

R 4: Summary: Integral Calculus

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1. THE RIEMANN INTEGRAL:

(a) *Supremum, Infimum:* If $A \subseteq \mathbb{R}$ is a subset of the set of real numbers \mathbb{R} , then we define $\sup(A)$ to be the least upper bound of A if A is bounded above and ∞ if it is not bounded above. Similarly, we define $\inf(A)$ to be the greatest lower bound of A if A is bounded below and $-\infty$ if it is not bounded below.

(b) *Partition:* Let $[a, b]$ be a closed interval. A subset $P := \{x_0, x_1, \dots, x_n\}$ of $[a, b]$ is called a *partition* of $[a, b]$ if

$$a = x_0 < x_1 < \dots < x_n = b.$$

(c) *Upper Sums, Lower Sums:*

i. Suppose the function f is bounded on $[a, b]$ and $P := \{x_0, x_1, \dots, x_n\}$ is a partition of $[a, b]$. Then we set

$$M_i(f) := \sup\{f(x) \mid x \in [x_{i-1}, x_i]\}$$

$$m_i(f) := \inf\{f(x) \mid x \in [x_{i-1}, x_i]\},$$

and $\Delta x_i := x_i - x_{i-1}$, and call the sums

$$U(f, P) := \sum_{i=1}^n M_i(f) \Delta x_i$$

$$L(f, P) := \sum_{i=1}^n m_i(f) \Delta x_i$$

upper sum and *lower sum* of f with respect to P , respectively.

ii. If P and Q are partitions and if $P \subseteq Q$ then $U(f, Q) \leq U(f, P)$ and $L(f, Q) \geq L(f, P)$.

iii. If P and Q are partitions, then $L(f, P) \leq U(f, Q)$.

iv. Let

$$\bar{I} := \inf\{U(f, Q) \mid Q \text{ partition of } [a, b]\}$$

$$\underline{I} := \sup\{L(f, Q) \mid Q \text{ partition of } [a, b]\},$$

then $\underline{I} \leq \bar{I}$.

(d) We say that a function f bounded on $[a, b]$ is *Riemann integrable* or simply *integrable* if $\underline{I} = \bar{I}$. In this case we define

$$\int_a^b f(x) dx := \underline{I} = \bar{I},$$

called the *definite (Riemann) integral* or simply the *integral* of f from a to b . Moreover, we define

$$\int_a^a f(x) dx := 0$$

$$\int_b^a f(x) dx := - \int_a^b f(x) dx.$$

(e) A bounded function f on $[a, b]$ is Riemann integrable if and only if for all $\varepsilon > 0$ there exists a partition P of $[a, b]$ such that

$$U(f, P) - L(f, P) < \varepsilon.$$

(f) If f is (piecewise) monotonic (either increasing or decreasing) on $[a, b]$ then f is Riemann integrable.

(g) If f is (piecewise) continuous on $[a, b]$ then f is Riemann integrable.

(h) Suppose f is Riemann integrable on $[a, b]$. Then for any $\varepsilon > 0$ there exists a partition $P = \{x_0, \dots, x_n\}$ of $[a, b]$ such that for any choice of $\xi_i \in [x_{i-1}, x_i]$

$$\left| \int_a^b f(x) dx - \sum_{i=1}^n f(\xi_i) \Delta x_i \right| < \varepsilon.$$

The sum $\sum_{i=1}^n f(\xi_i) \Delta x_i$ is called a *Riemann sum* of f with respect to the partition P .

2. ESTIMATING DEFINITE INTEGRALS:

Given an Riemann integrable function f defined on $[a, b]$, choose $n \in \mathbb{N}$ and subdivide $[a, b]$ in n subintervals of equal length $\Delta x := \frac{b-a}{n}$. Let $x_i := a + i \Delta x$.

