

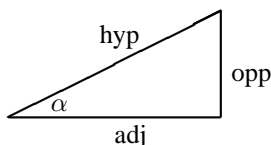
Review
Fundamental Concepts and Techniques of Calculus

R 2: Summary: Trigonometry

Last updated: 040730

1. Trigonometric Functions: Geometric definitions:

- (a) $\sin \alpha = \frac{\text{opp}}{\text{hyp}}$
- (b) $\cos \alpha = \frac{\text{adj}}{\text{hyp}}$
- (c) $\tan \alpha = \frac{\text{opp}}{\text{adj}}$
- (d) $\cot \alpha = \frac{\text{adj}}{\text{opp}}$
- (e) $\sec \alpha = \frac{\text{hyp}}{\text{adj}}$
- (f) $\csc \alpha = \frac{\text{hyp}}{\text{opp}}$



2. Basic Trigonometric Identities

- (a) $\sin(-x) = -\sin x$
- (b) $\cos(-x) = \cos x$
- (c) $\sin(x + 2\pi) = \sin x$
- (d) $\cos(x + 2\pi) = \cos x$
- (e) $\tan x = \frac{\sin x}{\cos x}$
- (f) $\cot x = \frac{\cos x}{\sin x}$
- (g) $\sec x = \frac{1}{\cos x}$
- (h) $\csc x = \frac{1}{\sin x}$
- (i) $\cot x = \frac{1}{\tan x}$

3. Pythagorean Identities

- (a) $\cos^2 x + \sin^2 x = 1$
- (b) $1 + \tan^2 x = \sec^2 x$
- (c) $1 + \cot^2 x = \csc^2 x$

4. Addition formulae

- (a) $\sin(x + y) = \sin x \cos y + \sin y \cos x$
- (b) $\cos(x + y) = \cos x \cos y - \sin x \sin y$
- (c) $\tan(x + y) = \frac{\tan x + \tan y}{1 - \tan x \tan y}$

5. Double-Angle Identities

- (a) $\sin 2x = 2 \sin x \cos x$
- (b) $\cos 2x = \cos^2 x - \sin^2 x$
- (c) $\tan 2x = \frac{2 \tan x}{1 - \tan^2 x}$

6. Power Identities

- (a) $\sin^2 x = \frac{1}{2}(1 - \cos 2x)$
- (b) $\cos^2 x = \frac{1}{2}(1 + \cos 2x)$

7. Products of Functions

- (a) $\sin x \sin y = \frac{1}{2} \cos(x - y) - \frac{1}{2} \cos(x + y)$
- (b) $\sin x \cos y = \frac{1}{2} \sin(x - y) + \frac{1}{2} \sin(x + y)$
- (c) $\cos x \cos y = \frac{1}{2} \cos(x - y) + \frac{1}{2} \cos(x + y)$

8. Half-Angle Identities

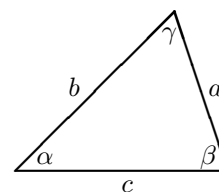
- (a) $\sin \frac{x}{2} = \pm \sqrt{\frac{1 - \cos x}{2}}$
- (b) $\cos \frac{x}{2} = \pm \sqrt{\frac{1 + \cos x}{2}}$
- (c) $\tan \frac{x}{2} = \frac{\sin x}{1 + \cos x} = \frac{1 - \cos x}{\sin x}$

9. Phase Shift

- (a) $\sin\left(\frac{\pi}{2} - x\right) = \cos x$
- (b) $\cos\left(\frac{\pi}{2} - x\right) = \sin x$
- (c) $\tan\left(\frac{\pi}{2} - x\right) = \cot x$

10. The Law of Sines:

- (a) $\frac{\sin \alpha}{\sin \beta} = \frac{a}{b}$,
- (b) $\frac{\sin \alpha}{\sin \gamma} = \frac{a}{c}$,
- (c) $\frac{\sin \beta}{\sin \gamma} = \frac{b}{c}$,



11. The Law of Cosines:

- (a) $a^2 = b^2 + c^2 - 2bc \cos \alpha$
- (b) $b^2 = a^2 + c^2 - 2ac \cos \beta$
- (c) $c^2 = a^2 + b^2 - 2ab \cos \gamma$

12. Values at Special Angles

x	$\sin x$	$\cos x$	$\tan x$	$\cot x$
0	0	1	0	∞
$\frac{\pi}{6}$	$\frac{1}{2}$	$\frac{1}{2}\sqrt{3}$	$\frac{1}{3}\sqrt{3}$	$\sqrt{3}$
$\frac{\pi}{4}$	$\frac{1}{2}\sqrt{2}$	$\frac{1}{2}\sqrt{2}$	1	1
$\frac{\pi}{3}$	$\frac{1}{2}\sqrt{3}$	$\frac{1}{2}$	$\sqrt{3}$	$\frac{1}{3}\sqrt{3}$
$\frac{\pi}{2}$	1	0	∞	0

