Review
Fundamental Concepts and Techniques of Calculus

R 3: Summary: Differential Calculus

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1. Definitions of limits:

   (a) *Two-sided limit:* Suppose the function f is defined in a neighborhood of $c \in \mathbb{R}$. We say that f *has limit* L *as* x *approaches* c and write
   $$\lim_{x\to c} f(x) = L$$
   if and only if for all $\varepsilon > 0$ there exists a $\delta > 0$ such that for all $x \in D_f$
   $$0 < |x - c| < \delta \Rightarrow |f(x) - L| < \varepsilon$$
   (D_f denotes the domain of f).

   (b) *Left-sided limit:* Suppose the function f is defined in a half-neighborhood of the form (a, c) of $c \in \mathbb{R}$. We say that f *has limit* L *as* x *approaches* c *from the left* and write
   $$\lim_{x\to c^-} f(x) = L$$
   if and only if for all $\varepsilon > 0$ there exists a $\delta > 0$ such that for all $x \in D_f$
   $$0 < c - x < \delta \Rightarrow |f(x) - L| < \varepsilon$$

   (c) *Right-sided limit:* Suppose the function f is defined in a half-neighborhood of the form (c, b) of $c \in \mathbb{R}$. We say that f *has limit* L *as* x *approaches* c *from the right* and write
   $$\lim_{x\to c^+} f(x) = L$$
   if and only if for all $\varepsilon > 0$ there exists a $\delta > 0$ such that for all $x \in D_f$
   $$0 < x - c < \delta \Rightarrow |f(x) - L| < \varepsilon$$

   (d) *Limit at* ∞*:* Suppose the function f is defined on an interval of the form (a, ∞). We say that f *has limit* L *as* x *approaches* ∞ and write
   $$\lim_{x\to\infty} f(x) = L$$
   if and only if for all $\varepsilon > 0$ there exists a $B > 0$ such that for all $x \in (a, \infty)$
   $$x > B \Rightarrow |f(x) - L| < \varepsilon$$

   (e) *Limit at* $-\infty$*:* Suppose the function f is defined on an interval of the form $(-\infty, b)$. We say that f *has limit* L *as* x *approaches* $-\infty$ and write
   $$\lim_{x\to-\infty} f(x) = L$$
   if and only if for all $\varepsilon > 0$ there exists a $B > 0$ such that for all $x \in (-\infty, b)$
   $$x < -B \Rightarrow |f(x) - L| < \varepsilon$$

   (f) *Positive Infinite Limit:* Suppose the function f is defined in a neighborhood of $c \in \mathbb{R}$. We say that f *approaches* ∞ *as* x *approaches* c and write
   $$\lim_{x\to c} f(x) = \infty$$
   if and only if for all $B > 0$ there exists a $\delta > 0$ such that for all $x \in D_f$
   $$0 < |x - c| < \delta \Rightarrow f(x) > B$$

   (g) *Negative Infinite Limit:* Suppose the function f is defined in a neighborhood of $c \in \mathbb{R}$. We say that f *approaches* $-\infty$ *as* x *approaches* c and write
   $$\lim_{x\to c} f(x) = -\infty$$
   if and only if for all $B > 0$ there exists a $\delta > 0$ such that for all $x \in D_f$
   $$0 < |x - c| < \delta \Rightarrow f(x) < -B$$

2. Limit theorems: Techniques for computing limits:

   (a) Limits and Algebraic Operations:
   Suppose the functions f and g have limits at c and suppose $k \in \mathbb{R}$, then

      i. $\lim\limits_{x\to c}\big(f(x)+g(x)\big) = \lim\limits_{x\to c} f(x) + \lim\limits_{x\to c} g(x)$.

      ii. $\lim\limits_{x\to c}\big(kf(x)\big) = k\lim\limits_{x\to c} f(x)$.

      iii. $\lim\limits_{x\to c}\big(f(x)g(x)\big) = \lim\limits_{x\to c} f(x)\lim\limits_{x\to c} g(x)$.

iv. $\lim_{x\to c}\dfrac{f(x)}{g(x)} = \dfrac{\lim_{x\to c} f(x)}{\lim_{x\to c} g(x)}$, provided that $\lim_{x\to c} g(x) \neq 0$.

v. If $\lim_{x\to c} f(x) \neq 0$ and $\lim_{x\to c} g(x) = 0$, then $\lim_{x\to c}\dfrac{f(x)}{g(x)}$ does not exist.