(a) *Left-Hand and Right-Hand Sums:*

$$\text{LHS}_f(n) := \sum_{i=0}^{n-1} f(x_i)\Delta x$$
$$\text{RHS}_f(n) := \sum_{i=1}^n f(x_i)\Delta x.$$

If the function f is differentiable on $[a, b]$ and $|f'(x)| \leq B$ for all $x \in [a, b]$ and some $B \in \mathbb{R}$, then

$$\left| \int_a^b f(x) dx - \text{LHS}_f(n) \right| \leq \frac{B(b-a)^2}{2n}$$
$$\left| \int_a^b f(x) dx - \text{RHS}_f(n) \right| \leq \frac{B(b-a)^2}{2n}.$$

(b) *Trapezoidal Estimate:*

$$\text{TRAP}_f(n) := \frac{1}{2}(\text{LHS}_f(n) + \text{RHS}_f(n)).$$

If the function f is twice differentiable on $[a, b]$ and $|f''(x)| \leq B$ for all $x \in [a, b]$ for some $B \in \mathbb{R}$, then

$$\left| \int_a^b f(x) dx - \text{TRAP}_f(n) \right| \leq \frac{B(b-a)^3}{12n^2}.$$

(c) *Midpoint Sum:*

Let $\mu_i := \frac{1}{2}(x_{i-1} + x_i) = a + (i - \frac{1}{2})\Delta x$ denote the midpoint of the i th subinterval of $[a, b]$. Then

$$\text{MPS}_f(n) := \sum_{i=1}^n f(\mu_i)\Delta x.$$

If the function is twice differentiable on $[a, b]$ and $|f''(x)| \leq B$ for all $x \in [a, b]$ for some $B \in \mathbb{R}$. then

$$\left| \int_a^b f(x) dx - \text{MPS}_f(n) \right| \leq \frac{B(b-a)^3}{24n^2}.$$

(d) *Simpson's Rule:*

$$\text{SIMP}_f(2n) := \frac{1}{3}(2 \cdot \text{MPS}_f(n) + \text{TRAP}_f(n))$$
$$= \sum_{i=1}^n (f(x_{i-1}) + 4f(\mu_i) + f(x_i)) \frac{\Delta x}{6}.$$

If the function f is four times differentiable on $[a, b]$ and $|f^{(4)}(x)| \leq B$ for all $x \in [a, b]$ for some $B \in \mathbb{R}$, then

$$\left| \int_a^b f(x) dx - \text{SIMP}_f(n) \right| \leq \frac{B(b-a)^5}{180n^4}.$$

3. PROPERTIES OF THE DEFINITE INTEGRAL:

(a) *Linearity:* Suppose f and g are integrable and $\alpha \in \mathbb{R}$, then

i. $\int_a^b \alpha f(x) dx = \alpha \int_a^b f(x) dx$

ii. $\int_a^b (f(x) + g(x)) dx$
 $= \int_a^b f(x) dx + \int_a^b g(x) dx$

(b) *Interval Property:* Suppose f is integrable on the interval I and $a, b, c \in I$, then

$$\int_a^c f(x) dx = \int_a^b f(x) dx + \int_b^c f(x) dx.$$

(c) *Order Properties:*

- i. If the function f is integrable on $[a, b]$ and $f(x) \geq 0$ for all $x \in [a, b]$, then $\int_a^b f(x) dx \geq 0$.
- ii. If the functions f and g are integrable on $[a, b]$ and $f(x) \leq g(x)$ for all $x \in [a, b]$, then $\int_a^b f(x) dx \leq \int_a^b g(x) dx$.
- iii. If f is integrable on $[a, b]$, then $|f|$ is integrable on $[a, b]$ and

$$\left| \int_a^b f(x) dx \right| \leq \int_a^b |f(x)| dx.$$

- iv. If f is integrable on $[a, b]$ and m the minimum and M the maximum value of f on $[a, b]$ then

$$m(b-a) \leq \int_a^b f(x) dx \leq M(b-a).$$

(d) *Symmetry:* Suppose f is integrable on the symmetric interval $[-a, a]$. Then,

- i. if f is odd on $[-a, a]$, then

$$\int_{-a}^a f(x) dx = 0;$$

- ii. if f is even on $[-a, a]$, then

$$\int_{-a}^a f(x) dx = 2 \int_0^a f(x) dx.$$

(e) *Average Value of a Function:* If f is integrable on $[a, b]$, then

$$\frac{1}{b-a} \int_a^b f(x) dx$$

is the *average value* of f on the interval $[a, b]$.

