

Review
Fundamental Concepts and Techniques of Calculus

Solutions to the Exercises: Basic Techniques of Algebra

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1. Perform Arithmetic Operations with Fractions:

- (a) 7
- (b) $\frac{3}{8}$
- (c) 48
- (d) $\frac{59}{484}$
- (b) $(x - 5)^2 = 0, x_1 = x_2 = 5$
- (c) $4\left(x + \frac{i}{2}\sqrt{2}\right)^2\left(x - \frac{i}{2}\sqrt{2}\right)^2 = 0,$
 $x_1 = x_2 = \frac{i}{2}\sqrt{2}, x_3 = x_4 = -\frac{i}{2}\sqrt{2}$
- (d) $(2x + 3)(2x + 3), x = -3/2$
- (e) $(x^2 - 4)(x^2 - 2), x = \pm\sqrt{2}, \pm 2$

2. Simplify (and Rationalize) Expressions Involving Exponents and Radicals:

- (a) 5^{-1} , or $1/5$
- (b) $-125z^3$
- (c) $-3/5$
- (d) $(-27)^{3/5}$
- (e) $\frac{x\sqrt{75}}{y^2}$
- (f) $x\sqrt[3]{y}$
- (g) $\sqrt[6]{10a^7b}$
- (h) $2\sqrt[3]{2} + 9\sqrt[3]{2}$
- (i) $\frac{1}{2}\sqrt{10}$
- (j) $\frac{1}{2}\sqrt[3]{100}$
- (k) $3(\sqrt{6} - \sqrt{5})$

3. Simplify Expressions Involving Absolute Value:

- (a) $\frac{1}{5}\sqrt{5} - 5\frac{1}{5}$
- (b) -1
- (c) -1

4. Simplify Expressions Involving Factorials:

- (a) 120
- (b) 1344
- (c) $\frac{2}{3}$

5. Manipulate Logarithmic Expressions:

- (a) 6
- (b) $\frac{3}{5}$
- (c) 6
- (d) 3
- (e) -2
- (f) $\log_8\left(\frac{y^{1/3}(y+4)^{2/3}}{y-1}\right)$
- (g) $-2 \ln 2 + 6 \ln 3$
- (h) $4 \ln 6 - 2 \ln 5$

6. Factor a Polynomial and Solve:

- (a) $(3x - 2)(3x + 2) = 0, x_1 = \frac{2}{3}, x_2 = -\frac{2}{3}$

7. Complete the Square and find the Vertex:

- (a) $(x - 2)^2 - 3$, vertex $V = (2, -3)$
- (b) $3(x - 1)^2 + 9$, vertex $V = (1, 9)$
- (c) $2(x + 5)^2 - 47$, vertex $V = (-5, -47)$

8. Solving Quadratic Equations by Factoring or Using the Quadratic Formula:

- (a) $x_1 = 3, x_2 = -3$
- (b) $x_1 = -1, x_2 = 2$
- (c) $x_1 = -\frac{1}{2}, x_2 = 3$
- (d) $x_1 = -6, x_2 = 7$
- (e) $x_1 = x_2 = 3$
- (f) $x_1 = 1 + i, x_2 = 1 - i$
- (g) $x_1 = 1 + 2i, x_2 = 1 - 2i$

9. Solving Inequalities:

- (a) $x > -2$
- (b) $-2 - \sqrt{2} \geq x \geq -2 + \sqrt{2}$
- (c) $-\frac{3}{2} < x < 0$, and $x > 3$
- (d) $-3 < x < -1$, and $x > 3$
- (e) $S := (1, 2) \cup (6, \infty)$
- (f) $x > 3$

10. Solving Inequalities Involving Absolute Values:

- (a) $1 < x < 3$
- (b) $0 \geq x \geq 3$
- (c) $(-\infty, \frac{65}{8}) \cup (\frac{79}{8}, \infty)$
- (d) $[\frac{2}{3}, \infty)$

11. Solve Exponential and Logarithmic Equations:

- (a) $x_1 = -5, x_2 = 2$
- (b) $x = \frac{1}{3}(4 - \log_2 3)$
- (c) $x_1 = \ln(5), x_2 = i\pi$
- (d) $x = \ln(7)$
- (e) $x = e^2 - 2$
- (f) $x = \sqrt{e+1} + 1$
- (g) $x = 1$
- (h) $\frac{2}{\sqrt{e-1}}$
- (i) $x = 2$
- (j) $x = \frac{25(5+\sqrt{73})^2}{4}$

12. Find the Equation of a Line:

- (a) $y = 3/4x + 1/4$
- (b) $y = -15/8x + 5/2$
- (c) $y = -1/6x + 11/12$
- (d) $y = 4x - 3$
- (e) $y = -2x + 10$