(b) Limits of inverse functions:
If $\lim_{x\to x_0} f(x) = y_0$ and f has an inverse near x_0 and $\lim_{x\to y_0} f^{-1}(y)$ exists. Then $\lim_{x\to y_0} f^{-1}(y) = x_0$.

(c) Substitution Rule:
If $\lim_{x\to x_0} f(x) = y_0$ and $\lim_{y\to y_0} g(y) = g(y_0)$, then

$$\lim_{x\to x_0} g\big(f(x)\big) = g\Big(\lim_{x\to x_0} f(x)\Big) = g(y_0)$$

(d) Limits and Inequalities:
Iff and g have limits at x_0 and if $f(x) \leq g(x)$ near x_0 then

$$\lim_{x\to x_0} f(x) \leq \lim_{x\to x_0} g(x).$$

(e) Squeeze Play: If $\lim_{x\to c} f(x) = \lim_{x\to c} g(x) = L$ and if $f(x) \leq h(x) \leq g(x)$ near c, then $\lim_{x\to c} h(x) = L$.

(f) If $\lim_{x\to c} f(x) = 0$ and if g is bounded around c, then $\lim_{x\to c} f(x)g(x) = 0$.

(g) L'Hôpital's Rules: (can be applied several times!)

i. "$\frac{0}{0}$": If f and g are differentiable in a neighborhood of $c \in \mathbb{R}\cup\{\pm\infty\}$, $\lim_{\to c} f(x) = \lim_{\to c} g(x) = 0$ and if

$$\frac{\lim_{x\to c} f'(x)}{\lim_{x\to c} g'(x)}$$

exists then

$$\frac{\lim_{x\to c} f(x)}{\lim_{x\to c} g(x)} = \frac{\lim_{x\to c} f'(x)}{\lim_{x\to c} g'(x)}$$

ii. "$\frac{\infty}{\infty}$": If f and g are differentiable in a neighborhood of $c \in \mathbb{R}\cup\{\pm\infty\}$, $\lim_{\to c} f(x) = \pm\lim_{\to c} g(x) = \pm\infty$ and if

$$\frac{\lim_{x\to c} f'(x)}{\lim_{x\to c} g'(x)}$$

exists then

$$\frac{\lim_{x\to c} f(x)}{\lim_{x\to c} g(x)} = \frac{\lim_{x\to c} f'(x)}{\lim_{x\to c} g'(x)}$$

(h) Half- and two-sided limits:

i. If $\lim_{\to c^-} f(x) = \lim_{\to c^+} f(x) = L$, then $\lim_{\to c} f(x) = L$.

ii. If $\lim_{\to c} f(x) = L$, then $\lim_{\to c^-} f(x) = \lim_{\to c^+} f(x) = L$.

iii. If $\lim_{\to c^-} f(x)$ or $\lim_{\to c^+} f(x)$ fail to exist, then $\lim_{\to c} f(x)$ fails to exist.

iv. If $\lim_{\to c^-} f(x) \neq \lim_{\to c^+} f(x)$, then $\lim_{\to c} f(x)$ fails to exist.

3. Fundamental limits:

(a) $\lim_{x\to c} k = k$.

(b) $\lim_{x\to c} x = c$

(c) If p is a polynomial, i.e. a function of the form $p(x) = a_0 + a_1x + a_2x^2 + \cdots + a_nx^n$, then $\lim_{x\to c} p(x) = p(c)$.