- (f) *The Mean-value Theorem for Integrals:* If f is continuous on $[a, b]$, then there exists a number $\xi \in [a, b]$ such that

$$f(\xi) = \frac{1}{b-a} \int_a^b f(x) dx.$$

- (g) *Second Mean-value Theorem for Integrals:* If f and g are continuous on $[a, b]$ and $g(x) \geq 0$ for all $x \in [a, b]$, then there exists a number $\xi \in [a, b]$ such that

$$\int_a^b f(x)g(x) dx = f(\xi) \int_a^b g(x) dx.$$

- (h) *Anti-derivative:* A function F is an *anti-derivative* or *primitive* or *indefinite integral* of the function f on an interval I if $F'(x) = f(x)$ for all $x \in I$.

4. THE FUNDAMENTAL THEOREM OF CALCULUS:

- (a) If f is continuous on $[a, b]$ then the function

$$F(x) := \int_a^x f(t) dt$$

is differentiable on (a, b) and

$$F'(x) = \frac{d}{dx} \left(\int_a^x f(t) dt \right) = f(x)$$

for all $x \in (a, b)$.

- (b) If the function f is continuous on $[a, b]$ and F any anti-derivative of f on $[a, b]$, then

$$\int_a^b f(x) dx = F(b) - F(a) = F(x) \Big|_a^b.$$

- (c) If f is continuous on $[a, b]$ and ψ_1 and ψ_2 are differentiable on (a, b) then

$$G(x) := \int_{\psi_1(x)}^{\psi_2(x)} f(t) dt$$

is differentiable on (a, b) and

$$\begin{aligned} G'(x) &= \frac{d}{dx} \left(\int_{\psi_1(x)}^{\psi_2(x)} f(t) dt \right) \\ &= f(\psi_2(x))\psi_2'(x) - f(\psi_1(x))\psi_1'(x). \end{aligned}$$

5. BASIC INTEGRATION FORMULAS:

- (a) $\int x^r dx = \begin{cases} \frac{x^{r+1}}{r+1} + C, & r \neq -1 \\ \ln|x| + C, & r = -1 \end{cases}$

$$(b) \int (g(x))^r g'(x) dx = \begin{cases} \frac{(g(x))^{r+1}}{r+1} + C, & r \neq -1 \\ \ln|g(x)| + C, & r = -1 \end{cases}$$

$$(c) \int \cos x dx = \sin x + C$$

$$\int \sin x dx = -\cos x + C$$

$$(d) \int \sec x dx = \ln|\sec x + \tan x| + C$$

$$\int \csc x dx = -\ln|\csc x + \cot x| + C.$$

$$(e) \int \sec^2 dx = \tan x + C$$

$$\int \csc^2 x dx = -\cot x + C$$

$$(f) \int \sec^3 x dx = \frac{1}{2} \sec x \tan x + \frac{1}{2} \ln|\sec x + \tan x| + C$$

$$\int \csc^3 x dx = -\frac{1}{2} \csc x \cot x - \frac{1}{2} \ln|\csc x + \cot x| + C$$

$$(g) \int \sec x \tan x dx = \sec x + C$$

$$\int \csc x \cot x dx = -\csc x + C$$

$$(h) \int e^x dx = e^x + C$$

$$\int a^x dx = \frac{a^x}{\ln a} + C$$

$$(i) \int \frac{1}{1+x^2} dx = \tan^{-1} x + C$$

$$(j) \int \frac{1}{\sqrt{1-x^2}} dx = \sin^{-1} x + C$$

$$(k) \int \frac{1}{x\sqrt{x^2-1}} dx = \sec^{-1} x + C$$

$$(l) \int \cosh x dx = \sinh x + C$$

$$\int \sinh x dx = \cosh x + C$$

$$(m) \int \operatorname{sech}^2 x dx = \tanh x + C$$

$$\int \operatorname{csch}^2 x dx = -\operatorname{coth} x + C$$

$$(n) \int \operatorname{sech} x \tanh x dx = -\operatorname{sech} x + C$$

$$\int \operatorname{csch} x \operatorname{coth} x dx = -\operatorname{csch} x + C$$

