1}{x}$ S

(f) $\lim_{x \rightarrow 0} x \sin \frac{1}{x}$ S

3. Limits at ∞ : Find the Following Limits if they Exist:

(a) $\lim_{x \rightarrow \infty} \frac{x + 1}{x - 3}$ S

(b) $\lim_{x \rightarrow \infty} \frac{x^2 + 1}{2x^2 + 4}$ HSD

(c) $\lim_{x \rightarrow \infty} \frac{1 - x^3}{x^3 + 4x + 3}$ S

(d) $\lim_{x \rightarrow \infty} \frac{x + 1}{x^2 - 3}$ S

5. L'Hopital's Rule: Find the Following Limits:

(a) $\lim_{x \rightarrow \infty} \frac{3^x}{\ln x}$ HSD

(b) $\lim_{x \rightarrow \infty} \left(1 + \frac{a}{x}\right)^x$ HSD

(c) $\lim_{x \rightarrow 0} \frac{\tan \pi x}{e^x - 1}$ S

(d) $\lim_{x \rightarrow 0^+} \frac{\ln x}{\cot x}$ S

(e) $\lim_{x \rightarrow 0} \frac{-6x}{\tan 4x}$ S

(f) $\lim_{x \rightarrow 0} \left(\frac{1}{\sin x} - \frac{1}{x}\right)$ S

*The letters "H","S","D" in the status box indicate whether a *hint* (H), a *solution* (S) or a *detailed solution* (D) is available.

- (g) $\lim_{x \rightarrow \infty} (\sqrt{x^2 + 2x} - x)$ S
- (h) $\lim_{x \rightarrow \infty} \left(\cos \frac{1}{x} \right)^x$ S
- (i) $\lim_{x \rightarrow \infty} (x^3 + 1)^{1/\ln x}$ S
- (j) $\lim_{x \rightarrow 0} |\sin x|^x$ S
- (k) $\lim_{x \rightarrow \pi/2} \frac{\ln(\sin x)}{x - \frac{\pi}{2}}$ S
- (l) $\lim_{x \rightarrow \infty} \frac{1}{x} \int_0^x \sin \left(\frac{1}{1+t} \right) dt$ S
- (m) $\lim_{x \rightarrow 2} \frac{-\arctan(4x - 8)}{\arcsin(7x - 14)}$ S

6. Continuity, Removable Discontinuities, and Jump Discontinuities:

- (a) Sketch the graph of the function

$$f(x) = \begin{cases} 1 & x \leq 0 \\ x^2 & 0 < x < 1 \\ 1 & 1 \leq x < 2 \\ x & 2 \leq x \end{cases}$$

and classify the discontinuities (if any) as being a removable discontinuity, a jump discontinuity, or an infinite discontinuity. S

- (b) Sketch the graph of the function

$$f(x) = \begin{cases} x + 7 & x < -3 \\ |x - 2| & -3 < x < -1 \\ x^2 - 2x & -1 < x < 3 \\ 2x - 3 & 3 \leq x \end{cases}$$

and classify the discontinuities (if any) as being a removable discontinuity, a jump discontinuity, or an infinite discontinuity. S

- (c) Find where the function $f(x) = \frac{x+1}{x^2+5x+4}$ is continuous. S
- (d) Find where the function $f(x) = \frac{x+3}{x-3}$ is continuous and classify any points of discontinuity. S
- (e) Let $f(x) = \frac{x-2}{x^2-4}$ with $D:(-\infty, \infty)$. Is it possible to choose $f(2)$ such that the resulting function is continuous at 2? S
- (f) Let $f(x) = \frac{x-2}{x^2+3x-10}$ with $D:(-\infty, \infty)$. Is it possible to choose $f(2)$ such that the resulting function is continuous at 2? S
- (g) Let $f(x) = \frac{x-3}{x^2+8x+12}$ with $D:(-\infty, \infty)$. Is it possible to choose $f(-6)$ such that the resulting function is continuous at -6 ? S

7. Recall and interpret the Intermediate Value Theorem: The real valued, continuous function f is defined on the closed and bounded interval $[a, b]$. Which of the following must be true?

