Review
Fundamental Concepts and Techniques of Calculus

Solutions to the Exercises: Differential Calculus

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1. Two-Sided Limits: Find the Following Limits if they Exist:
   (a) 2
   (b) 0
   (c) $-\frac{1}{3}$
   (d) -1
   (e) 3
   (f) $\frac{1}{8}\sqrt{2}$
   (g) $\frac{2}{3}$
   (h) 0
   (i) -1
   (j) $\frac{2}{3}$
   (k) -4
   (l) 0
   (m) 0

2. One-Sided Limits: Find the Following Limits if they Exist:
   (a) -1
   (b) 0
   (c) ∞
   (d) -∞
   (e) 0
   (f) 0

3. Limits at ∞: Find the Following Limits if they Exist:
   (a) 1
   (b) $\frac{1}{2}$
   (c) -1
   (d) 0
   (e) $\frac{5}{4}$
   (f) ∞
   (g) 2
   (h) 1
   (i) 0

4. Squeeze Play, Pinching Theorem: Find the Following Limits:
   (a) 0
   (b) 0
   (c) 0
   (d) 0
   (e) 0
   (f) 0

5. L'Hopital's Rule: Find the Following Limits:
   (a) ∞
   (b) e^a
   (c) $\frac{\pi}{\ln(e)}$
   (d) 0
   (e) -$\frac{3}{2}$
   (f) 0
   (g) 1
   (h) 1
   (i) e^3
   (j) 1
   (k) 0
   (l) 0
   (m) -$\frac{4}{7}$

6. Continuity, Removable Discontinuities, and Jump Discontinuities:
   (a) no discontinuities
   (b) infinite discontinuity at $x = -3$, removable discontinuity at $x = -1$
   (c) infinite discontinuity at $x = -4$
   (d) infinite discontinuity at $x = 3$
   (e) no, division by zero is invalid
   (f) no, division by zero is invalid
   (g) no, division by zero is invalid

7. Recall and interpret the Intermediate Value Theorem: The real valued, continuous function f is defined on the closed and bounded interval $[a, b]$. Which of the following must be true?
   (a) true
   (b) false
   (c) true
   (d) false
   (e) false

- (f) false
- (g) true
- (h) false

8. Pick the correct Statement of the Mean Value theorem:
   - (a) false
   - (b) true
   - (c) false
   - (d) false
   - (e) false
   - (f) false
   - (g) false
   - (h) true
   - (i) false
   - (j) false

9. Recall and Interpret the Extreme Value Theorem The real valued, continuous function f is defined on the closed and bounded interval $[a, b]$. Which of the following must be true?
   - (a) false
   - (b) false
   - (c) false
   - (d) false
   - (e) false
   - (f) false
   - (g) false
   - (h) false
   - (i) true

10. Find the Equation for the Lines Tangent and Normal (Perpendicular) to the given Curve at the given Point
    - (a) Tangent line: $T(x) = \left(\frac{1}{2}\sqrt{3}\right)x + \left(\frac{1}{2} - \frac{\pi}{12}\sqrt{3}\right)$
      Normal line: $N(x) = \left(-\frac{2}{3}\sqrt{3}\right)x + \left(\frac{1}{2} + \frac{\pi}{9}\sqrt{3}\right)$.
    - (b) Tangent line: $T(x) = \frac{1}{6}x + \frac{3}{2}$
      Normal line: $N(x) = -6x + 16$.
    - (c) Tangent line: $T(x) = x$
      Normal line: $N(x) = -x$.
    - (d) Tangent line: $T(x) = \frac{1}{3}x + \ln 3$
      Normal line: $N(x) = -3x + (10 + \ln 3)$.

11. The Derivative:
    - (a) We say that f is *differentiable* at c if the limit

      $$\lim_{x \to c} \frac{f(x) - f(c)}{x - c}$$

      exists. If the limit exists, it is called the *derivative* of f at c and denoted by $f'(c)$.
    - (b) Use the Definition to compute the Derivative of the following Functions at x_0:
      - i. $\frac{1}{x+1} - \frac{x}{(x+1)^2}; \frac{1}{9}$
      - ii. $\frac{-1}{2x\sqrt{x}}; \frac{-1}{16}$
      - iii. $\cos(x); 1$
      - iv. $2x\sin(\frac{1}{x}) - \cos(\frac{1}{x}) - 2x + 1$
    - (c) Determine whether each Function is (α) differentiable (β) continuous at x_0:
      - i. differentiable, continuous
      - ii. differentiable, continuous
      - iii. continuous