$$(o) \int \cos^n x dx = \frac{1}{n} \cos^{n-1} x \sin x + \frac{n-1}{n} \int \cos^{n-2} x dx$$

$$\int \sin^n x dx = -\frac{1}{n} \sin^{n-1} x \cos x + \frac{n-1}{n} \int \sin^{n-2} x dx$$

$$(p) \int \tan^n x dx = \frac{1}{n-1} \tan^{n-1} x - \int \tan^{n-1} x dx$$

$$\int \cot^n x dx = -\frac{1}{n-1} \cot^{n-1} x - \int \cot^{n-1} x dx$$

$$(q) \int \sec^n x dx = \frac{1}{n-1} \sec^{n-2} x \tan x + \frac{n-2}{n-1} \int \sec^{n-2} x dx$$

$$\int \csc^n x dx = -\frac{1}{n-1} \csc^{n-2} x \cot x + \frac{n-2}{n-1} \int \csc^{n-2} x dx$$

(STEP 2:) Decompose $p(x)/q(x)$ by replacing each linear factor of q which has the form $(ax + b)^k$ (with multiplicity $k \geq 1$) by

$$\frac{A_1}{ax + b} + \frac{A_2}{(ax + b)^2} + \cdots + \frac{A_k}{(ax + b)^k}$$

and each quadratic factor of q which has the form $(ax^2 + bx + c)^k$ (with multiplicity $k \geq 1$) by

$$\frac{A_1x + B_1}{ax^2 + bx + c} + \cdots + \frac{A_kx + B_k}{(ax^2 + bx + c)^k}$$

(STEP 3:) Compute the undetermined coefficients A_j, B_j in the numerators using “comparing coefficients”, limit techniques etc.

(STEP 4:) Solve the resulting integrals which are of the form

$$\int \frac{A_j}{(ax + b)^j} dx, \int \frac{A_jx + b_j}{(ax^2 + bx + c)^j} dx$$

6. INTEGRATION RULES:

(a) *Linearity:*

$$i. \int \alpha f(x) dx = \alpha \int f(x) dx$$

$$ii. \int (f(x) + g(x)) dx = \int f(x) dx + \int g(x) dx.$$

(b) *Substitution Rule:*

$$i. \int_a^b f(g(x))g'(x) dx = \int_{g(a)}^{g(b)} f(y) dy$$

$$ii. \int f(g(x))g'(x) dx = \int f(y) dy$$

iii. *Trigonometric Substitutions:*

If the integrand involves $\sqrt{a^2 - x^2}$, $\sqrt{x^2 - a^2}$ or $\sqrt{a^2 + x^2}$, substitute $x = a \sin \theta$, $x = a \sec \theta$, $x = a \tan \theta$, respectively.

iv. *Rationalizing Substitutions:*

If the integrand involves an expression of the form $\sqrt[n]{f(x)}$ substitute $y = \sqrt[n]{f(x)}$ or equivalently, $y^n = f(x)$.

If the integrand is a rational expression in $\sin x$ and $\cos x$, substitute $y = \tan(x/2)$, which implies

$$\cos x = \frac{1 - y^2}{1 + y^2}, \quad \sin x = \frac{2y}{1 + y^2}$$

$$dx = \frac{2 dy}{1 + y^2}$$

(c) *Integration by Parts:*

$$i. \int_a^b f'(x)g(x) dx = f(x)g(x) \Big|_a^b - \int_a^b f(x)g'(x) dx$$

$$ii. \int f'(x)g(x) dx = f(x)g(x) - \int f(x)g'(x) dx$$

(d) *Integration by Partial Fractions:* If the integrand is a proper rational function of the form $p(x)/q(x)$ where p and q are polynomials (if it is not proper, first use long division) follow this procedure:

(STEP 1:) Factor q into a product of linear and irreducible (over \mathbb{R}) quadratic factors.