(d) If r is a rational function (quotient of polynomials), and $c \in D_r$ (i.e. not a zero of the denominator of r) then $\lim_{x\to c} r(x) = r(c)$.

(e) $\lim_{x\to c} x^{\frac{1}{n}} = c^{\frac{1}{n}}$, provided $x \geq 0$ if n is even.

(f) $\lim_{x\to c} \sin x = \sin c$.

(g) $\lim_{x\to c} \cos x = \cos c$

(h) $\lim_{x\to c} e^x = e^c$.

(i) $\lim_{x\to c} \ln x = \ln c, \quad x > 0$.

(j) $\lim_{x\to 0} \dfrac{\sin x}{x} = 1$.

(k) $\lim_{x\to 0} \dfrac{\cos x - 1}{x} = 0$

(l) $\lim_{x\to\pm\infty} \dfrac{1}{x} = 0$

(m) $\lim_{x\to 0^+} \dfrac{1}{x} = \infty, \quad \lim_{x\to 0^-} \dfrac{1}{x} = -\infty$.

(n) $\lim_{x\to\infty} e^x = \infty, \quad \lim_{x\to-\infty} e^x = 0$.

(o) $\lim_{x\to\infty} \ln x = \infty, \quad \lim_{x\to 0^+} \ln x = -\infty$.

4. *Extrema*

(a) *(Global) Maximum, Minimum:* We say that the function f has a *(global) maximum* at $c \in D_f$ if $f(x) \leq f(c)$ for all $x \in D_f$; $f(c)$ is then called the *maximum value* of f. Similarly, we say that f has a *(global) minimum* at $c \in D_f$ if $f(x) \geq f(c)$ for all $x \in D_f$; $f(c)$ is then called the *minimum value* of f.

(b) *Local Maximum, Minimum:* We say that the function f has a *local maximum* at c if there is an interval (a,b) containing c such that $f(x) \leq f(c)$ for all $x \in (a,b)$. Similarly, we say that f has a *local minimum* at c if there is an interval (a,b) containing c such that $f(x) \geq f(c)$ for all $x \in (a,b)$.

(c) *Extremum:* A *(local or global) extremum* of a function f is a (local or global) maximum or minimum of f.

5. Continuity:

(a) *Continuous:* A function f is *continuous at* c if and only if for any $\varepsilon > 0$ there is a $\delta > 0$ such that for all $x \in D_f$ (D_f denotes the domain of f)

$$|x-c| < \delta \Rightarrow |f(x)-f(c)| < \varepsilon$$

If $S \subseteq D_f$ is a subset of the domain D_f of f, then we say that f is *continuous on* S if and only if f is continuous at every point $c \in S$.

(b) If f is an interior point of D_f then

$$f \text{ is continuous at } c \Leftrightarrow \lim_{x\to c} f(x) = f(c)$$

(c) *Algebraic Operations:* If f and g are continuous at c, then

i. $f+g$ is continuous at c

ii. $f \cdot g$ is continuous at c

iii. $\alpha \cdot f$ is continuous at c for any $\alpha \in \mathbb{R}$

iv. f/g is continuous at c provided $g(c) \neq 0$

(d) *Composition:* If f is continuous at c and g is continuous at $f(c)$ then the composition $g \circ f$ is continuous at c.

(e) *Discontinuities:* A point $c \in \mathbb{R}$ is called a *point of disontinuity* of the function f if and only if f is not continuous at c.

Moreover, if $\lim_{x\to c} f(x)$ exists, then we call the point c a *removable* discontinuity (or removable singularity) of f otherwise it is called an *essential* discontinuity of f.

An essential discontinuity c of f is called a *jump discontinuity* of f if $\lim_{x\to c^-} f(x)$ and $\lim_{x\to c^+} f(x)$ exist (and are distinct) otherwise it is called an *infinite* discontinuity.