12. Taking Derivatives Using the Differentiation Rules:
    - (a) Polynomials and Rational Functions
      - i. $16x^3 - 12x^2 + 2x$
      - ii. $-\frac{1}{x^2\sqrt{3+2x^2}} - \frac{2}{(3+2x^2)^{\frac{3}{2}}}$
      - iii. $\frac{2x}{x^3+1} - \frac{3(x^2-1)x^2}{(x^3+1)^2}$
      - iv. $4((\frac{1}{x} + \frac{2}{x^2})^3(3(\frac{1}{x} + \frac{2}{x^2})^2(-\frac{1}{x^2} - \frac{4}{x^3}) - \frac{9}{x^4})$
      - v. $\frac{2x+1}{x^2-x+1} - \frac{(x^2+x+1)(2x-1)}{(x^2-x+1)^2}$
      - vi. $\frac{2t+5}{t^2+t-20} - \frac{(t^2+5t+4)(2t+1)}{(t^2+t-20)^2}$
      - vii. $\frac{5x^4-2x+3x^2}{x^4+1} - \frac{4(x^5+x^3-x^2-1)x^3}{(x^4+1)^2}$
      - viii. $\frac{3}{x^2} - \frac{2}{x^3}$
      - ix. $5(x^2 - \frac{1}{x^3+1})^4(2x + \frac{3x^2}{(x^3+1)^2})$
      - x. $\frac{12(4x+3)^2}{(5x-2)^3} - \frac{15(4x+3)^3}{(5x-2)^4}$
    - (b) Trigonometric Functions
      - i. $\cos^2(x) - \sin^2(x)$
      - ii. $2x\cos(x) - x^2\sin(x)$
      - iii. $-\frac{\sec(x)\tan(X)}{\tan^2(x)+1}$
      - iv. $\frac{2}{3}\sec(\tan(x))^{(2/3)}\tan(\tan(x))(\tan^2(x) + 1)$
      - v. $(1+\tan^2(\sec(x+1)))\sec(x+1)\tan(x+1)$
      - vi. $4x\tan(x)\sec(x) + 2x^2(\tan^2(x) + 1)\sec(x) + 2x^2\tan^2(x)\sec(x)$
      - vii. $-\sin(\sin(x))\cos(x)$
      - viii. $\frac{\cos(x)}{3(1+\sin(x))^{(2/3)}}$
      - ix. $\frac{1}{4}\sec^{\frac{1}{4}}(\tan(x))\tan(\tan(x))(\tan^2(x)+1)$
      - x. $\frac{6\cot(2\sqrt{3x+1})(-1-\cot^2(2\sqrt{3x+1}))}{\sqrt{3x+1}}$
      - xi. $2x\sec(\frac{1}{x^3}) - \frac{3\sec(\frac{1}{x^3})\tan(\frac{1}{x^3})}{x^2}$
      - xii. $-6\cos^2(2x)\tan(4x)\sin(2x)+\cos^3(2x)(4+4\tan^2(4x))$
    - (c) Exponential and Logarithmic Functions

i. $52^{(5x)}\ln(2)\cdot 3^{\ln(x)}+\frac{2^{(5x)}3^{\ln(x)}\ln(3)}{x}$
ii. $\frac{2}{(2x+1)}$
iii. $(x+1)^x(\ln(x+1)+\frac{x}{x+1})$
iv. $x^{x^x}(\ln(x^x)+x\ln(x)+x)$

(d) Misc.

i. $-\sin(2^x+2^{-x})(2^x\ln(2)-2^{-x}\ln(2))$
ii. $\tan(x)^{\ln(x^x)}[(\ln(x)+1)\ln(\tan(x))+\frac{\ln(x^x)(\tan^2(x)+1)}{\tan(x)}]$
iii. $\sin(x)^{\cos(x)}[-\sin(x)\ln(\sin(x))+\frac{\cos^2(x)}{\sin(x)}]$
iv. $\frac{2xe^{(x^2+1)}}{\sqrt{-x^4-2x^2}}+2\arcsin(x^2+1)\cdot e^{(x^2+1)}x\ln(e)$
v. no solution
vi. $\frac{\sec(x)\tan(x)+1+\tan^2(x)}{|\sec(x)+\tan(x)|}$
vii. $-\frac{3}{2}[1+\tan^2(\sqrt{e^{-3x}})]\sqrt{e^{-3x}}$
viii. $2\sec(e^{\tan(4+x^2)})\tan(e^{\tan(4+x^2)})e^{\tan(4+x^2)}(1+\tan^2(4+x^2))x$
ix. $\frac{4e^{4\ln(x)}}{x}$
x. $2e^{\sin(2x)}\cdot\ln(\cos(e^{2x}))+e^{\sin(2x)}\cdot-\frac{2\sin(e^{2x})e^{2x}}{\cos(e^{2x})}$
xi. $\frac{1}{x(\ln(x))^3\sqrt{1-\frac{1}{(\ln(x))^2}}}$
xii. $-\frac{-3+8x}{3\sqrt{1-9x^2+24x^3-16x^4}}$
xiii. $-\frac{1}{x(\ln(x))^2}$
xiv. $-\frac{e^{\text{arccot}(x)}ln(e)}{x^2+1}$
xv. $\frac{1-x^2}{x^2+(x^2+1)^2}$

