

**Review**  
Fundamental Concepts and Techniques of Calculus

**Exercises: \* Integral Calculus**

Last updated: 041012

1. Integration By Parts:

(a)  $\int x e^x dx$  HSD

(b)  $\int x^3 3^x dx$  S

(c)  $\int x \sin x dx$  HSD

(d)  $\int e^x \sin x dx$  HSD

(e)  $\int \ln x dx$  S

(f)  $\int x^n \ln x dx$  S

(g)  $\int x \cos \pi x dx$  S

(h)  $\int \cos(\ln x) dx$  S

(i)  $\int x \tan^{-1}(x^2) dx$  S

(j)  $\int \frac{1}{x} \sin^{-1}(\ln x) dx$  S

(k)  $\int \cos \sqrt{x} dx$  S

(l)  $\int x \ln(x+1) dx$  S

(m)  $\int \frac{\sin^{-1} 2x}{\sqrt{1-4x^2}} dx$  S

(n)  $\int x^2 \cosh 2x dx$  S

(o)  $\int x^2(2x-1)^{-7} dx$  S

(p)  $\int x \tan^{-1}(x^2) dx$  S

(q)  $\int \sin^{-1} 2x dx$  S

(r)  $\int x^2 \tan^{-1}(x^2) dx$  S

(s)  $\int \frac{1}{x} \sin^{-1}(\ln x^2) dx$  S

(t)  $\int \sin \sqrt{x} dx$  S

2. Trigonometric Integrals:

(a)  $\int \tan^3 x \sec^3 x dx$  HSD

(b)  $\int \sin^4(3t) \cos(3t) dt$  S

(c)  $\int \sin^4 x \cos^3 x dx$  HSD

(d)  $\int \sec^3(6x) dx$  S

(e)  $\int \cos^6 x dx$  HSD

(f)  $\int \sin^4 x \cos^4 x dx$  HSD

(g)  $\int (\cos 3x - \sin x)^2 dx$  S

(h)  $\int \tan^4 x dx$  S

(i)  $\int \tan x \sec^{\frac{3}{2}} x dx$  S

(j)  $\int \cos^3 x \cos^2 x dx$  S

(k)  $\int \csc^3(\frac{1}{2}x) dx$  S

(l)  $\int \sin 5x \cos 2x dx$  S

(m)  $\int \sec^5 x dx$  S

(n)  $\int \cot^2 x \csc^2 x dx$  S

3. Substitution:

(a)  $\int \sqrt{1 + \sqrt{x}} dx$  HSD

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\*The letters "H", "S", "D" in the status box indicate whether a *hint* (H), a *solution* (S) or a *detailed solution* (D) is available.

