Review
Fundamental Concepts and Techniques of Calculus

R 5: Summary: Sequences and Series

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1. SEQUENCES OF REAL NUMBERS, DEFINITIONS:
   (a) *Sequence:* A real-valued function $a : \mathbb{N} \to \mathbb{R}$ from the set of positive integers $\mathbb{N}$ into the real numbers $\mathbb{R}$ is called a *sequence of real numbers* or a *real sequence.* It is customary to denote $a(n)$ simply by a_n and the whole sequence by $(a_n)_{n\in\mathbb{N}} = (a_1, a_2, a_3, \ldots)$ or by $\{a_n\}_{n\in\mathbb{N}} = \{a_1, a_2, a_3, \ldots\}$. Often the notation is simplified to (a_n) or $\{a_n\}$.
   (b) *Boundedness:* A (real) sequence $(a_n)_{n\in\mathbb{N}}$ is said to be
      i. *bounded above* if there exists some constant $B \in \mathbb{R}$ such that $a_n \le B$ for all $n \in \mathbb{N}$.;
      ii. *bounded below* if there exists some constant $b \in \mathbb{R}$ such that $a_n \ge b$ for all $n \in \mathbb{N}$;
      iii. *bounded* if there exists some constant $C \in \mathbb{R}$ such that $|a_n| \le C$ for all $n \in \mathbb{N}$.
   (c) *Monotonic sequences:* A (real) sequence $(a_n)_{n\in\mathbb{N}}$ is said to be
      i. *increasing* if there exists some $k \in \mathbb{N}$ such $a_n < a_{n+1}$ for all $n \in \mathbb{N}$ with $n > k$.
      ii. *non-decreasing* if there exists some $k \in \mathbb{N}$ such $a_n \le a_{n+1}$ for all $n \in \mathbb{N}$ with $n > k$.
      iii. *decreasing* if there exists some $k \in \mathbb{N}$ such $a_n > a_{n+1}$ for all $n \in \mathbb{N}$ with $n > k$.
      iv. *non-increasing* if there exists some $k \in \mathbb{N}$ such $a_n \ge a_{n+1}$ for all $n \in \mathbb{N}$ with $n > k$.
   (d) *Limits of sequences:* A (real) sequence $(a_n)_{n\in\mathbb{N}}$ has *limit* $L \in \mathbb{R}$, written

$$\lim_{n\to\infty} a_n = L, \quad a_n \to L,$$

   if and only if, for any $\varepsilon > 0$ there exists some constant $N \in \mathbb{N}$ such that for all $n \in \mathbb{N}$:

$$n > N \Rightarrow |a_n - L| < \varepsilon.$$

2. SOME LIMIT THEOREMS:
   (a) Suppose $(a_n)_{n\in\mathbb{N}}$ and $(b_n)_{n\in\mathbb{N}}$ are convergent sequences and $\alpha \in \mathbb{R}$, then
      i. $\lim\limits_{n\to\infty}(a_n + b_n) = \lim\limits_{n\to\infty} a_n + \lim\limits_{n\to\infty} b_n$
      ii. $\lim\limits_{n\to\infty}(\alpha a_n) = \alpha \lim\limits_{n\to\infty} a_n$
      iii. $\lim\limits_{n\to\infty}(a_n b_n) = \big(\lim\limits_{n\to\infty} a_n\big)\big(\lim\limits_{n\to\infty} b_n\big)$
      iv. $\lim\limits_{n\to\infty} \dfrac{a_n}{b_n} = \dfrac{\lim\limits_{n\to\infty} a_n}{\lim\limits_{n\to\infty} b_n}$, if $b_n \ne 0$ for all $n \in \mathbb{N}$ and $\lim_{n\to\infty} b_n \ne 0$.
   (b) *Continuous Functions Preserve Limits:* Suppose the function $f : D_f \to \mathbb{R}$ is continuous and $(a_n)_{n\in\mathbb{N}}$ a convergent sequence such that $a_n \in D_f$ for all $n \in \mathbb{N}$. Then the sequence of images $\big(f(a_n)\big)_{n\in\mathbb{N}}$ also converges and

$$\lim_{n\to\infty} f(a_n) = f\Big(\lim_{n\to\infty} a_n\Big).$$

   (c) *Squeeze Play:* Suppose $a_n \le b_n \le c_n$ for all $n \in \mathbb{N}$ with $n > k$ for some $k \in \mathbb{N}$. If $\lim_{n\to\infty} a_n = \lim_{n\to\infty} c_n = L$, then $\lim_{n\to\infty} b_n = L$.
   (d) Every convergent sequence is bounded.
   (e) Every bounded monotonic sequence is convergent.

