Review
Fundamental Concepts and Techniques of Calculus

Exercises:* Sequences and Series

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1. Investigate whether the following sequences are bounded, monotonic, convergent:

   (a) $\left(\frac{n^2-1}{n^2}\right)_n$ ☐

   (b) $\left((-1)^n \frac{2}{n}\right)_n$ ☐

   (c) $\left(\frac{n^2-1}{n-1}\right)_n$ ☐

   (d) $\left(\frac{\ln n}{n-1}\right)_n$ ☐

   (e) $\left(\frac{\ln n}{\sqrt{n}}\right)_n$ ☐

   (f) $\left(\frac{2^n}{(n+1)!}\right)_n$ ☐

2. Determine the following limits if they exist:

   (a) $\lim_{n\to\infty} (n+2)^{\frac{1}{n}}$ ☐

   (b) $\lim_{n\to\infty} \frac{5^{n+1}}{4^{2n-1}}$ ☐

   (c) $\lim_{n\to\infty} \left(-\frac{1}{3}\right)^n$ ☐

   (d) $\lim_{n\to\infty} \left(\frac{1}{n} - n\right)$ ☐

   (e) $\lim_{n\to\infty} 2^n$ ☐

   (f) $\lim_{n\to\infty} \ln \frac{1}{n}$ ☐

   (g) $\lim_{n\to\infty} \left(1 + \frac{1}{3n}\right)^n$ ☐

   (h) $\lim_{n\to\infty} \frac{x^{50n}}{n!}$ ☐

   (i) $\lim_{n\to\infty} \frac{\ln n^2}{n}$ ☐

   (j) $\lim_{n\to\infty} \ln\left(\frac{n-1}{n}\right)$ ☐

   (k) $\lim_{n\to\infty} \frac{2n!}{3n}$ ☐

   (l) $\lim_{n\to\infty} \left(1 + \frac{1}{n}\right)^{2n^2}$ ☐

   (m) $\lim_{n\to\infty} \left(\sqrt{n+2} - \sqrt{n}\right)$ ☐

   (n) $\lim_{n\to\infty} \left(\sqrt{n^2+n-1} - n\right)$ ☐

   (o) $\lim_{n\to\infty} \frac{1+2+3+\cdots+n}{n^2}$ ☐

3. Find two divergent sequences whose product converges! ☐

4. Verify: If $|a_n| \to 0$, then $a_n \to 0$. ☐

5. The Finacci sequence $(a_n)_n$ is defined recursively by $a_1 := 1$, $a_2 := 1$ and $a_{n+1} := a_n + a_{n-1}$ for $n = 2, 3, 4 \ldots$. Calculate the first 20 terms of the sequence $\left(a_{n+1}/a_n\right)_n$. Does this sequence converge? What is its limit? ☐

6. Find the sum of the infinite series if it exists:

   (a) $\sum_{n=1}^{\infty} \frac{1}{n(n+3)}$ ☐

   (b) $\sum_{n=0}^{\infty} e^{5-3n}$ ☐

   (c) $\sum_{n=1}^{\infty} 5\left(\frac{3}{7}\right)^n$ ☐

   (d) $\sum_{n=4}^{\infty} \frac{3^n + 5^n}{3^n 5^n}$ ☐

   (e) $\sum_{n=1}^{\infty} \frac{10\,000}{3^n}$ ☐

   (f) $\sum_{n=0}^{\infty} (-1)^n \frac{5^n}{3^{2n+3}}$ ☐

   (g) $\sum_{n=0}^{\infty} \left(\frac{1}{n+1} - \frac{1}{n+5}\right)$ ☐

7. Determine whether the following series converge. Justify your conclusion!

   (a) $\sum_{n=0}^{\infty} \sin n$ ☐

*The letters "H","S", "D" in the status box indicate whether a *hint* (H), a *solution* (S) or a *detailed solution* (D) is available.