(b) $\int t^5 \sqrt{1+t^2} dt$	<input type="checkbox"/> S	(z) $\int x^2 e^{-x^3} dx$	<input type="checkbox"/> HS
(c) $\int x^2 (e^x - 1) dx$	<input type="checkbox"/> S	4. Partial Fractions:	
(d) $\int \frac{\sin t}{1 + \cos t} dt$	<input type="checkbox"/> S	(a) $\int \frac{7}{(x-2)(x+5)} dx$	<input type="checkbox"/> S
(e) $\int \frac{1 + \sqrt{x+1}}{\sqrt{x+1}} dx$	<input type="checkbox"/> S	(b) $\int \frac{x}{(x+1)(x+2)(x+3)} dx$	<input type="checkbox"/> S
(f) $\int \frac{\tan(\ln t)}{t} dt$	<input type="checkbox"/> S	(c) $\int \frac{dx}{(x-1)^3}$	<input type="checkbox"/> S
(g) $\int \frac{\sin^{-1} 2x}{\sqrt{1-4x^2}} dx$	<input type="checkbox"/> HSD	(d) $\int \frac{x^5}{(x-2)^2} dx$	<input type="checkbox"/> S
(h) $\int \frac{x}{\sqrt{x^2-4}} dx$	<input type="checkbox"/>	(e) $\int \frac{2x^2+3}{x^2(x-1)} dx$	<input type="checkbox"/> S
(i) $\int \frac{x}{\sqrt{4-x^2}} dx$	<input type="checkbox"/>	(f) $\int \frac{x^2+3}{x^2-3x+2} dx$	<input type="checkbox"/> S
(j) $\int \frac{1}{x^2 \sqrt{x^2-a^2}} dx$	<input type="checkbox"/>	(g) $\int \frac{-3x}{x^2-x-12} dx$	<input type="checkbox"/> S
(k) $\int \frac{x}{\sqrt{6x-x^2}} dx$	<input type="checkbox"/>	(h) $\int \frac{x-3}{x^2+4x+1} dx$	<input type="checkbox"/> S
(l) $\int (x^2+3)(x^3+9x+1)^{\frac{1}{3}} dx$	<input type="checkbox"/> S	(i) $\int \frac{7+6x}{(6x+1)^2} dx$	<input type="checkbox"/> S
(m) $\int \frac{e^{\sqrt{x}}}{\sqrt{x}} dx$	<input type="checkbox"/> S	(j) $\int \frac{x^3+x^2+x+3}{(x^2+1)(x^2+3)} dx$	<input type="checkbox"/> S
(n) $\int x \sqrt{x^2+6x} dx$	<input type="checkbox"/>	(k) $\int \frac{x^2-3x-1}{x^3+x^2-2x} dx$	<input type="checkbox"/> S
(o) $\int x \tan^{-1} x dx$	<input type="checkbox"/>	(l) $\int \frac{x^4-x^3-x-1}{x^3-x^2} dx$	<input type="checkbox"/> S
(p) $\int x \sin^{-1} x dx$	<input type="checkbox"/>	(m) $\int e^{2x} \tan^2(e^{2x}) \sec^2(e^{2x}) dx$	<input type="checkbox"/> S
(q) $\int \frac{\ln(x+1)}{\sqrt{x+1}} dx$	<input type="checkbox"/> S	(n) $\int \frac{1-\cos x}{1+\sin x} dx$	<input type="checkbox"/> S
(r) $\int \frac{1}{x^2} \left( \frac{x}{x+1} \right)^{\frac{1}{3}} dx$	<input type="checkbox"/> S	(o) $\int \frac{25x^2+21+2x}{(x+3)(5x^2+3)} dx$	<input type="checkbox"/> S
(s) $\int \frac{e^x}{\sqrt{9-e^{2x}}} dx$	<input type="checkbox"/> HSD	(p) $\int \frac{dx}{[(2x+1)^2+9]^2}$	<input type="checkbox"/>
(t) $\int \frac{x+3}{\sqrt{x^2+4x+13}} dx$	<input type="checkbox"/> HSD	(q) $\int \frac{dx}{(x^2+2x+2)^2}$	<input type="checkbox"/>
(u) $\int \frac{x dx}{\sqrt{4-x^2}}$	<input type="checkbox"/> S	(r) $\int \frac{dx}{x^4-16}$	<input type="checkbox"/>
(v) $\int \sqrt{6x-x^2-8} dx$	<input type="checkbox"/> S	(s) $\int \frac{dx}{1-\sqrt{x}}$	<input type="checkbox"/>
(w) $\int \frac{x}{\sqrt{x-1}} dx$	<input type="checkbox"/> S	(t) $\int \frac{dx}{x(x^{1/3}-1)}$	<input type="checkbox"/>
(x) $\int \frac{\tan^2 x}{x-\tan x} dx$	<input type="checkbox"/> S	(u) $\int \frac{\sqrt{x}}{\sqrt{x}-1} dx$	<input type="checkbox"/>
(y) $\int \frac{dx}{x^2 \sqrt{a^2+x^2}}$	<input type="checkbox"/> HSD	(v) $\int \frac{x+1}{x \sqrt{x-2}} dx$	<input type="checkbox"/>

(w)  $\int \frac{1}{1 + \cos x - \sin x} dx$

(x)  $\int \frac{1}{2 + \cos x} dx$

(y)  $\int \frac{1}{\sin x + \tan x} dx$

5. Improper Integrals:

(a)  $\int_1^\infty \frac{dx}{x^2}$

(b)  $\int_1^\infty \frac{dx}{4 + x^2}$

(c)  $\int_0^8 \frac{dx}{x^{2/3}}$

(d)  $\int_0^1 \frac{dx}{\sqrt{1 - x^2}}$

(e)  $\int_e^\infty \frac{\ln x}{x} dx$

(f)  $\int_{1/3}^3 \frac{dx}{\sqrt[3]{3x - 1}}$

6. Applications of Integration:

(a) Area Between Two Curves:

- i.  $y = 2x, y = 4 - x$ , and  $y = -x/2$
- ii. Under  $|x^2 - 1|$  over  $[-3, 1]$
- iii. Between  $y = \cos x$  and  $y = \sin x$  over  $[0, \pi/4]$

(b) Volume By Slicing:

- i. Under  $y = \sqrt{1 - x^2}$ , rotated about the line  $y = -1$
- ii. Under  $y = \sqrt{1 - x^2}$ , rotated about the line  $x = 1$
- iii. The base of the solid is the circular region  $x^2 + y^2 \leq 1$ . A cross-section perpendicular to the  $y$ -axis is a square.
- iv. The base of the solid is the semi-circular region under  $y = \sqrt{1 - x^2}$  over  $[-1, 1]$ . A cross-section perpendicular to the  $y$ -axis is an equilateral triangle.
- v. Two cylinders of radius  $r$  intersect at right angles, their axes meeting. Find the volume of the intersection. What is the shape of a cross-section parallel to the plane of the two axes?

(c) Volume By Shells:

- i.  $\{(x, y) \mid 0 \leq x \leq 2, 0 \leq y \leq x^3\}$ , rotated about the line  $x = 3$
- ii.  $\{(x, y) \mid 3 \leq x \leq 5, \sqrt{x} \leq y \leq x^2\}$ , rotated about the line  $x = 1$

- iii.  $\{(x, y) \mid 3 \leq x \leq 5, \sqrt{x} \leq y \leq x^2\}$ , rotated about the line  $x = 5$
- iv.  $\{(x, y) \mid 3 \leq x \leq 5, \sqrt{x} \leq y \leq x^2\}$ , rotated about the line  $y = 1$
- v. Find the volume generated by revolving an isosceles right triangle with leg  $a$  about a line perpendicular to the hypotenuse and through a vertex (not the right-angle vertex).

(d) Arc Length:

- i.  $C : y = x^2, 0 \leq x \leq 2$
- ii.  $C : y = \ln(\cos x), 0 \leq x \leq \pi/4$

(e) Surface Area:

- i.  $C := \{(x, y) \mid x^2 + y^2 = 1, x \geq 0, y \geq 0\}$  revolved about the line  $y = 1$
- ii.  $C := \{(x, y) \mid x^2 + y^2 = 1, x \geq 0, y \geq 0\}$  revolved about the line  $x = 2$
- iii.  $C : y = \cos x, 0 \leq x \leq \pi/2$
- iv. A square with side  $s$  is revolved about a line parallel to a diagonal and a distance  $r$  from the nearest vertex.

(f) Mass From Density:

- i. Find the mass of a rod of length  $\ell$  whose density varies as the square of the distance from one end.
- ii. Find the mass of a semi-circular disc of radius  $a$  in which density varies as the square of the distance from the diameter.
- iii. Find the mass of the solid circular cylinder of radius  $r$  and length  $\ell$  whose density varies as the distance from the axis of symmetry.
- iv. Find the mass of a circular disc of radius 4 in which density varies as the distance from the center.

(g) Center of Mass and Centroid:

- i. Find the center of mass of a rod of length 2 whose density varies as the distance from one end.
- ii. Find the center of mass of a plate covering the region below  $y = x^2$  over  $[0, 2]$  whose density varies as the distance from the line  $x = 3$ .
- iii. Find the center of mass of a wire covering the unit circle whose density varies as the distance from the line  $x = 2$ .
- iv. Find the centroid of a semi-circular disc of radius  $a$
- v. Find the centroid of a solid cone.

- vi. Find the centroid of a solid parabolic cone obtained by revolving the region below  $y = \sqrt{x}$  over  $[0, a]$  about the  $x$ -axis.
- vii. Find the centroid of a hemispherical bowl of radius  $a$ .
- viii. Find the centroid of the arc  $y = \ln x, 1 \leq x \leq e$ .
- ix. Use Pappus' Theorem to find the volume of the solid generated by revolving the region below  $y = \sin x$  over  $[0, \pi]$  about the line  $x = 2\pi$ .

(h) *Fluid Pressure:*

- i. A water tank is a cylinder 6 feet in diameter, standing on end. If water fills the tank to a depth of 4 feet, how much force is exerted on the sides of the tank?
- ii. The face of a dam is an isosceles trapezoid of height 20 feet and bases 10 feet and 40 feet. It is situated vertically with the smaller base at the bottom. Find the force on the dam if the water level is 2 feet below the top of the dam.