   More precisely: Every non-decreasing sequence that is bounded above converges to its least upper bound, every non-increasing sequence that is bounded below converges to its greatest lower bound.
   (f) If the sequence $(a_n)_{n\in\mathbb{N}}$ is monotonic and unbounded, then

$$\lim_{n\to\infty} \frac{1}{a_n} = 0.$$

   (g) If $\lim_{n\to\infty} a_n = 0$ and $(b_n)_{n\in\mathbb{N}}$ is bounded, then

$$\lim_{n\to\infty} a_n b_n = 0.$$

3. SOME IMPORTANT LIMITS:
   (a) $\lim\limits_{n\to\infty} c = c,\ c \in \mathbb{R}$;
   (b) $\lim\limits_{n\to\infty} x^{1/n} = 1,\ x > 0$;

(c) *Geometric Sequence:*

$$\lim_{n\to\infty} x^n = \begin{cases} 0 & |x| < 1 \\ 1 & x = 1 \\ \text{diverges} & \text{else;} \end{cases}$$

(d) $\displaystyle\lim_{n\to\infty} \frac{1}{n^p} = 0,\ p > 0;$

(e) $\displaystyle\lim_{n\to\infty} \frac{x^n}{n!} = 0;$

(f) $\displaystyle\lim_{n\to\infty} \frac{\ln n}{n} = 0;$

(g) $\displaystyle\lim_{n\to\infty} n^{1/n} = 1;$

(h) $\displaystyle\lim_{n\to\infty} \left(1 + \frac{x}{n}\right)^n = e^x;$

4. INFINITE SERIES:

(a) *Infinite Series:* Let $(a_n)_{n\in\mathbb{N}}$ be a (real) sequence. The *infinite series* $\sum_{k=1}^{\infty} a_k$ is defined to be the sequence $(S_n)_{n\in\mathbb{N}}$ of n*-th partial sums*

$$S_n := \sum_{k=1}^{n} a_k.$$

The coefficients a_n are called the *terms* of the series.

(b) *Series of Positive Terms:* The series $\sum_{k=1}^{\infty} a_k$ is called a *series of positive terms* if $a_k > 0$ for all $k \in \mathbb{N}$.

(c) *Convergent Series:* Let $\sum_{k=1}^{\infty} a_k$ be an infinite series and $S \in \mathbb{R}$. The series $\sum_{k=1}^{\infty} a_k$ *converges to* S if the sequence $(S_n)_{n\in\mathbb{N}}$ of partial sums converges to S. If $\lim_{n\to\infty} S_n = S$, we call S the *value* of the infinite series and write

$$\sum_{k=1}^{\infty} a_k = S.$$

(d) *Absolute Convergence:* The series $\sum_{k=1}^{\infty} a_k$ is said to be *absolutely convergent* if the series $\sum_{k=1}^{\infty} |a_k|$ converges.

(e) *Conditional Convergence:* If the series $\sum_{k=1}^{\infty} a_k$ converges, but not absolutely, then it is called *conditionally convergent.*

(f) *Combinations of Series:* If $\sum_{k=1}^{\infty} a_k$ and $\sum_{k=1}^{\infty} b_k$ are convergent series and $\alpha \in \mathbb{R}$, then

  i. $\displaystyle\sum_{k=1}^{\infty}(a_k + b_k) = \sum_{k=1}^{\infty} a_k + \sum_{k=1}^{\infty} b_k;$

  ii. $\displaystyle\sum_{k=1}^{\infty} \alpha a_k = \alpha \sum_{k=1}^{\infty} a_k.$

5. CONVERGENCE TESTS:

(a) *Divergence Test:* If the series $\sum_{n=1}^{\infty} a_n$ converges, then $\lim_{n\to\infty} a_n = 0$. Equivalently: If $\lim_{n\to\infty} a_n \neq 0$ then $\sum_{n=1}^{\infty} a_n$ diverges.

(b) *Integral Test:* If f is a positive, continuous and decreasing function on the interval $[K, \infty)$ for some constant $K \in \mathbb{N}$, then

$$\sum_{k=K}^{\infty} f(k)\ \text{conv.} \Leftrightarrow \int_K^{\infty} f(x)\,dx\ \text{conv.}$$

(c) *Comparison Test:* Suppose the series of positive terms $\sum_{k=1}^{\infty} b_k$ dominates the series of positive terms $\sum_{k=1}^{\infty} a_k$, i.e. $b_k \geq a_k$ for all $k > K$ for some $K \in \mathbb{N}$. Then

  i. If $\sum_{k=1}^{\infty} b_k$ converges, then $\sum_{k=1}^{\infty} a_k$ converges;

  ii. If $\sum_{k=1}^{\infty} a_k$ diverges, then $\sum_{k=1}^{\infty} b_k$ diverges;

