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IDENTIFICATION OF COMPLIANT PSEUDO-RIGID-BODY MECHANISM CONFIGURATIONS RESULTING IN BISTABLE BEHAVIOR

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ABSTRACT

Bistable mechanisms, which have two stable equilibria within their range of motion, are important parts of a wide variety of systems, such as closures, valves, switches, and clasps. Compliant bistable mechanisms present design challenges because the mechanism's energy storage and motion characteristics are strongly coupled and must be considered simultaneously. This paper studies compliant bistable mechanisms which may be modeled as four-link mechanisms with a torsional spring at one joint. Theory is developed to predict compliant and rigid-body mechanism configurations which guarantee bistable behavior. With this knowledge, designers can largely uncouple the motion and energy storage requirements of a bistable mechanism design problem. Examples demonstrate the power of the theory in bistable mechanism design.

INTRODUCTION

A bistable mechanism has two stable equilibrium positions within its range of motion. This behavior is desirable for a variety of applications. However, bistable mechanism design presents a number of challenges, particularly since the mechanisms' motion and energy storage characteristics are strongly coupled. This is especially true for bistable compliant mechanisms, in which the motion and energy storage generally both take place within the same flexible segments [1]. This paper addresses the need for a simple bistable mechanism design procedure by exploring the fundamental relationships between mechanism motion and bistable behavior.

Several authors have discussed various bistable mechanism characteristics, including the design of particular examples of bistable mechanisms [1-4]. Particular interest has emerged recently in bistable micro-mechanisms, where power requirements may be greatly reduced by using bistable mechanisms, which require energy only to switch states, while requiring no energy to maintain state [5]. Bistable microvalves [6-9], microswitches and micro-relays [10-13], and even a bistable fiber-optic switch [14] have been demonstrated. A bistable system which would provide the spring force for assembling microparts has also been suggested [15]. This paper, rather than presenting examples of bistable mechanisms, develops theory that identifies mechanism configurations that guarantee bistable behavior.

EXPLORATION OF THE PROBLEM

Each of the bistable mechanism examples referenced above stores and releases energy during motion. In fact, all bistable systems require some form of energy storage because stable positions occur at local minima of potential energy. Mechanical bistable systems typically rely on strain energy storage to gain bistable behavior. Compliant bistable mechanisms represent an elegant way to achieve bistable behavior because the flexible members allow both motion and energy storage to be incorporated into one element. In addition, compliance offers several other advantages, such as diminished part count, reduced friction, and no backlash or wear [16,17].

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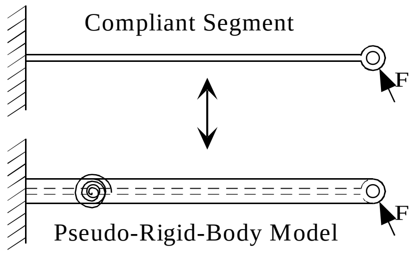


Figure 1: The concept of the pseudo-rigid-body model. Compliant segments are modeled as rigid segments joined by pin joints, with torsional springs at the joints.

However, the design of compliant bistable mechanisms is not straight-forward, requiring the simultaneous analysis of both the motion and energy storage of the mechanism [1]. To avoid this problem, most of the bistable systems presented above use a simple buckled beam to gain the bistable behavior. While this approach is simple, it gives the mechanism designer little flexibility or control in the specification of the bistable snapping force or the location of stable states. This is especially true for microbeams, which rely on residual film stress, a highly variable parameter, to induce buckling [18,19].

The pseudo-rigid-body model [20-22] provides an easy way to model the complex, nonlinear deflections of many compliant mechanisms. The model approximates the force-deflection characteristics of a compliant segment using two or more rigid segments joined by pin joints, with torsional springs at the joints modeling the segment's stiffness, as illustrated in Fig. 1. The lengths of the pseudo-rigid links, as well as the stiffnesses of the torsional springs, are found using simple equations.