7. APPLICATIONS:

(a) *Area between Two Curves:* If f and g are continuous functions on $[a, b]$ such that $g(x) \leq f(x)$ for all $x \in [a, b]$, then the region $\{(x, y) \mid a \leq x \leq b, g(x) \leq y \leq f(x)\}$ is called the *region between f and g over $[a, b]$* . Then the *area differential* at $x \in [a, b]$ is given by

$$dA(x) = (f(x) - g(x)) dx$$

and the total area by

$$A = \int_a^b dA = \int_a^b (f(x) - g(x)) dx.$$

(b) *Volume by Slicing:*

i. *Cavalieri's Principle:* Given a solid Ω with continuous cross-sections $A(x)$ perpendicular to a coordinate axis whose points are labeled by x . Then the *volume differential* of Ω at the point x is given by

$$dV(x) = A(x) dx$$

and if $a < b$ are points on the axis, then the volume of Ω between a and b is given by

$$V = \int_a^b dA = \int_a^b A(x) dx.$$

ii. *Disc/Washer Method:* Let $c \in \mathbb{R}$ and f and g continuous on $[a, b]$ such that $c \leq g(x) \leq f(x)$ for all $x \in [a, b]$. Moreover, let Ω denote the solid generated by revolving the region between f and g over $[a, b]$

about the axis $y = c$. Then the volume differential of Ω at the point $x \in [a, b]$ is given by

$$dV(x) = \pi \left[(f(x) - c)^2 - (g(x) - c)^2 \right] dx$$

and the total volume of Ω by

$$\begin{aligned} V &= \int_a^b dV \\ &= \pi \int_a^b \left[(f(x) - c)^2 - (g(x) - c)^2 \right] dx \end{aligned}$$

- (c) *Volume by Shells:* Let $c \in \mathbb{R}$, $c \leq a < b$, and let f and g be continuous on $[a, b]$ such that $g(x) \leq f(x)$ for all $x \in [a, b]$. Moreover, let Ω denote the solid generated by revolving the region between f and g over $[a, b]$ about the axis $x = d$. Then the volume differential of Ω at $x \in [a, b]$ is given by

$$dV(x) = 2\pi(x - d)(f(x) - g(x))dx,$$

and the total volume by

$$\begin{aligned} V &= \int_a^b dV \\ &= 2\pi \int_a^b (x - d)(f(x) - g(x))dx, \end{aligned}$$

- (d) *Arc Length:*

- i. *Arc, Smooth Curve:* If f is continuously differentiable on $[a, b]$ (i.e. f' is continuous) then the curve

$$C := C_f := \{(x, f(x)) \mid a \leq x \leq b\}$$

is called an *arc* or a *smooth curve*.

- ii. *Arc Length:* Let C_f be an arc, defined by the function f on $[a, b]$, and let $P := \{x_0, \dots, x_n\}$ be a partition of $[a, b]$. Moreover, let

$$\begin{aligned} \Delta s_j &:= \sqrt{(x_j - x_{j-1})^2 + (f(x_j) - f(x_{j-1}))^2} \\ &= \sqrt{1 + (f'(\xi_j))^2} \Delta x_j \end{aligned}$$

for some $\xi_j \in (x_{j-1}, x_j)$. Then,

$$s(P) := \sum_{j=1}^n \Delta s_j$$

gives the length of the polygonal arc inscribed in C_f relative to the partition P .