(i) *Work:*

- i. Find the work done in moving an object from  $x = 1$  to  $x = 5$  if the force acting at point  $x$  is given by  $F(x) = \sqrt{x^2 + 1}$ .
- ii. A vertical cylindrical tank of radius 2 feet and height 6 feet is full of water. Find the work done in pumping out the water (a) to an outlet at the top of the tank; (b) to a level 5 feet above the top of the tank. (Assume that the water weighs 62.5 pounds per cubic foot.)
- iii. A conical container (vertex down) of radius  $r$  feet and height  $h$  feet is full of a liquid weighing  $\sigma$  pounds per cubic foot. Find the work done in pumping out the top  $\frac{1}{2}h$  feet of liquid: (a) to the top of the tank; (b) to a level  $k$  feet about the top of the tank.
- iv. A chain that weighs 15 pounds per foot is hanging from the top of an 80-foot building to the ground. How much work is done in pulling the chain to the top of the building?

7. Parametric Curves:

- (a) Express the Parametric Curve by an Equation in  $x$  and  $y$ :
  - i.  $x(t) = 3t - 1, y(t) = 5 - 2t$
  - ii.  $x(t) = 2t - 1, y(t) = 8t^3 - 5$
  - iii.  $x(t) = e^{2t}, y(t) = e^{2t} - 1$

- iv.  $x(t) = e^{6t}, y(t) = e^{3t} - 3$
- v.  $x(t) = 2 \sin t, y(t) = \cos t$

(b) Find a Parametrization  $x = x(t), y = y(t), t \in [0, 1]$  for

- i. The line segment from  $(3, 7)$  to  $(8, 5)$ .
- ii. The line segment from  $(2, 6)$  to  $(6, 3)$ .
- iii. The parabolic arc  $x = 1 - y^2$  from  $(0, -1)$  to  $(0, 1)$ .
- iv. The curve  $y^2 = x^3$  from  $(4, 8)$  to  $(1, 1)$ .

(c) Find a parametrization  $x = x(t), y = y(t)$  for the hyperbola  $x^2 - y^2 = 1$ .

(d) Find the slope of the given curve at the given point and give an equation of the tangent line:

- i.  $x(t) = 2 \cos t, y(t) = 3 \sin t, t = \pi/3$
- ii.  $x(t) = 1 + t, y(t) = 1 - t^2, t = 2$

(e) Find the Length of the Parametric Arc:

- i.  $x(t) = t, y(t) = t^2, t \in [0, 3]$
- ii.  $x(t) = 2 \sin t, y(t) = 2 \cos t, t \in [0, \pi/2]$

8. Polar Coordinates:

(a) Write the Equation in Polar Coordinates:

- i.  $x = 2$
- ii.  $x^2 + (y - 1)^2 = 4$
- iii.  $y = mx$

(b) Write the Equation in Cartesian Coordinates:

- i.  $r \sin \theta = 4$
- ii.  $\tan \theta = 2$
- iii.  $r = 4 \sin(\theta + \pi)$

(c) Sketch the Polar Curves:

- i.  $\theta = -\frac{1}{4}\pi$
- ii.  $r = -2 \sin \theta$
- iii.  $r = 1 - \cos \theta$
- iv.  $r = \cos 2\theta$
- v.  $r = \sin 3\theta$
- vi.  $r = -1 + 2 \cos \theta$

(d) Calculate the Area enclosed by the Polar Curve:

- i.  $r = a \cos \theta, \theta \in [-\frac{1}{2}\pi, \frac{1}{2}\pi]$
- ii.  $r = a(1 + \cos 3\theta), \theta \in [-\frac{1}{3}\pi, \frac{1}{3}\pi]$
- iii. Outside  $r = 2$ , but inside  $r = 4 \sin \theta$
- iv. Inside  $r = 4$ , and between the lines  $\theta = \frac{1}{2}\pi$  and  $r = 2 \sec \theta$

v. Inside the inner loop of  $r = 1 - 2 \sin \theta$

(e) Find the Slope of the Polar Curve:

i.  $r = a, [a, \pi/3]$

ii.  $r = \sin 2\theta, [\sqrt{3}/2, \pi/6]$

iii.  $r = 1 + \cos \theta$ ; where are the tangent lines  
either horizontal or vertical?

(f) Find the Length of the Polar Curve:

i.  $r = 3 \cos \theta$

ii.  $r = a(1 + \cos \theta)$