The usefulness of the pseudo-rigid-body model in allowing accurate analysis and synthesis of mechanism motion and energy storage characteristics has been abundantly demonstrated [1,23-27]. For the purpose of the analysis presented here, however, it is sufficient to realize that several types of compliant segments may be represented by links joined by pin joints with torsional springs. Therefore, the remainder of this paper will use rigid-body mechanism models with torsional springs at one or more joints to examine compliant mechanism motion and stability. The results of this work may then be applied to either rigid-body or compliant mechanisms, depending on the desired mechanism performance and the designer's wishes.

The Stability of Compliant Mechanisms

Deflection of compliant segments or torsional springs within a mechanism requires the application of forces to the mechanism. A mechanism is at an equilibrium position when no external forces are required to maintain the mechanism's position. An equilibrium position is stable if the mechanism returns to that position after small disturbances, but it is unstable if small disturbances cause the mechanism to change to another position. The potential energy storage can be related to the stability of the mechanism using the Lagrange-Dirichlet theorem, which states that an equilibrium position is stable if it corresponds to a local minimum of potential energy [28]. This theorem leads to a more formal definition of a bistable mechanism: a bistable mechanism is a mechanism which contains two locations of local potential energy minima within its range of motion.

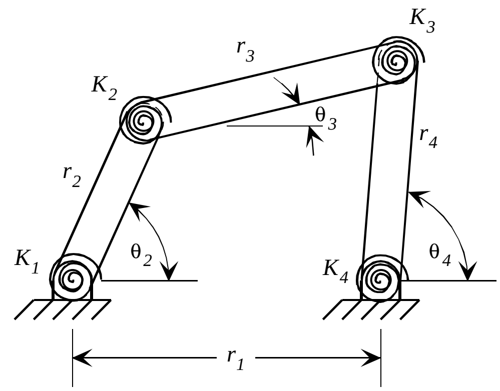


Figure 2: A four-link mechanism with a torsional spring at each joint.

Using the pseudo-rigid-body model, the potential energy equation of a compliant mechanism can easily be found. For a segment modeled using a torsional spring, the potential energy V stored in the segment is

$$V = \frac{1}{2}K\Theta^2 \tag{1}$$

where K is the torsional spring constant, and Θ is the pseudo-rigid-body angle, or the angle of deflection of the compliant segment. The total potential energy in the mechanism is the sum of the potential energy stored in each compliant segment. Equilibrium positions may be found by locating mechanism positions where the first derivative of the potential energy is zero. The sign of the second derivative at these points determines the stability of the equilibrium position, with positive corresponding to a stable position, and negative corresponding to an unstable position.

Approach to Mechanism Analysis

The model of an arbitrary fully compliant four-link mechanism is shown in Fig. 2. The model has four links, with link lengths r_1, r_2, r_3, and r_4, and four torsional springs, with spring constants K_1, K_2, K_3, and K_4. The angle of each link with respect to the horizontal is given by θ_2, θ_3, and θ_4, with link one being defined as a horizontal ground link. The torsional springs are considered to be undeflected in the fabrication position determined by link angles θ_{20}, θ_{30}, and θ_{40}. The bistable mechanism design problem consists of finding mechanism configurations which will always be bistable. To do this, each possible torsional spring location may be examined independently to determine whether a spring placed at that point in the mechanism causes bistable behavior. This is done by choosing its spring constant to be non-zero while all other spring constants are set equal to zero. The resulting potential energy equation may be differentiated, and its derivative set equal to zero. Solutions to this equation determine equilibrium locations. Therefore, the problem to be solved may be stated: *Find the torsional spring locations in a general pseudo-rigid-body four-link mechanism which produce two stable positions within the allowable motions of the mechanism.*

The solution to this problem represents an elegant and easily-applied set of design tools for bistable compliant mechanisms. It will be presented as a series of theorems governing bistable mechanism behavior, with the theorem proofs demonstrating the solution method outlined above.