- (a) If either $f(a)$ or $f(b)$ is larger than c , then there exists $x \in [a, b]$ such that $f(x) = c$. S
- (b) The function has either a maximum or minimum value but we cannot say which. S
- (c) If either $f(a)$ or $f(b)$ is smaller than c , then there exists $x \in [a, b]$ such that $f(x) = c$. S
- (d) If $f(a)$ is smaller than c and $f(b)$ is larger than c , then there exists $x \in (a, b)$ such that $f(x) = c$. S
- (e) If $f(a)$ and $f(b)$ are both larger than c , then there exists $x \in [a, b]$ such that $f(x) = c$. S
- (f) The function has a derivative on the open interval (a, b) . S
- (g) If there exists $x \in [a, b]$ such that $f(x) = c$, then either $f(a) < c < f(b)$ or $f(b) < c < f(a)$. S
- (h) None or the above. S

8. Pick the correct Statement of the Mean Value theorem:

- (a) Let f be continuous on $[a, b]$ and suppose f' is continuous on (a, b) . Then there exists $c \in [a, b]$ such $f'(c)(f(b) - f(a)) = b - a$. S
- (b) If f is continuous on $[a, b]$ and f' exists on (a, b) , then there exists a point $c \in (a, b)$ such that $f'(c)(b - a) = f(b) - f(a)$. S
- (c) If f is continuous on $[a, b]$, then f is differentiable on (a, b) and $f'(c)(b - a) = f(b) - f(a)$. S
- (d) If f is differentiable on (a, b) , then f is continuous on $[a, b]$ and $f'(c)(b - a) = f(b) - f(a)$. S
- (e) If $f'(x) = 0$ for all $x \in (a, b)$, then f equals a constant. S
- (f) Let f be differentiable on (a, b) . Then $f'(c)(b - a) = f(b) - f(a)$ for all $c \in (a, b)$. S
- (g) Let f be differentiable on (a, b) and continuous on $[a, b]$. Then for all $c \in (a, b)$, we have $f'(c)(b - a) = f(b) - f(a)$. S
- (h) If for some $c \in (a, b)$ we have $f'(c)(b - a) = f(b) - f(a)$, then it follows f is differentiable on (a, b) and continuous on $[a, b]$. S
- (i) For some $c \in (a, b)$ it follows $f'(c)(b - a) = f(a) - f(b)$ provided f is continuous on $[a, b]$ and differentiable on (a, b) . S

- (j) None of the above. S
9. Recall and Interpret the Extreme Value Theorem
The real valued, continuous function f is defined on the closed and bounded interval $[a, b]$. Which of the following must be true?
- (a) If $f(a)$ and $f(b)$ are both larger than c , then there exists $x \in [a, b]$ such that $f(x) = c$. S
- (b) The function has a derivative on the open interval (a, b) . S
- (c) There exists $x \in [a, b]$ such that $f(x) \geq f(y)$ for all $y \in [a, b]$. S
- (d) If either $f(a)$ or $f(b)$ is larger than c , then there exists $x \in [a, b]$ such that $f(x) = c$. S
- (e) The function has either a maximum or minimum value but we cannot say which. S
- (f) There exists $x \in [a, b]$ such that $f(x) = f(y)$ for all $y \in [a, b]$. S
- (g) If either $f(a)$ or $f(b)$ is smaller than c , then there exists $x \in [a, b]$ such that $f(x) = c$. S
- (h) If there exists $x \in [a, b]$ such that $f(x) = c$, then either $f(a) < c < f(b)$ or $f(b) < c < f(a)$. S
- (i) None of the above. S
10. Find the Equation for the Lines Tangent and Normal (Perpendicular) to the given Curve at the given Point
- (a) $f(x) = \sin x, (\frac{\pi}{6}, \frac{1}{2})$ HSD
- (b) $f(x) = \sqrt{x+1}, (3, 2)$ HSD
- (c) $f(x) = 1 - e^{-x}, (0, 0)$ HSD
- (d) $f(x) = 1 + \ln x, (3, 1 + \ln 3)$ HSD
11. The Derivative:
- (a) Recall the Definition of a Derivative HS
- (b) Use the Definition to compute the Derivative of the following Functions at x_0 :
- i. $f(x) := \frac{x}{x+1}, x_0 = 2$ S
- ii. $f(x) = \frac{1}{\sqrt{x}}, x_0 = 4$ S
- iii. $f(x) = \sin x, x_0 = 0$ S
- iv. $f(x) = \begin{cases} x^2 \sin(\frac{1}{x}) + x(1-x) & x \neq 0 \\ 0 & \text{else.} \end{cases}, x_0 = 0.$ S
- (c) Determine whether each Function is (α) differentiable (β) continuous at x_0 :
- i. $f(x) = \begin{cases} x^2 & x < 1 \\ 2x - 1 & x \geq 1 \end{cases}, x_0 = 1.$ S

$$\text{ii. } f(x) = \begin{cases} x^2 + 4 & x \leq -1 \\ 3 - 2x & x > -1 \end{cases}, x_0 = -1.$$