(d) *Limit Comparison Test:* Suppose $\sum_{k=1}^{\infty} a_k$ and $\sum_{k=1}^{\infty} b_k$ are series of positive terms and $0 < \displaystyle\lim_{n\to\infty} \frac{a_n}{b_n} < \infty$, then

$$\sum_{k=1}^{\infty} a_k\ \text{conv.}\ \Leftrightarrow \sum_{k=1}^{\infty} b_k\ \text{conv.}$$

(e) *Alternating Series Test:* If $(a_n)_{n\in\mathbb{N}}$ is decreasing, positive and $\lim_{n\to\infty} a_n = 0$, then

  i. $\sum_{n=1}^{\infty}(-1)^{n+1} a_n$ converges;

  ii. $|S - S_n| < a_{n+1}$ for all $n \in \mathbb{N}$, where S denotes the value of the alternating series.

(f) *Ratio Test:* Let $\sum_{n=1}^{\infty} a_n$ be a series and let $\varrho := \displaystyle\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right|$, then

  i. If $\varrho < 1$ then $\sum_{n=1}^{\infty} a_n$ converges absolutely;

  ii. If $\varrho > 1$ then $\sum_{n=1}^{\infty} a_n$ diverges;

  iii. If $\varrho = 1$ then the test is inconclusive.

(g) *Root Test:* Let $\sum_{n=1}^{\infty} a_n$ be a series and let $\varrho := \displaystyle\lim_{n\to\infty} |a_n|^{1/n}$, then

  i. If $\varrho < 1$ then $\sum_{n=1}^{\infty} a_n$ converges absolutely;

  ii. If $\varrho > 1$ then $\sum_{n=1}^{\infty} a_n$ diverges;

  iii. If $\varrho = 1$ then the test is inconclusive.

6. IMPORTANT SERIES, ETC. (FOR COMPARISONS ETC.):

   (a) *Geometric Series:*

   $$\sum_{n=0}^{\infty} r^n = \begin{cases} \dfrac{1}{1-r} & |r| < 1 \\ \text{diverges} & \text{else} \end{cases}$$

   (b) p-*series:*

   $$\sum_{n=1}^{\infty} \frac{1}{n^p} \text{ converges } \Leftrightarrow \ p > 1$$

   (c) *Telescoping Series:*

   $$\sum_{n=1}^{\infty} \frac{1}{n(n+1)} = 1$$

   (d) *Useful Estimate:* For all $k \in \mathbb{N}$ there exists a constant b_k such that for $x \in (b_k, \infty)$

   $$\ln x \leq x^{1/k}$$

7. TAYLOR SERIES, POWER SERIES:

   (a) *Taylor Series (Expansion):* Suppose f possessess derivatives of any order on the interval I containing x_0. Then the series

   $$P(x) := \sum_{j=0}^{\infty} \frac{f^{(j)}(x_0)}{j!}(x - x_0)^j$$

   is called the *Taylor series* or *Taylor expansion* of f centered at x_0;

   (b) *Taylor Polynomials:* The n-th partial sum

   $$P_n(x) = \sum_{j=0}^{n} \frac{f^{(j)}(x_0)}{j!}(x - x_0)^j$$

   of the Taylor series expansion of f at x_0 is called the n-th *Taylor polynomial* of f centered at x_0 (which exists if f has derivatives up to order n on I).

   (c) *Taylor's Remainder Term:* Suppose P_n is the n-th Taylor polynomial of f centered at x_0. Then the term

   $$R_n(x) := f(x) - P_n(x)$$

   for $x \in I$ is called the n-the *Taylor's remainder* for f at x.

   (d) *Taylor's Remainder Theorem:* If f possesses derivatives up to order $n+1$ on the open interval I containing x_0, then for each $x \in I$

   $$R_n(x) = \frac{1}{n!} \int_{x_0}^{x} f^{(n+1)}(t)(x-t)^n \, dt.$$

   (e) *Lagrange's Form of Taylor's Remainder:* If f possesses derivatives up to order $n+1$ on the open interval I containing x_0, then for each $x \in I$ there exists a number ξ between x and x_0 such that

   $$R_n(x) = \frac{f^{(n+1)}(\xi)}{(n+1)!}(x - x_0)^{n+1}$$

   (f) *Power Series:*

      i. *Power Series in x:* An infinite series of the form

      $$\sum_{n=0}^{\infty} b_n x^n$$

      is called a *power series* in x.

      ii. *More General Power Series:* If g is any function, then an infinite series of the form

      $$\sum_{n=0}^{\infty} b_n \big(g(x)\big)^n$$

      is called a *power series* in $g(x)$.