THEOREMS GOVERNING BISTABLE MECHANISM BEHAVIOR

Four-link mechanisms may be classified according to Grashof's criterion [29-31] as Grashof or non-Grashof mechanisms. Grashof's criterion is stated mathematically as

$$s + l \le p + q \tag{2}$$

where s, l, p, and q are the lengths of the shortest, longest, and two intermediate-length links, respectively. Grashof's criterion, Eq. (2), allows classification of four-link mechanisms as Grashof mechanisms (those that satisfy the inequality) and non-Grashof mechanisms (those that do not satisfy it). In addition, change-point mechanisms are a subset of Grashof mechanisms for which the left and right sides of Eq. (2) are equal. In this paper, change-point mechanisms will be treated differently than all other types of Grashof mechanisms, so that the three mechanism classes treated here are Grashof (not including change-point), change-point, and non-Grashof mechanisms.

Grashof Mechanisms

Theorem 1. A compliant mechanism whose pseudo-rigid-body model behaves like a Grashof four-link mechanism with a torsional spring placed at one joint will be bistable if and only if the torsional spring is located opposite the shortest link and the spring's undeflected state does not correspond to a mechanism position in which the shortest link and the other link opposite the spring are collinear.

Corollary 1.1. A rigid-link Grashof four-link mechanism with one torsional spring placed at one joint will be bistable if and only if the torsional spring is located opposite the shortest link and the spring's undeflected state does not correspond to a mechanism position in which the shortest link and the other link opposite the spring are collinear.

Proof. Theorem 1 will be proven by analyzing the potential energy equation for a general four-link mechanism with a spring at one joint. Solutions for potential energy minima will then be analyzed to determine whether mechanism motion allows each minimum to be reached. Because of the previously-demonstrated accuracy of the pseudo-rigid-body model, the results apply equally well for either compliant mechanisms or rigid-body mechanisms. Therefore, the same proof applies to both Theorem 1 and Corollary 1.1.

Analysis of the Energy Equation. For any four-link mechanism, the energy equation is found by summing the potential energy in each spring, giving

$$V = \frac{1}{2}(K_1\psi_1^2 + K_2\psi_2^2 + K_3\psi_3^2 + K_4\psi_4^2) \tag{3}$$

where

$$\begin{aligned} \psi_1 &= \theta_2 - \theta_{20} \\ \psi_2 &= \theta_2 - \theta_{20} - (\theta_3 - \theta_{30}) \\ \psi_3 &= \theta_4 - \theta_{40} - (\theta_3 - \theta_{30}) \\ \psi_4 &= \theta_4 - \theta_{40} \end{aligned} \tag{4}$$

Choosing θ_2 as the generalized coordinate, the first derivative is

$$\frac{dV}{d\theta_2} = 0 = K_1\psi_1 + K_2\psi_2\left(1 - \frac{d\theta_3}{d\theta_2}\right) + K_3\psi_3\left(\frac{d\theta_4}{d\theta_2} - \frac{d\theta_3}{d\theta_2}\right) + K_4\psi_4\frac{d\theta_4}{d\theta_2} \tag{5}$$

Because this mechanism may be inverted so that any of its links is ground, only one spring position needs to be analyzed, and the results may then be applied to any of the four spring positions. Position 4 is chosen because the equations are somewhat simpler, and because θ_2, the generalized coordinate, does not appear in the expression for ψ_4 given in Eq. (4). If K_4 is exclusively non-zero, Eq. (5) becomes

$$0 = K_4(\theta_4 - \theta_{40})\frac{d\theta_4}{d\theta_2} \tag{6}$$

The first part of this equation, $\theta_4 - \theta_{40} = 0$, provides two solutions corresponding to the two ways that the mechanism can be assembled. That is, for any given link lengths r_1, r_2, r_3, and r_4, and the initial angle of the fourth link, θ_{40}, two different mechanism positions exist, assuming that θ_{40} does not correspond to an extreme value and that the mechanism can be assembled. An example is shown in Fig. 3. The exact positions may be found by solving the Freudenstein equations [32]:

$$\begin{aligned} r_2\cos\theta_2 + r_3\cos\theta_3 &= r_1 + r_4\cos\theta_{40} \\ r_2\sin\theta_2 + r_3\sin\theta_3 &= r_4\sin\theta_{40} \end{aligned} \tag{7}$$