(b) $\displaystyle\sum_{n=1}^{\infty} \frac{2}{3+\ln n}$ ☐

(c) $\displaystyle\sum_{n=0}^{\infty} \frac{1+n-n^2}{2+3n^2}$ ☐

(d) $\displaystyle\sum_{n=0}^{\infty} \frac{1}{1+2+3+\cdots+n}$ ☐

(e) $\displaystyle\sum_{n=0}^{\infty} \left(1+\frac{1}{n}\right)^n$ ☐

(f) $\displaystyle\sum_{n=1}^{\infty} \frac{n!}{n^n}$ ☐

8. Rewrite each repeating decimal as a fraction.

(a) $0.333\ldots$ ☐

(b) $1.175175175\ldots$ ☐

(c) $3.282828\ldots$ ☐

(d) $0.1636363\ldots$ ☐

9. Test the following infinite series for convergence.

(a) $\displaystyle\sum \frac{3n}{n^2+5}$ ☐

(b) $\displaystyle\sum \frac{3n}{n^3+5}$ ☐

(c) $\displaystyle\sum \frac{2}{2^n-1}$ ☐

(d) $\displaystyle\sum \frac{3}{n-3^n}$ ☐

(e) $\displaystyle\sum \frac{1}{\sqrt{n}\ln n}$ ☐

(f) $\displaystyle\sum \frac{\sqrt{n}}{\ln n}$ ☐

(g) $\displaystyle\sum \frac{3}{n^2\ln n}$ ☐

(h) $\displaystyle\sum \frac{\ln n}{n}$ ☐

(i) $\displaystyle\sum_{k=1}^{\infty} \frac{k!}{k^k}$ ☐

(j) $\displaystyle\sum \frac{\ln n}{n\sqrt{n}}$ ☐

(k) $\displaystyle\sum \frac{\ln n}{n^2\sqrt{n}}$ ☐

(l) $\displaystyle\sum \frac{\ln n}{n^{1.005}}$ ☐

(m) $\displaystyle\sum \frac{2n-4}{2n}$ ☐

(n) $\displaystyle\sum \frac{\sin^2}{n\sqrt{n}}$ ☐

(o) $\displaystyle\sum \frac{n-1}{n^2 2^n}$ ☐

(p) $\displaystyle\sum \sin^2(\pi n)$ ☐

10. Determine whether each infinite series converges absolutely, conditionally or diverges.

(a) $\displaystyle\sum \frac{(-1)^n}{n^2-2n}$ ☐

(b) $\displaystyle\sum_{n=3}^{\infty} \frac{\cos n\pi}{\sqrt{n}}$ ☐

(c) $\displaystyle\sum \frac{3^n}{2^{-1}}$ ☐

(d) $\displaystyle\sum \frac{n^{5/2}}{n^2-1}$ ☐

(e) $\displaystyle\sum \frac{(-1)^n \ln n}{2n-3}$ ☐

(f) $\displaystyle\sum \frac{n!}{(3n)!}$ ☐

(g) $\displaystyle\sum_{n=1}^{\infty} (-1)^n \frac{n+2}{n^2+3n+5}$ ☐

(h) $\displaystyle\sum_{n=1}^{\infty} (-1)^n \frac{1\times 3\times 5\times\cdots\times(2n+1)}{2\times 5\times 8\times\cdots\times(3n+2)}$ ☐