 S

$$\text{iii. } f(x) = \begin{cases} 1 - x^2 & x < 2 \\ -3 & x = 2 \\ x^2 - 8x + 9 & x > 2 \end{cases}, x_0 = 2.$$

 S

12. Taking Derivatives Using the Differentiation Rules:

(a) Polynomials and Rational Functions

i. $f(x) = (x - 2x^2)^2$ S

ii. $f(x) = \frac{1}{x\sqrt{3+2x^2}}$ S

iii. $f(x) = \frac{x^2 - 1}{x^3 + 1}$ S

iv. $f(x) = \left((x^{-1} + 2x^{-2})^3 + 3x^{-3} \right)^4$ S

v. $g(x) = \frac{x^2 + x + 1}{x^2 - x + 1}$ S

vi. $g(t) = \frac{t^2 + 5t + 4}{t^2 + t - 20}$ S

vii. $f(x) = \frac{x^3 - 1}{x^4 + 1} (x^2 + 1)$ S

viii. $f(x) = \left(1 - \frac{1}{x} \right) \left(2 - \frac{1}{x} \right)$ S

ix. $f(x) = \left(x^2 - \frac{1}{x^3 + 1} \right)^5$ S

x. $f(x) = \left(\frac{4x + 3}{5x - 2} \right)^3$ S

(b) Trigonometric Functions

i. $f(x) = \sin x \cos x$ S

ii. $f(x) = x^2 \cos x$ S

iii. $f(x) = \frac{\sec x}{\tan^2 x + 1}$ S

iv. $f(x) = \sqrt[3]{\sec^2(\tan(x))}$ S

v. $f(x) = \tan(\sec(x+1))$ S

vi. $f(x) = 2x^2 \tan x \sec x$ S

vii. $f(x) = \cos(\sin x)$ S

viii. $f(x) = (1 + \sin x)^{\frac{1}{3}}$ S

ix. $f(x) = (\sec(\tan x))^{\frac{1}{4}}$ S

x. $f(x) = \cot^2(2\sqrt{3x+1})$ S

xi. $f(x) = x^2 \sec(x^{-3})$ S

xii. $f(x) = \cos^3(2x) \tan(4x)$ S

(c) Exponential and Logarithmic Functions

i. $f(x) = 2^{5x} 3^{\ln x}$ S

ii. $f(x) = \ln(\ln(2x+1))$ S

iii. $f(x) = (x + 1)^x$ S

iv. $f(x) = x^{x^x}$ S

(d) Misc.

i. $f(x) = \cos(2^x + 2^{-x})$ S

ii. $f(x) = (\tan x)^{\ln(x^x)}$ S

iii. $f(x) = (\sin x)^{\cos x}$ S

iv. $f(x) = \arcsin(x^2 + 1)e^{x^2+1}$ S

v. $f(x) = \frac{(x+1)(2+x)\sin 3x}{e^x(\sqrt[5]{x^2+2x+1})(\operatorname{arcsec}(3^x))}$ S

vi. $f(x) = \ln|\sec x + \tan x|$ S

vii. $f(x) = \tan \sqrt{e^{-3x}}$ S

viii. $f(x) = \sec(e^{\tan(4x^2)})$ S

ix. $f(x) = e^{4 \ln x}$ S

x. $f(x) = e^{\sin(2x)} \ln(\cos e^{2x})$ S

xi. $f(x) = \sin(\operatorname{arcsec}(\ln x))$ S

xii. $f(x) = \frac{1}{3} \arcsin(3x - 4x^2)$ S

xiii. $f(x) = \cos(\operatorname{arcsec}(\ln x))$ S

xiv. $f(x) = e^{\operatorname{arccot} x}$ S

xv. $f(x) = \arctan\left(\frac{x}{1+x^2}\right)$ S

13. Implicit Differentiation

(a) Find dy/dx in terms of x and y :