If the set $\{s(P) \mid P \text{ partition of } [a, b]\}$ is bounded above, then C_f is called *rectifiable* and we define the *length* s of C_f by

$$s := \sup\{s(P) \mid P \text{ partition of } [a, b]\}.$$

- iii. If f is continuously differentiable, then

$$ds(x) := \sqrt{1 + (f'(x))^2} dx$$

is the *arc length differential* of the curve C_f at x . The total length of the curve C_f is given by

$$s = \int_a^b ds = \int_a^b \sqrt{1 + (f'(x))^2} dx.$$

- (e) *Area of a Surface of Revolution:* Let $c \in \mathbb{R}$ and let f be continuously differentiable on $[a, b]$ and $f(x) \geq c$ for all $x \in [a, b]$. Moreover, let Γ denote the surface generated by revolving the curve C_f about the axis $y = c$. Then the surface area differential of Γ at $x \in [a, b]$ is given by

$$\begin{aligned} dS &= 2\pi(f(x) - c) ds(x) \\ &= 2\pi(f(x) - c) \sqrt{1 + (f'(x))^2} dx \end{aligned}$$

and the total surface area by

$$\begin{aligned} S &= \int_a^b d\sigma \\ &= \int_a^b 2\pi(f(x) - c) \sqrt{1 + (f'(x))^2} dx. \end{aligned}$$

- (f) *Mass from Density:*

- i. *Mass of a Rod:* Suppose the projection of the material solid Ω onto the real number line is given by the interval $[a, b]$, and suppose $\delta(x)$ is the density of the solid at $x \in [a, b]$ (units: mass per unit length). Then the mass differential of Ω at $x \in [a, b]$ is given by

$$dm(x) = \delta(x) dx,$$

and the total mass of Ω by

$$m = \int_a^b dm = \int_a^b \delta(x) dx.$$

- ii. *The Mass of a Plate:* Suppose the projection of the material solid Ω in the plane is the region between f and g over the interval $[a, b]$. Moreover, suppose that the density of the solid is given by the (1-dimensional) function $\delta : [a, b] \rightarrow \mathbb{R}$ (units: mass per unit area); note: we assume the density is constant along the lines $x = c$ for $c \in [a, b]$.

Then the mass differential of Ω at $x \in [a, b]$ is given by

$$dm(x) = (f(x) - g(x))\delta(x) dx,$$

and the total mass of Ω by

$$\begin{aligned} m &= \int_a^b dm \\ &= \int_a^b (f(x) - g(x))\delta(x) dx. \end{aligned}$$

- iii. *The Mass of a Wire:* Suppose the axis of the wire W is a smooth curve C_f defined by the function f continuously differentiable on $[a, b]$. Moreover, suppose that $\delta(x)$ is the density of W at $x \in [a, b]$ (units: mass per unit length). Then the mass differential of W at $x \in [a, b]$ is given by

$$\begin{aligned} dm(x) &= \delta(x) ds(x) \\ &= \delta(x)\sqrt{1 + (f'(x))^2} dx, \end{aligned}$$

and the total mass of W by

$$\begin{aligned} m &= \int_a^b dm \\ &= \int_a^b \delta(x) ds \\ &= \int_a^b \delta(x)\sqrt{1 + (f'(x))^2} dx. \end{aligned}$$

- iv. *Three Dimensions:* Suppose Ω is a material solid described by the 3-dimensional region B in \mathbb{R}^3 with density function δ assigning to each $(x, y, z) \in B$ its density $\delta(x, y, z)$. Then the mass differential of Ω at (x, y, z) is given by

$$dm(x, y, z) = \delta(x, y, z) dV(x, y, z)$$

and the the total mass of Ω by

$$m = \int_B dm = \int_B \delta(x, y, z) dV.$$

(g) *Center of Mass and Centroid:*

- i. *The Center of Mass of a Rod:* Given a rod Ω with density function $\delta : [a, b] \rightarrow \mathbb{R}$. We consider the moment of Ω with respect to the point $x = 0$. The moment differential of Ω at $x \in [a, b]$ (the moment of the x -th slice of Ω) is given by

$$dM(x) = x dm(x) = x\delta(x) dx$$

and the total moment of Ω by

$$\begin{aligned} M &= \int_a^b dM \\ &= \int_a^b x dm(x) = \int_a^b x\delta(x) dx. \end{aligned}$$