13. Implicit Differentiation

(a) Find dy/dx in terms of x and y:

i. $4\sqrt{x+y}\cdot\cos^2(\sqrt{x+y})-1$
ii. ± 1
iii. $-\frac{\sqrt[3]{y}}{\sqrt[3]{x}}$
iv. $\frac{cos^2(xy)-y}{x}$
v. $\frac{36x-8xy+3y-2x}{8yx^2-3x}$

(b) Find Equations for the Tangent and Normal Lines at the indicated Point, respectively:

i. $y=-\frac{3}{2}x+6$; $y=\frac{2}{3}x+\frac{5}{3}$
ii. $y=-x+3$; $y=x-1$
iii. $y=\frac{1}{3}[2\sqrt{3}x+\pi-\sqrt{3}]$; $y=\frac{\sqrt{3}x}{2}-\frac{\sqrt{3}}{4}+\frac{\pi}{3}$

14. Find the Critical Numbers and Classify the Extreme Values (as Local/Global):

(a) critical point at (2,-3), global minimum
(b) critical points at $(\pm\frac{1}{\sqrt{2}},3.75)$, local minimums; critical point at $(0,4)$, local maximum
(c) critical point at $(4,0)$, local minimum; critical point at $(\frac{12}{5},35.4)$, local maximum
(d) critical point at (0,0), global minimum
(e) inflection point exists at $(\frac{1}{6},24)$inflection point exists at $(\frac{1}{6},24)$
(f) critical point at $(-2,-\frac{1}{4})$, global minimum; critical point at $(1,\frac{1}{5})$, global maximum
(g) critical point at $(-\sqrt{2},2)$, global minimum; critical point at $(\sqrt{2},2)$, global maximum
(h) critical point at $(\frac{\pi}{2}+2\pi n)$, local maximum; critical point at $(\frac{3\pi}{2}+2\pi n)$, local minimum; n is any positive integer
(i) critical point at $(-3,2)$ and (0,2), local maximums; critical point at (-2,2), global minimum

15. Describe the Concavity of the Graph of f and find the Points of Inflection:

(a) f is concave up $x>0$, f is concave down $x<0$, with an inflection point at $x=0$
(b) f is concave up $-1<x<1$ and concave down $-1>x>1$, with inflection points at $x=-1,0,$ and 1
(c) f is concave up $x>-1/2$, f is concave down $x<-1/2$, with an inflection point at $x=-1/2$
(d) f is concave up $-2<x<0$ and concave down $0<x<2$, with an inflection point at $x=0$
(e) f has inflection points at $x=\pi\pm 2\pi n$, where n is any integer
(f) f is concave down $(-\infty,0)$, concave up $(0,1)$, and linear $[1,\infty)$, inflection point at $x=0$

16. Find the Intervals on which f increases and the Interval on which f decreases:

(a) f increases $(-\infty,-1)$ and $(1,\infty)$, f decreases $(-1,1)$
(b) f increases $(-\infty,-\frac{14}{9})$ and $(0,\infty)$, f decreases $(\frac{-14}{9},0)$
(c) f increases $(-\infty,-1)$ and $(-1,0)$, f decreases $(0,1)$ and $(1,\infty)$
(d) f increases for all x

17. Sketch the Graph of the Function f:

(a) inflection point at (1,1)
(b) asymptote at $y=0$ and $x=\pm 0$
(c) asymptote at $x=-1$, increases $(-\infty,-1)$ and $(0,\infty)$, decreases $(-1,0)$
(d) local maximum at $(\pm 1,3)$, local minimum at (0,2)

(e) decreases $(-\infty, \frac{5}{6})$, increases $(\frac{5}{6}, \infty)$

(f) local maximum at (0,1); local minimum at $(\pm 2, \frac{1}{2})$

18. Maximun/Minimun Problems

(a) $h = 2\sqrt{3}$ units, $b = 5$ units

(b) $(\pm\sqrt{2}, 2)$

(c) $2\sqrt{6} + \frac{12}{\sqrt{6}}$ ft.

(d) 106 x 250 ft.

(e) -$\frac{\sqrt{3}}{3}$

(f) $\frac{3\sqrt{3}}{4}r^2$ units^2

(g) $\frac{3(1+\sqrt{3})}{r\sqrt{2}}$ units^2 paper/volume

(h) 4.0 cubic feet