6. *Extreme Value Theorem:* If the function f is continuous on the closed interval $[a,b]$, then there exist numbers c_1 and c_2 in $[a,b]$ such that f has a maximum at c_1 and a minimum at c_2.

7. *Intermediate-value Theorem:* If the function f is continuous on the closed interval $[a,b]$ and d is any value between the minimum and maximum value of f then there exists a number $c \in [a,b]$ such that $f(c) = d$.

8. The derivative:

(a) *Differentiable, Derivative:* Suppose the function f is defined in an open interval containing the point c. We say that f is *differentiable* at c if the limit

$$\lim_{x\to c} \frac{f(x)-f(c)}{x-c}$$

exists. If the limit exists, it is called the *derivative* of f at c and denoted by $f'(c)$.

If f is differentiable at all $c \in S \subseteq D_f$, then we say, f is *differentiable on* the set S. If f is differentiable on its domain, we simply say f is *differentiable.*

(b) *Differentiability Implies Continuity:* If the function f is differentiable at c then it is continuous at c.

(c) *Differential:* Given a differentiable function $y = f(x)$, the *differential of* x is denoted by dx, the *differential of* y is defined by

$$df = dy := f'(x)dx.$$

9. Differentiation Rules:

(a) *Summation Rule:* Suppose f and g are differentiable at x then $f+g$ is differentiable at x and

$$(f+g)'(x) = f'(x) + g'(x)$$

(b) *Scalar Multiple Rule:* Suppose f is differentiable at x and $\alpha \in \mathbb{R}$, then αf is differentiable at x and

$$(\alpha f)'(x) = \alpha f'(x)$$

(c) *Product Rule:* Suppose f and g are differentiable at x then $f \cdot g$ is differentiable at x and

$$(fg)'(x) = f'(x)g(x) + f(x)g'(x)$$

(d) *Quotient Rule:* Suppose f and g are differentiable at x and $g(x) \neq 0$, then f/g is differentiable at x and

$$\left(\tfrac{f}{g}\right)'(x) = \frac{f'(x)g(x) - f(x)g'(x)}{\big(g(x)\big)^2}$$

(e) *The Chain Rule:* If the function f is differentiable at x and the function g is differentiable at $f(x)$, then the compositon $g \circ f$ is differentiable at x and

$$(g \circ f)'(x) = g'\big(f(x)\big) \cdot f'(x)$$

(f) *The Inverse Function Rule:* If f is differentiable with inverse function f^{-1} then f^{-1} is differentiable at x if $f'\big(f^{-1}(x)\big) \neq 0$ and

$$\left(f^{-1}\right)'(x) = \frac{1}{f'\big(f^{-1}(x)\big)}$$

10. *Basic Derivatives:*
    (a) $\left(x^r\right)' = rx^{r-1}$, $(r \in \mathbb{R})$
    (b) $\sin' x = \cos x$,
    $\cos' x = -\sin x$
    (c) $\tan' x = \sec^2 x$,
    $\cot' x = -\csc^2 x$
    (d) $\sec' x = \sec x \tan x$,
    $\csc' x = -\csc x \cot x$
    (e) $\left(e^x\right)' = e^x$,
    $\left(a^x\right)' = a^x \ln a$
    (f) $\ln' x = \dfrac{1}{x}$,
    $\log_a' x = \dfrac{1}{x \ln a}$
    (g) $\sinh' x = \cosh x$,
    $\cosh' = \sinh x$
    (h) $\tanh' x = \operatorname{sech}^2 x$,
    $\coth' x = -\operatorname{csch}^2 x$
    (i) $\operatorname{sech}' x = -\operatorname{sech} x \tanh x$,
    $\operatorname{csch}' x = -\operatorname{csch} x \coth x$
    (j) $\arcsin' x = \dfrac{1}{\sqrt{1-x^2}}$,
    $\arccos' x = \dfrac{-1}{\sqrt{1-x^2}}$
    (k) $\arctan' x = \dfrac{1}{1+x^2}$,
    $\operatorname{arccot}' x = \dfrac{-1}{1+x^2}$
    (l) $\operatorname{arcsec}' x = \dfrac{1}{|x|\sqrt{x^2-1}}$,
    $\operatorname{arccsc}' x = \dfrac{-1}{|x|\sqrt{x^2-1}}$

11. *Rolle's Theorem:* If the function f is continuous on the interval $[a, b]$, differentiable in the interval (a, b) and $f(a) = f(b)$, then there exists at least one point $c \in (a, b)$ such that $f'(c) = 0$.