13. Euler identities

- (a) $e^{ix} = \cos x + i \sin x$
- (b) $\cos x = \frac{1}{2}(e^{ix} + e^{-ix})$
- (c) $\sin x = \frac{1}{2i}(e^{ix} - e^{-ix})$

14. Hyperbolic Functions

- (a) $\cosh x = \frac{1}{2}(e^x + e^{-x})$
- (b) $\sinh x = \frac{1}{2}(e^x - e^{-x})$
- (c) $\tanh x = \frac{\sinh x}{\cosh x}$
- (d) $\coth x = \frac{\cosh x}{\sinh x}$
- (e) $\operatorname{sech} x = \frac{1}{\cosh x}$
- (f) $\operatorname{csch} x = \frac{1}{\sinh x}$

15. Basic Identities for hyperbolic functions:

- (a) $\sinh(-x) = -\sinh x$
- (b) $\cosh(-x) = \cosh x$

16. Pythagorean Identities for hyperbolic functions:

- (a) $\cosh^2 x - \sinh^2 x = 1$
- (b) $\tanh^2 x + \operatorname{sech}^2 x = 1$

17. Addition formulae for hyperbolic functions:

- (a) $\sinh(x + y) = \sinh x \cosh y + \sinh y \cosh x$
- (b) $\cosh(x + y) = \cosh x \cosh y + \sinh x \sinh y$
- (c) $\tanh(x + y) = \frac{\tanh x + \tanh y}{1 + \tanh x \tanh y}$

18. Double-Angle Identities for hyperbolic functions:

- (a) $\sinh 2x = 2 \sinh x \cosh x$
- (b) $\cosh 2x = \cosh^2 x + \sinh^2 x$
- (c) $\tanh 2x = \frac{2 \tanh x}{1 + \tanh^2 x}$

19. Power Identities for hyperbolic functions:

- (a) $\sinh^2 x = \frac{1}{2}(\cosh 2x - 1)$
- (b) $\cosh^2 x = \frac{1}{2}(\cosh 2x + 1)$

20. Half-Angle Identities for hyperbolic functions:

- (a) $\sinh \frac{x}{2} = \operatorname{sgn}(x) \sqrt{\frac{\cosh x - 1}{2}}$
- (b) $\cosh \frac{x}{2} = \sqrt{\frac{\cosh x + 1}{2}}$
- (c) $\tanh \frac{x}{2} = \frac{\sinh x}{\cosh x + 1} = \frac{\cosh x - 1}{\sinh x}$

21. Inverse functions

- (a) $\sinh^{-1}(x) = \ln(x + \sqrt{x^2 + 1})$
- (b) $\cosh^{-1}(x) = \ln(x \pm \sqrt{x^2 - 1})$,
(+: $x \geq 1$; -: $x \leq -1$)

22. Polar coordinates:

- (a) The point P has polar coordinates $[r, \theta]$ if and only if it has distance $|r|$ from the pole and lies on the ray θ if $r \geq 0$, and on the ray $\theta + \pi$ if $r < 0$.
- (b) If P has polar coordinates $[r, \theta]$, then its cartesian coordinates are $(r \cos \theta, r \sin \theta)$.
- (c) If P has cartesian coordinates (x, y) then its polar coordinates are $[\sqrt{x^2 + y^2}, \theta]$, where

$$\theta = \begin{cases} 0 & \text{if } x = y = 0 \\ \frac{\pi}{2} & \text{if } x = 0 \text{ and } y > 0 \\ -\frac{\pi}{2} & \text{if } x = 0 \text{ and } y < 0 \\ \tan^{-1}\left(\frac{y}{x}\right) & \text{if } x > 0 \\ \tan^{-1}\left(\frac{y}{x}\right) + \pi & \text{if } x < 0 \end{cases}$$

23. n -th roots:

- (a) Let $a \in \mathbb{C}$. Then any solution of the equation $z^n = a$ is called a n -th root of a .
- (b) The n n -th roots of the number $a \in \mathbb{C}$ are given by

$$z_k = \sqrt[n]{|a|} e^{\frac{\arg(a) + 2\pi k i}{n}}$$

for $k = 0, 1, \dots, n-1$. Here $\arg(a)$ denotes the argument of a (polar angle).