The *center of mass* of Ω is the unique point $\bar{x} \in \mathbb{R}$ such that $\bar{x}m = M$ where m denotes the total mass of Ω . Thus

$$\bar{x} = \frac{\int_a^b dM}{\int_a^b dm} = \frac{\int_a^b x\delta(x) dx}{\int_a^b \delta(x) dx}.$$

- ii. *The Centroid of a Rod:* If the density of the rod Ω is constant, it is called a *uniform* or *homogeneous rod*. Its center of mass \bar{x} is called the *centroid* of the rod. Clearly,

$$\bar{x} = \frac{1}{2}(a + b).$$

- iii. *The Center of Mass of a Plate:* Suppose the Plate Ω is given by the continuous functions f and g on $[a, b]$ and the density function δ on $[a, b]$. We consider the moment M_x about the x -axis and the moment M_y about the y -axis. Then the moment differentials of Ω at x with respect to the x -axis and y -axis are given by

$$\begin{aligned} dM_x(x) &= \hat{y} dm(x) \\ &= \frac{1}{2}(f(x) + g(x)) dm(x), \\ dM_y(x) &= \hat{x} dm(x) = x dm(x), \end{aligned}$$

respectively, where \hat{x} and \hat{y} denote the x - and y -coordinate of the center of mass of the (uniform rod) mass differential $dm(x)$ at x . The total moments of Ω with respect to the x - and y -axis are given by

$$\begin{aligned} M_x &= \int_a^b dM_x \\ &= \frac{1}{2} \int_a^b ([f(x)]^2 - [g(x)]^2)\delta(x) dx, \\ M_y &= \int_a^b dM_y \\ &= \int_a^b x(f(x) - g(x))\delta(x) dx, \end{aligned}$$

respectively. The *center of mass* of Ω is the unique point $(\bar{x}, \bar{y}) \in \mathbb{R}^2$ such that

$\bar{x}m = M_y$ and $\bar{y}m = M_x$, where m denotes the total mass of Ω . Thus

$$\bar{x} = \frac{\int dM_y}{\int dm} = \frac{\int \hat{x} dm}{\int dm},$$

$$\bar{y} = \frac{\int dM_x}{\int dm} = \frac{\int \hat{y} dm}{\int dm},$$

respectively.

iv. *The Centroid of a Region:* If the density of the plate Ω is constant, we call its center of mass, the *centroid of the region* covered by the plate.

v. *The Center of Mass of a Wire:* Suppose the axis of the wire W is the smooth curve C_f , f continuously differentiable on $[a, b]$ with density δ on $[a, b]$. The mass differential $dm(x)$ has center of mass $(\hat{x}, \hat{y}) = (x, y)$. Then the moment differential of W at x with respect to the x -axis and y -axis is given by

$$dM_x(x) = \hat{y} dm(x)$$

$$= y dm(x) = y \delta(x) ds,$$

$$dM_y(x) = \hat{x} dm(x)$$

$$= x dm(x) = x \delta(x) ds,$$

respectively, and the total moment of W with respect to the x -axis and y -axis by

$$M_x = \int_a^b dM_x = \int_a^b y \delta(x) ds(x).$$

$$M_y = \int_a^b dM_y = \int_a^b x \delta(x) ds(x),$$

respectively. The center of mass (\bar{x}, \bar{y}) of W is therefore given by

$$\bar{x} = \frac{\int dM_y}{\int dm} = \frac{\int \hat{x} dm}{\int dm} = \frac{\int_a^b x \delta(x) ds}{\int_a^b \delta(x) ds},$$

$$\bar{y} = \frac{\int dM_x}{\int dm} = \frac{\int \hat{y} dm}{\int dm} = \frac{\int_a^b y \delta(x) ds}{\int_a^b \delta(x) ds},$$

vi. *The Centroid of an Arc:* If the density of a wire is constant, it is called the *centroid of the arc* covered by the wire.