The solutions to these equations are

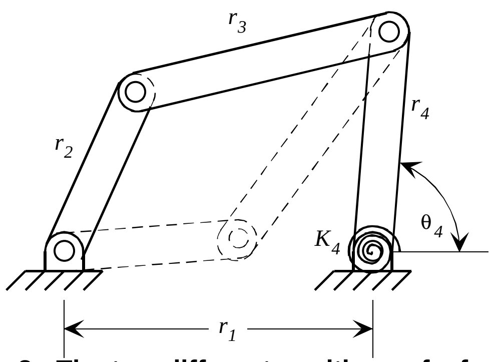


Figure 3: The two different positions of a four-link mechanism for a given angle θ_4.

$$\begin{array}{ccc} \theta_2 = \theta_{20} & & \theta_2 = 2\theta_u - \theta_{20} \\ \text{and} & \text{or} & \text{and} \\ \theta_3 = \theta_{30} & & \theta_3 = 2\theta_u - \theta_{30} \end{array} \tag{8}$$

where

$$\tan\theta_u = \frac{r_4 \sin\theta_{40}}{r_1 + r_4 \cos\theta_{40}} \tag{9}$$

and θ_{20} and θ_{30} are the initial angles of the second and third links, respectively. Note, however, that if θ_{20} is equal to θ_u, then these two solutions are identical to each other. This is the case of an extreme value for θ_{40}.

The second part of Eq. (6), the derivative, may be written

$$\frac{d\theta_4}{d\theta_2} = \frac{r_2 \sin(\theta_3 - \theta_2)}{r_4 \sin(\theta_3 - \theta_4)} = 0 \tag{10}$$

If $\sin(\theta_3 - \theta_4) \neq 0$, then this equation has two solutions:

$$\begin{aligned} \theta_2 &= \theta_3 \\ \theta_2 &= \theta_3 + \pi \end{aligned} \tag{11}$$

Therefore, the derivative term will be zero when links two and three are collinear, unless the denominator of Eq. (10) is also zero at this point. However, if the denominator is zero, it implies that links three and four are also collinear, which indicates that the mechanism is a change-point mechanism. This case will be examined later.

Interpretation of Solutions. The analysis presented above has shown that four solutions exist to the first derivative of the energy equation for a spring placed at any link of a four-link mechanism. The first two solutions, given in Eq. (8), are stable positions of the mechanism, while the two solutions in Eq. (11) are unstable positions unless θ_{40} is extreme-valued, as defined above. If this is the case, then the single solution to Eq. (7) will be stable, and it will be equivalent to one of the two solutions given in Eq. (11). Therefore, the potential energy equation will have at most two extrema over the mechanism's motion — one stable position and one unstable position. This proves that a four-link mechanism with a spring at one joint will not be bistable if the two links opposite it are collinear in the initial position.

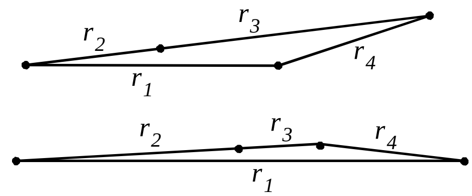


Figure 4: A graphical representation near the limits required for θ_2 to be equal to θ_3.