12. *The Mean Value Theorem:* If the function f is continuous on the interval $[a, b]$ and differentiable on the interval (a, b), then there exist at least one point $c \in (a, b)$ such that

$$f'(c) = \frac{f(b) - f(a)}{b - a}$$

13. *Applications:*
    (a) *Monotonicity:* The function f defined on the interval (a, b) is said to be
        i. *increasing* in (a, b) if for all $x_1, x_2 \in (a, b)$: $x_1 < x_2 \Rightarrow f(x_1) < f(x_2)$.
        ii. *non-decreasing* in (a, b) if for all $x_1, x_2 \in (a, b)$: $x_1 < x_2 \Rightarrow f(x_1) \leq f(x_2)$
        iii. *decreasing* in (a, b) if for all $x_1, x_2 \in (a, b)$: $x_1 < x_2 \Rightarrow f(x_1) > f(x_2)$
        iv. *non-increasing* in (a, b) if for all $x_1, x_2 \in (a, b)$: $x_1 < x_2 \Rightarrow f(x_1) \geq f(x_2)$
    (b) *Monotonicity of differentiable functions:*
    If f is differentiable on the interval (a, b), then
        i. If $f'(x) > 0$ for all$x \in (a, b)$, then f is increasing on (a, b);
        ii. If $f'(x) < 0$ for all $x \in (a, b)$, then f is decreasing on (a, b);
        iii. If $f'(x) = 0$ for all $x \in (a, b)$, then f is constant on (a, b);
    (c) *Concavity:* If the function f is differentiable on the interval (a, b), then we say that f is
        i. *concave upward* on (a, b) if f' is increasing on
        ii. *concave downward* on (a, b) if f' is decreasing on (a, b). (a, b).
        iii. *Inflection Point:* A point $c \in (a, b)$ at which the concavaity of f changes is called an *inflection point* of f.
    (d) Suppose the function f is twice differentiable on the interval (a, b). Then
        i. If $f''(x) > 0$ for all $x \in (a, b)$, then f is concave upward on (a, b).
        ii. If $f''(x) < 0$ for all $x \in (a, b)$, then f is concave downward on (a, b).
    (e) If f has a local extremum at c then $f'(c) = 0$ or $f'(c)$ does not exist.
    (f) *Critical Points:* Given a function f with domain D_f. The point $c \in D_f$ is called a *critical point* of f if either $f'(c) = 0$ or $f'(c)$ does not exist.
    (g) *The First-Derivative Test:* Suppose c is a critical point of f and f is continuous at c. If there exists a$\delta > 0$ such that:
        i. $f'(x) > 0$ for all $x \in (c - \delta, c)$ and $f'(x) < 0$ for all $x \in (c, c + \delta)$, then $f(c)$ is a local maximum.
        ii. $f'(x) < 0$ for all $x \in (c - \delta, c)$ and $f'(x) > 0$ for all $x \in (c, c + \delta)$, then $f(c)$ is a local minimum
        iii. If f keeps constant sign on $(c-\delta, c) \cup (c, c+\delta)$ then $f(c)$ is not a local extreme value.
    (h) *The Second-Derivative Test:* Suppose that $f'(c) = 0$ and that $f''(c)$ exists. Then
        i. If $f''(c) > 0$ then $f(c)$ is a local minimum.
        ii. If $f''(c) < 0$ then $f(c)$ is a local maximum.