vii. *Center of Mass in Three Dimensions:* Suppose Ω is a material solid described by the 3-dimensional region B in \mathbb{R}^3 with density function δ assigning to each $(x, y, z) \in B$ its density $\delta(x, y, z)$. Let $(\hat{x}, \hat{y}, \hat{z}) \in \mathbb{R}^3$ denote the center of mass of the mass differential dm at $(x, y, z) \in B$, then the center of mass $(\bar{x}, \bar{y}, \bar{z})$ of Ω is given by

$$\bar{x} = \frac{\int_B \hat{x} dm}{\int dm} = \frac{\int_B \hat{x} \delta(x, y, z) dV}{\int_B \delta(x, y, z) dV}$$

$$\bar{y} = \frac{\int_B \hat{y} dm}{\int dm} = \frac{\int_B \hat{y} \delta(x, y, z) dV}{\int_B \delta(x, y, z) dV}$$

$$\bar{z} = \frac{\int_B \hat{z} dm}{\int dm} = \frac{\int_B \hat{z} \delta(x, y, z) dV}{\int_B \delta(x, y, z) dV}$$

Let Γ denote the surface of B and let again $(\hat{x}, \hat{y}, \hat{z})$ denote the center of mass of the mass differential $dm = \delta(x, y, z) dS$ at $(x, y, z) \in \Gamma$. Then the center of mass $(\bar{x}, \bar{y}, \bar{z})$ of Γ is given by

$$\bar{x} = \frac{\int_S \hat{x} dm}{\int dm} = \frac{\int_S \hat{x} \delta(x, y, z) dS}{\int_S \delta(x, y, z) dS}$$

$$\bar{y} = \frac{\int_S \hat{y} dm}{\int dm} = \frac{\int_S \hat{y} \delta(x, y, z) dS}{\int_S \delta(x, y, z) dS}$$

$$\bar{z} = \frac{\int_S \hat{z} dm}{\int dm} = \frac{\int_S \hat{z} \delta(x, y, z) dS}{\int_S \delta(x, y, z) dS}$$

viii. *Symmetry and Centroid:*

If the (1-, 2-, 3-dimensional) region Ω has axis of symmetry ℓ then the centroid \bar{z} of Ω lies on ℓ .

(h) *Fluid Pressure:*

i. The pressure p exerted by a fluid with weight-density ω at depth h is given by

$$p = \omega h.$$

ii. The force F exerted by a fluid on a (horizontal) surface of area A at depth h is

$$F = pA = \omega hA.$$

(i) *Work:*

- i. The work W performed by an object moved a distance d along a straight line by a constant force F is

$$W = Fd.$$

(j) *Money Stream:*

- i. A *money stream* is a continuous function representing the flow of money as a function of time (units: dollars per unit time).

ii. *Future Value:*

- A. Suppose an amount P is deposited into an account at r percent interest compounded continuously. Then its value B after T years is

$$B = Pe^{rT},$$

called the *future value* of P after T years at rate r .

- B. Now suppose a money stream $P(t)$ is deposited into an account at r percent interest compounded continuously. Then the “future value differential” $dB(t)$ at time t is

$$dB(t) = P(t) dt e^{r(T-t)}$$

and the future value of the entire money stream $P(t)$ at time T is

$$B = \int_0^T dB(t) = \int_0^T P(t)e^{r(T-t)} dt.$$

8. THE THEOREMS OF PAPPUS:

- (a) *The First Theorem of Pappus:* Suppose the solid Ω is generated by revolving the region R about the line ℓ that does not intersect R . Let \bar{r} denote the distance of the centroid of R from ℓ . Then

$$V = 2\pi\bar{r}A,$$

where A denotes the area of R .

- (b) *The Second Theorem of Pappus:* Suppose the surface Γ is generated by revolving the arc C about the line ℓ that does not intersect C . Let \bar{r} denote the distance of the centroid of C from ℓ . Then

$$S = 2\pi\bar{r}s,$$

where s denotes the length of C .