While the two stable positions are possible for any configuration of link lengths and one torsional spring, except for the extreme value case previously discussed, the unstable positions can not be reached in some configurations. In other words, a mechanism can always be assembled in either stable position, but it may not be able to toggle between the stable positions after assembly. To demonstrate this, consider a mechanism in either unstable position, when the two links opposite the spring are collinear. For a mechanism to reach the point where $\theta_2 = \theta_3$, two inequalities must be satisfied, as shown in Fig. 4. These are

$$\begin{aligned} r_1 + r_4 &\geq r_2 + r_3 \\ |r_1 - r_4| &\leq r_2 + r_3 \end{aligned} \tag{12}$$

Similarly, if θ_2 and θ_3 differ by π radians, the following two conditions must be met:

$$\begin{aligned} r_1 + r_4 &\geq |r_2 - r_3| \\ |r_1 - r_4| &\leq |r_2 - r_3| \end{aligned} \tag{13}$$

The second condition of Eq. (12) and the first condition of Eq. (13) can both be proven for any four-link mechanism by showing that the difference of the lengths of any two links is less than or equal to the sum of the lengths of the other two links. To prove this, consider the inequality which must be satisfied for a mechanism to be assembled. For four given link lengths, the length of the longest link must be less than or equal to the sum of the lengths of the other links. Mathematically,

$$s + p + q \geq l \tag{14}$$

where s, l, p, and q are as defined in Eq. (2). Algebra gives

$$\begin{aligned} l - q &\leq s + p \\ l - p &\leq s + q \\ l - s &\leq p + q \end{aligned} \tag{15}$$

In addition, because l is the length of the longest link, the following inequalities result:

$$\begin{aligned} p - s &< l + q \\ q - s &< l + p \\ |p - q| &< l + s \end{aligned} \tag{16}$$

These six inequalities prove that the difference of any two link lengths is less than or equal to the sum of the other two link lengths for any four-link mechanism, so that the second condition of Eq. (12) and the first condition of Eq. (13) are satisfied. However, for the mechanism to be bistable, it must be able to satisfy at least one of the other two inequalities in Eq. (12) or (13), showing that it is able to reach one of the two unstable positions to toggle into the other stable position. To determine which mechanism configurations are bistable, every possible configuration of link lengths will be considered.

A Few Intermediate Results. Before presenting proofs for each mechanism configuration's ability to reach an unstable position, three useful relations will be stated. The first two state that the sum of the lengths of the longest link and one intermediate-length link is greater than or equal to the sum of the lengths of the other two links:

$$l + p \geq q + s \tag{17}$$

and

$$l + q \geq p + s \tag{18}$$

The third useful relation expresses the fact that the difference between l and s will always be greater than or equal to the difference between p and q:

$$l - s \geq |q - p| \tag{19}$$

Eqs. (17), (18), and (19) will be used extensively in the determination of which mechanism configurations can reach the unstable positions.

The material presented up to this point proves that for a spring placed at any of its four joints, a four-link mechanism may be assembled in one of two stable positions. However, it will only be able to toggle between the two positions if one of the two unstable positions can be reached. These unstable positions correspond to the positions where the two links opposite the spring are collinear, or, in other words, when they have the same angle or their angles differ by π radians. For the mechanism to reach the position where the two opposite links' angles are identical, the first condition of Eq. (12) must be met:

$$r_{a1} + r_{a2} \geq r_{o1} + r_{o2} \qquad \text{Condition One (C1)} \tag{20}$$

where r_{a1} and r_{a2} are the lengths of the two links adjacent to the spring, and r_{o1} and r_{o2} are the lengths of the two links opposite the spring. We will call this Condition One (C1) for a four-link bistable mechanism. Similarly, for the mechanism to reach the position where the two opposite links' angles differ by π radians, the second condition of Eq. (13) must be satisfied:

$$|r_{a1} - r_{a2}| \leq |r_{o1} - r_{o2}| \qquad \text{Condition Two (C2)} \tag{21}$$

We will call this Condition Two (C2) for a four-link bistable mechanism. For a complete analysis of which spring positions result in a bistable mechanism, each spring position must be examined to determine if either or both of C1 and C2 are satisfied. If both are satisfied, then that spring position results in a bistable mechanism that can reach its two stable positions by rotation in either direction. If exactly one is satisfied, then that position gives a bistable mechanism that can reach its two stable positions by toggling through just one of the two unstable states. If neither is satisfied, then that spring position does not result in a bistable mechanism.