i. $\tan \sqrt{x+y} = 2x$ S

ii. $|x| + |y| = 1$ S

iii. $x^{2/3} + y^{2/3} = 1$ S

iv. $\tan(xy) = x$ S

v. $x^2 - 3xy + 4y^2x^2 = 18x^2$ S

(b) Find Equations for the Tangent and Normal Lines at the indicated Point, respectively:

i. $9x^2 + 4y^2 = 72, (2, 3)$ S

ii. $y^2 + 2xy + x^2 = 9, (2, 1)$ S

iii. $x = \cos y, (\frac{1}{2}, \frac{\pi}{3})$ S

14. Find the Critical Numbers and Classify the Extreme Values (as Local/Global):

(a) $f(x) = x^2 - 4x + 1, x \in [0, 3]$ S

(b) $f(x) = x^4 - x^2 + 4$ S

(c) $f(x) = x^3(4-x)^2$ S

(d) $f(x) = \frac{x^2}{1+x^2}, x \in [-2, 6]$ S

(e) $f(x) = 6 + 216x^2 + \frac{2}{x}, x \in [0, \infty)$ S

(f) $f(x) = \frac{x}{4+x^2}, x \in [-3, 1]$ S

(g) $f(x) = x\sqrt{4-x^2}$ S

(h) $f(x) = 2\cos^3 x + 3\sin x, x \in [0, \pi]$ S

(i) $f(x) = \begin{cases} |x+1| & -3 \leq x < 0 \\ x^2 - 4x + 2 & 0 \leq x < 3 \\ 2x - 7 & 3 \leq x < 4 \end{cases}$ S

15. Describe the Concavity of the Graph of f and find the Points of Inflection:

(a) $f(x) = x^3 - 3x + 2$ S

(b) $f(x) = \frac{6x}{x^2+1}$ S

(c) $f(x) = 2x^3 + 3x^2$ S

(d) $f(x) = x\sqrt{4-x^2}$ S

(e) $f(x) = x + \sin x$ S

(f) $f(x) = \begin{cases} x^3 & x < 1 \\ 3x - 2 & x \geq 1 \end{cases}$ S

16. Find the Intervals on which f increases and the Interval on which f decreases:

(a) $f(x) = x + \frac{1}{x}$ S

(b) $f(x) = 3x^3 + 7x^2 + 7$ S

(c) $f(x) = \frac{x^2+1}{x^2-1}$ S

(d) $f(x) = x + \sin x$ S

17. Sketch the Graph of the Function f :

(a) $f(x) = x^3 - 2x^2 + x + 1$ S

(b) $f(x) = \frac{x-4}{x^2}$ S

(c) $f(x) = \frac{2x^2}{x+1}$ S

(d) $f(x) = 3 - |x^2 - 1|$ S

(e) $f(x) = x(x-1)^{\frac{1}{5}}$ S

(f) $f''(x) > 0$ if $|x| > 2, f''(x) < 0$ if $|x| < 2, f'(0) = 0, f'(x) > 0$ if $x < 0$ and $f'(x) < 0$ if $x > 0, f(0) = 1, f(-2) = f(2) = \frac{1}{2}, f(x) > 0$ for all x, f is an even function and continuous

S

18. Maximun/Minimum Problems

(a) Find the dimensions of the isosceles triangle of largest area with perimeter 12. S

(b) Find the point on the parabola $y = x^2$ which is closest to the point $(0, 3)$. S

(c) A rectangular garden 12 square feet in area is to be fenced off against rabbits. Find the least possible length of fencing if one side of the garden is already protected by a barn. S

- (d) A rectangular playground is to be enclosed by a fence and divided in 4 pieces by 3 fences parallel to one side of the playground. 1060 feet of fencing is used. Find the dimension of the playground which will have the largest total area.
- (e) A triangle is formed by the coordinate axes and a line through the point $(2, 5)$. Determine the slope of this line if the area of the triangle is to be a maximum.
- (f) What is the maximum possible area for a triangle inscribed in a circle of radius r .
- (g) Conical paper cups are usually made so that the depth is $\sqrt{2}$ times the radius of the rim. Show that this design requires the least amount of paper per unit volume.
- (h) What is the maximum volume for a rectangular box (square base, no top) made from 12 square feet of cardboard?