11. Find the radius of convergence and the interval of convergence of the power series.

(a) $\displaystyle\sum_{k=1}^{\infty} \frac{k-1}{k} x^k$ ☐

(b) $\displaystyle\sum_{k=2}^{\infty} \frac{1}{\ln k} x^k$ ☐

(c) $\displaystyle\sum_{n=1}^{\infty} n^3 x^n$ ☐

(d) $\displaystyle\sum_{n=2}^{\infty} \frac{n(n-2)}{2} x^n$ ☐

(e) $\displaystyle\sum_{k=2}^{\infty} \ln(k)\, x^k$ ☐

(f) $\displaystyle\sum_{n=1}^{\infty} (-1)^n \frac{1}{n^3} x^{n+3}$ ☐

(g) $\displaystyle\sum_{n=1}^{\infty} \frac{n(n+2)}{(n+1)(3n+1)} x^n$ ☐

(h) ☐

(i) $\displaystyle\sum_{n=2}^{\infty} \frac{1}{\ln n}(x-3)^n$ ☐

(j) $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n}{\ln(n+1)}(x-1)^n$ ☐

(k) $\displaystyle\sum_{n=1}^{\infty} x^{2n+1}$ ☐

(l) $\displaystyle\sum_{n=0}^{\infty} \frac{n^3+1}{2^n} x^{5n}$ ☐

(m) $\displaystyle\sum_{n=0}^{\infty} \Big(2-\sqrt{n+1}\Big)(x+1)^{3n}$ ☐

(n) $\displaystyle\sum_{n=0}^{\infty} \frac{9^n\sqrt{n+1}}{2+n}(x-1)^{2n+1}$ ☐

12. Find the radius of convergence.

(a) $\displaystyle\sum_{n-1}^{\infty} \frac{n^n}{n!} x^n$ ☐

(b) $\displaystyle\sum_{n=1}^{\infty} \frac{n^{2n}}{(2n)!} x^n$ ☐

(c) $\displaystyle\sum_{n=1}^{\infty} \frac{1\times 3\times 5\times\cdots\times(2n-1)}{2^n[1\times 4\times 7\times\cdots\times(3n-2)]} x^n$ ☐

13. Find the sum of each power series and give its interval of convergence.

(a) $\displaystyle\sum_{n=0}^{\infty} n(n-1)(n-2)x^{n-3}$ ☐

(b) $\displaystyle\sum_{n=0}^{\infty} (-1)^n n x^n$ ☐

(c) $\displaystyle\sum_{n=0}^{\infty} (-1)^n n(n-1) x^n$ ☐

(d) $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n}{n+1} x^{n+1}$ ☐

(e) $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n}{(n+1)(n+2)} x^{n+2}$ ☐

14. Establish the following indenties

(a) $\dfrac{\pi}{4} = 1 - \dfrac{1}{3} + \dfrac{1}{5} - \dfrac{1}{7} + \dfrac{1}{9} + \cdots$ ☐

(b) $\ln 2 = \dfrac{1}{1\cdot 2} + \dfrac{1}{2\cdot 2^2} + \dfrac{1}{3\cdot 2^3} + \dfrac{1}{4\cdot 2^4} + \cdots$ ☐

(c) $\displaystyle\int e^{x^2}\,dx = \sum_{n=0}^{\infty} \frac{x^{2n+1}}{(2n+1)n!} + C$ ☐

15. Sum the series.

(a) $1 - \dfrac{2}{2} + \dfrac{3}{4} - \dfrac{4}{8} + \dfrac{5}{16} - \cdots$ ☐

(b) $1 + \dfrac{1}{2\cdot 2} + \dfrac{1}{3\cdot 2^2} + \dfrac{1}{4\cdot 2^3} + \dfrac{1}{5\cdot 2^4} + \cdots$ ☐

(c) $\dfrac{1\cdot 2}{1} + \dfrac{2\cdot 3}{3} + \dfrac{3\cdot 4}{3^2} + \dfrac{4\cdot 5}{3^3} + \dfrac{5\cdot 6}{3^4} + \cdots$ ☐

16. Find a power series expansion for each function and give its interval of convergence.

(a) $f(x) = \dfrac{1}{1-x^4}$ ☐

(b) $f(x) = \dfrac{1}{1+x^3}$ ☐

(c) $f(x) = \dfrac{1}{(1+x)^3}$ ☐

(d) $f(x) = \ln(1+x^2)$ ☐

(e) $f(x) = \dfrac{1}{1+2x}$ ☐

(f) $f(x) = \tan^{-1} 3x^2$ ☐

(g) $f(x) = \sin(3x)$ ☐

(h) $f(x) = x^3 \cos x$ ☐

(i) $f(x) = x^3 e^{-3x^3}$ ☐

17. Expand the following functions as indicated.

(a) $f(x) = \dfrac{1}{x}$ in powers of $x+1$. ☐

(b) $f(x) = \dfrac{1}{(1+2x)^4}$ in powers of $x-2$. ☐

(c) $f(x) = \sqrt{x}$ in powers of $x-1$. ☐

(d) $f(x) = \ln x$ in powers of $x-2$. ☐