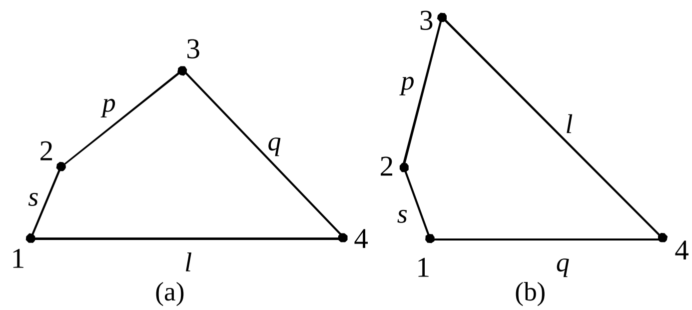


Figure 5: The two kinematic chains which form a four-link mechanism.

For either a Grashof, a change-point, or a non-Grashof mechanism, the mechanism can form one of two kinematic chains, or basic ways that the mechanism can be formed. These are illustrated in Fig. 5. In Fig. 5(a), the shortest and longest links are adjacent, and in Fig. 5(b) they are opposite. Each basic chain will be considered.

Conclusion of Proof. The material presented to this point applies equally to any four-link mechanism. This last section of the proof, however, applies only to Grashof mechanisms. We will first consider a mechanism with a spring at position 1. For a Grashof mechanism of the type shown in Fig. 5(a) with a spring placed at position 1,

$$s + l < p + q \tag{22}$$

which violates C1 because the sum of the lengths of the two adjacent links is less than the sum of the lengths of the two opposite links. Similarly, by Eq. (19), C2 is also violated. For a Grashof mechanism of the type shown in Fig. 5(b) with a spring at position 1,

$$q - s > l - p \tag{23}$$

which violates C2. By Eq. (17), C1 is violated. Hence, a Grashof mechanism with a spring at position 1 will not be bistable for either kinematic chain.

By following the same method, each spring position can be analyzed to determine whether it results in bistable behavior. The results for Grashof mechanisms are shown in Table 1. In this table, spring position 1a means position 1 in Fig. 5(a), position 1b means position 1 in Fig. 5(b), and so on. The table shows that for either kinematic chain, the mechanism will be bistable if the spring is placed at positions 3 or 4. This means that a Grashof mechanism will be bistable if a spring is placed at either of the

Table 1: Analysis of the eight spring positions in Fig. 5 for a Grashof mechanism. The inequality proving that the condition is met or not met is shown, along with the source of the inequality (Grash. = Grashof's law, otherwise, the equation number is given).

	C1 met?	Proof	Source	C2 met?	Proof	Source
1a	No	$s+l<p+q$	Grash.	No	$l-s\geq\|q-p\|$	(19)
1b	No	$q+s\leq l+p$	(17)	No	$q-s>l-p$	Grash.
2a	No	$p+s\leq l+q$	(18)	No	$p-s>l-q$	Grash.
2b	No	$p+s\leq l+q$	(18)	No	$p-s>l-q$	Grash.
3a	Yes	$p+q>l+s$	Grash.	Yes	$\|q-p\|\leq l-s$	(19)
3b	Yes	$l+p\geq q+s$	(17)	Yes	$l-p<q-s$	Grash.
4a	Yes	$l+q\geq p+s$	(18)	Yes	$l-q<p-s$	Grash.
4b	Yes	$l+q\geq p+s$	(18)	Yes	$l-q<p-s$	Grash.

two joints that are not adjacent to the shortest link. In addition, any Grashof mechanism that satisfies one condition satisfies the other, meaning that the mechanism can toggle through either unstable position to reach the second stable position. Therefore, we have proven Theorem 1 and Corollary 1.1.