   (g) *Radius of Convergence, Interval of Convergence:* Let $\sum_{n=0}^{\infty} b_n x^n$ be a power series in x. Then there exists a unique r with $0 \leq r \leq \infty$ (note: r can be ∞ !) such that

      i. $\sum_{n=0}^{\infty} b_n x^n$ converges absolutely for all $x \in (-r, r)$;

      ii. $\sum_{n=0}^{\infty} b_n x^n$ diverges for all $x \notin [-r, r]$.

      This r is called the *radius of convergence* of $\sum_{n=0}^{\infty} b_n x^n$. The interval

      $$\Big\{x \in \mathbb{R} \,\Big|\, \sum_{n=0}^{\infty} b_n x^n \text{ converges}\Big\}$$

      is called the *interval of convergence* of the power series $\sum_{n=0}^{\infty} b_n x^n$.

   (h) *Computing the Radius of Convergence:* If $\sum_{n=0}^{\infty} b_n x^n$ is a power series in x, then its radius of convergence is given by

      i. $r = \lim\limits_{n\to\infty} \left|\dfrac{b_n}{b_{n+1}}\right|$

      ii. $r = \dfrac{1}{\overline{\lim\limits_{n\to\infty}} \sqrt[n]{|b_n|}}$ (Hadamard)

   (i) *Differentiation and Integration:* Suppose $\sum_{n=0}^{\infty} b_n x^n$ is a power series with radius of convergence $r > 0$. Let $f(x) = \sum_{n=0}^{\infty} b_n x^n$. Then

i. f is differentiable on $(-r,r)$ and its derivative f' has a power series representation with the same radius of convergence. Moreover,

$$f'(x) = \sum_{n=1}^{\infty} n b_n x^{n-1}$$

for all $x \in (-r,r)$.

ii. f is integrable on $(-r,r)$ and its integral $\int f(x)\,dx$ has a power series representation with the same radius of convergence. Moreover,

$$\int f(x)\,dx = \sum_{n=0}^{\infty} \frac{b_n}{n+1}\, x^{n+1} + C$$

for all $x \in (-r,r)$.

iii. f has continuous derivatives of all orders on $(-r,r)$ and

$$b_n = \frac{f^{(n)}(0)}{n!}$$

for all $n \in \{0,1,2,\ldots\}$.

iv. *Identity Theorem:* Suppose

$$\sum_{n=0}^{\infty} a_n x^n = \sum_{n=0}^{\infty} b_n x^n$$

for all x in the intersection of the intervals of convergence of the two series. Then $a_n = b_n$ for all $n \in \{0,1,2,\ldots,\}$.

(j) *Algebraic Operations:*
Given two power series $f(x) = \sum_{k=0}^{\infty} a_k x^k$ and $g(x) = \sum_{k=0}^{\infty} b_k x^k$ with positive radius of convergence. Let $r > 0$ be chosen such that both power series converge for all $|x| < r$. Then

i. $f(x) + g(x) = \sum_{k=0}^{\infty} \big(a_k + b_k\big) x^k$.

ii. $(\alpha f)(x) = \sum_{k=0}^{\infty} (\alpha a_k) x^k$.

iii. $f(x) \cdot g(x) = \sum_{k=0}^{\infty} \left(\sum_{j=0}^{k} a_j b_{k-j} \right) x^k$.

(k) *Important Taylor Series Expansions:*

i. *Exponential Series:*
$$e^x = \sum_{k=0}^{\infty} \frac{x^k}{k!}, \quad |x| < \infty$$

ii. *Sine Series:*
$$\sin x = \sum_{k=0}^{\infty} (-1)^k \frac{x^{2k+1}}{(2k+1)!}, \quad |x| < \infty$$

iii. *Cosine Series:*
$$\cos x = \sum_{k=0}^{\infty} (-1)^k \frac{x^{2k}}{(2k)!}, \quad |x| < \infty$$

iv. *Hyperbolic Sine Series:*
$$\sinh x = \sum_{k=0}^{\infty} \frac{x^{2k+1}}{(2k+1)!}, \quad |x| < \infty$$

v. *Hyperbolic Cosine Series:*
$$\cosh x = \sum_{k=0}^{\infty} \frac{x^{2k}}{(2k)!}, \quad |x| < \infty$$

vi. *Geometric Series:*
$$\frac{1}{1-x} = \sum_{k=0}^{\infty} x^k, \quad |x| < 1$$

vii. *Binomial Series:*
$$(1+x)^\alpha = \sum_{k=0}^{\infty} \binom{\alpha}{k} x^k, \quad |x| < 1, \alpha \in \mathbb{R}$$

viii. *Logarithmic Series:*
$$\ln(x+1) = \sum_{k=1}^{\infty} (-1)^{k+1} \frac{x^k}{k}, \quad |x| < 1$$

ix. $\arctan(x) = \sum_{k=0}^{\infty} (-1)^k \frac{x^{2k+1}}{2k+1}, \quad |x| < 1$