13. Find the Distance and Midpoint between Two Points:

- (a) $d= 10, m= (5, 4)$
- (b) $d= \sqrt{289}, m= (0, 5/2)$
- (c) $d= \sqrt{37/9}, m= (-3, 7/6)$

14. Graph the Quadratic Equation (Conic):

- (a) $y = -\frac{1}{2}(x-3)^2$; parabola, opened downwardly with vertex at $(3, 0)$
- (b) $(x - \frac{1}{2})^2 + (y + \frac{3}{4})^2 = 1$; circle of radius $r = 1$ and center at $(\frac{1}{2}, -\frac{3}{4})$.
- (c) $(x - 1)^2 + y^2 = 32$; circle of radius $4\sqrt{2}$ and center at $(1, 0)$.
- (d) $\frac{(x+\frac{4}{3})^2}{\frac{28}{9}} - \frac{(y-1)^2}{7} = 1$; hyperbola with asymptotes $y = 1 \pm \frac{3}{2}(x + \frac{4}{3})$, foci $F_1 = (-\frac{4}{3} - \frac{1}{3}\sqrt{91}, 1)$, $F_2 = (-\frac{4}{3} + \frac{1}{3}\sqrt{91}, 1)$, vertices $V_1 = (-\frac{4}{3} - \frac{2}{3}\sqrt{7}, 1)$, $V_2 = (-\frac{4}{3} + \frac{2}{3}\sqrt{7}, 1)$ and center $(-\frac{4}{3}, 1)$.

15. Find the Composition of Two Functions:

- (a) $(g \circ f)(x) = \frac{16}{x^3} + \frac{8}{x^2} + 1, (f \circ g)(x) = \frac{1}{x^3+x^2+1/2}$
- (b) $(g \circ f)(x) = x, (f \circ g)(x) = x$

16. Find the Inverse of a Function:

- (a) $\frac{3x-3}{2-3x}$
- (b) $f^{-1}(x) = \sqrt{4-x^2}$
- (c) $\frac{-2x-1}{1-x}$

(d) f is not injective (one-to-one) on \mathbb{R} , since, for example, $f(\frac{1}{2}) = f(-\frac{19}{2})$, as can be easily verified. Therefore, f is not invertible on \mathbb{R} . However, the restriction of f to $[-\frac{9}{2}, \infty)$ is a bijection from $[-\frac{9}{2}, \infty)$ onto $[-\frac{29}{4}, \infty)$ whose inverse $f^{-1} : [-\frac{29}{4}, \infty) \rightarrow [-\frac{9}{2}, \infty)$ is given by $f^{-1}(x) = -\frac{9}{2} + \sqrt{x + \frac{29}{4}}$ for all $x \in [-\frac{29}{4}, \infty)$.

- (e) $\frac{\sqrt[3]{x+1}}{4}$

17. Find the Domain and Range of a Function:

- (a) $D:(-\infty, \infty); R:[0, \infty)$
- (b) $D:[3, \infty]; R:[-1, \infty)$
- (c) $D:[-2, 2]; R:[0, 2]$
- (d) $D:(-\infty, 3/5), (3/5, \infty); R:(-\infty, \infty)$
- (e) $D_f = [-3, 3] \setminus \{1, 2\},$
 $\text{range}(f) = (-\infty, -6\sqrt{3}] \cup [0, \infty)$
- (f) $D:(-\frac{1}{2}, \frac{1}{2}); R:[1, \infty)$

18. Determine Whether a Function is Even, Odd or Neither:

- (a) even
- (b) odd
- (c) even
- (d) odd
- (e) odd
- (f) odd
- (g) neither
- (h) neither

19. Graph Exponential and Logarithmic Functions:

- (a) $D:(-5, \infty), R:(-\infty, \infty)$
- (b) $D:(-\infty, \infty), R:[4, \infty)$
- (c) $D:(5, \infty), R:(-\infty, \infty)$

20. Manipulate Complex Numbers:

- (a) $6 - 22i$
- (b) -10
- (c) $4 + i$
- (d) $1 - i$
- (e) $\frac{62}{949} + \frac{297}{949}i$

21. Finding Powers and Roots of Complex Numbers Using D'Moivre's Theorem:

- (a) 8
- (b) -8

(c) $-i$

(d) Let $\varphi = \arg(16+12i)$, then the two square roots of $16 + 12i$ are given by

$$z_0 = 2\sqrt{5}e^{\frac{\varphi}{2}i}$$

$$z_1 = 2\sqrt{5}e^{\frac{\varphi}{2}i+\pi i}$$

(e) The six roots of 64 are

$$z_0 = 2$$

$$z_1 = 1 + i\sqrt{3}$$

$$z_2 = -1 + i\sqrt{3}$$

$$z_3 = -2$$

$$z_4 = -1 - i\sqrt{3}$$

$$z_5 = 1 - i\sqrt{3}.$$