Non-Grashof Mechanisms

Theorem 2. A compliant mechanism whose pseudo-rigid-body model behaves like a non-Grashof four-link mechanism with a torsional spring at any one joint will be bistable if and only if the spring's undeflected state does not correspond to a mechanism position in which the two links opposite the spring are collinear.

Corollary 2.1. A non-Grashof rigid-link four-link mechanism with a torsional spring at any one joint will be bistable if and only if the spring's undeflected state does not correspond to a mechanism position in which the two links opposite the spring are collinear.

Proof. Once again, the proven accuracy of the pseudo-rigid-body model allows us to prove Theorem 2 and Corollary 2.1 simultaneously. All of the material presented in the preceding proof, except for the last section, applies equally to Grashof or non-Grashof mechanisms. Therefore, Theorem 2 and Corollary 2.1 can be proven by showing that a spring at any position in either mechanism configuration shown in Fig. 5 satisfies at least one of the two conditions in Eq. (20) and Eq. (21) (C1 and C2). The material already presented proves that the mechanism will not be bistable if the two links opposite the spring are collinear in the undeflected position.

For example, if a spring is placed at position 1a, following the nomenclature used earlier, then Grashof's law gives the inequality

$$s + l > p + q \tag{24}$$

which proves that the non-Grashof mechanism satisfies C1.

Table 2: Analysis of each spring position in Fig. 5 for a non-Grashof mechanism. The conventions used in Table 1 are repeated for this table.

	C1 met?	Proof	Source	C2 met?	Proof	Source
1a	Yes	$s+l>p+q$	Grash.	No	$l-s\geq\|q-p\|$	(19)
1b	No	$q+s\leq l+p$	(17)	Yes	$q-s<l-p$	Grash.
2a	No	$p+s\leq l+q$	(18)	Yes	$p-s<l-q$	Grash.
2b	No	$p+s\leq l+q$	(18)	Yes	$p-s<l-q$	Grash.
3a	No	$p+q<l+s$	Grash.	Yes	$\|q-p\|\leq l-s$	(19)
3b	Yes	$l+p\geq q+s$	(17)	No	$l-p>q-s$	Grash.
4a	Yes	$l+q\geq p+s$	(18)	No	$l-q>p-s$	Grash.
4b	Yes	$l+q\geq p+s$	(18)	No	$l-q>p-s$	Grash.

However, by Eq. (19), C2 is not satisfied. If a spring is placed at position 1b, then Eq. (17) proves that C1 is not met. Also, Grashof's law gives

$$q - s < l - p \tag{25}$$

which proves that C2 is met. Results for all other spring positions are shown in Table 2. Exactly one of the two conditions is satisfied for every possible spring position. This means that a spring placed at any of the four positions will cause a non-Grashof mechanism to be bistable, unless the spring is undeflected when the two opposite links are collinear. Therefore, Theorem 2 and Corollary 2.1 are proven.

A Note Regarding Non-Grashof Mechanisms. While a non-Grashof mechanism with a spring at one joint will always be able to reach one of the two unstable positions, Table 2 proves that it will not be able to reach the other unstable equilibrium position because just one of the two conditions is satisfied for each spring location. The information in the table also allows determination of which direction a given mechanism will be able to move to reach toggle. Notice that springs placed at 1b, 2a, 2b, and 3a result in mechanisms which only meet C2, meaning that the angles of the two links opposite the spring must differ by π radians. The other spring locations - 1a, 3b, 4a, and 4b - result in mechanisms which require the two opposite links to reach the same angle. A close look at Fig. 5 reveals that each of these positions which satisfy condition 1 is adjacent to the longest link, while each position which satisfies condition 2 is not adjacent to the longest link. This information is valuable in some design problems because meeting C2 requires the two opposite links to be able to cross each other. In situations where the two links are coplanar, as is often the case with surface micromachined MEMS, this is usually not possible.

Change-Point Mechanisms

Theorem 3. A compliant mechanism whose pseudo-rigid-body model behaves like a change-point four-link mechanism with a torsional spring placed at any one joint will be bistable if and only if the spring's undeflected state does not correspond to a mechanism position in which the two links opposite the spring are collinear.

Corollary 3.1. A rigid-body change-point four-link mechanism with a torsional spring placed at any one joint will be bistable if and only if the spring's undeflected state does not correspond to a mechanism position in which the two links opposite the spring are collinear.

Proof. Theorem 3 and Corollary 3.1 will again be proven together. For a spring placed at position 4, as shown in Fig. 3, we previously noted that when links 2 and 3 are collinear, the derivative term in Eq. (10) may result in both the numerator and denominator being zero. This is because the position where all links are collinear in a change-point mechanism is a singular position - at this point, the mechanism can move in two different ways. If it moves one direction, then $|\theta_4 - \theta_{40}|$ becomes larger; if it moves the other direction, then $|\theta_4 - \theta_{40}|$ becomes smaller. Thus, movement in one direction means that the derivative of θ_4 changes sign; in the other direction, its sign remains the same. If its sign changes, then the singular position represents a relative maximum in potential energy, while no sign change means that the potential energy continues to increase. This is true regardless of which link is shortest or longest because the change-point position may always be reached for a change-point mechanism [30]. When the mechanism reaches this position, the spring will tend to force the mechanism to move in the direction which will reduce potential energy, resulting in bistable behavior. Thus, for a change-point mechanism, a spring placed at any of the four locations will result in a mechanism with bistable behavior unless the spring is undeflected when the two opposite links are collinear, as was previously discussed. Note that because all links are collinear at the change-point position, a mechanism which has the change-point position as its initial state will not be bistable.

A Note Regarding Change-Point Mechanisms. While the argument above proves Theorem 3 and Corollary 3.1, more information about change-point mechanisms may be gained by pursuing the same analysis procedure used above for Grashof or non-Grashof mechanisms. Table 3 shows the results of examining each spring position from Fig. 5. Note that a spring at any position will cause bistable behavior when the mechanism moves through the change-point position. However, the mechanism will only be able to reach toggle in either direction if the spring is placed opposite the shortest link (spring positions 3 and 4 in the table). Therefore, a change-point mechanism behaves like a hybrid between a Grashof and a non-Grashof mechanism. A spring located at any of the four joints will cause bistable behavior, but the mechanism will be able to move in either direction toward toggle only if the spring is located opposite the shortest link. In addition, for a spring located adjacent to the shortest link, the mechanism can only reach the unstable position in which the opposite links are at the same angle if the spring is also adjacent to the longest link. If the spring is not also adjacent to the longest link, then the mechanism will only be able to reach the unstable position in which the opposite links' angles differ by π radians.

Table 3: Analysis of each spring position in Fig. 5 for a change-point mechanism. The conventions used in Table 1 are repeated for this table.

	C1 met?	Proof	Source	C2 met?	Proof	Source
1a	Yes	$s+l=p+q$	Grash.	No	$l-s\geq\|q-p\|$	(19)
1b	No	$q+s\leq l+p$	(17)	Yes	$q-s=l-p$	Grash.
2a	No	$p+s\leq l+q$	(18)	Yes	$p-s=l-q$	Grash.
2b	No	$p+s\leq l+q$	(18)	Yes	$p-s=l-q$	Grash.
3a	Yes	$p+q=l+s$	Grash.	Yes	$\|q-p\|\leq l-s$	(19)
3b	Yes	$l+p\geq q+s$	(17)	Yes	$l-p=q-s$	Grash.
4a	Yes	$l+q\geq p+s$	(18)	Yes	$l-q=p-s$	Grash.
4b	Yes	$l+q\geq p+s$	(18)	Yes	$l-q=p-s$	Grash.

Table 4: The spring locations which cause bistable behavior in four-link mechanisms.

Mechanism Class	Location of Springs for Bistable Mechanism
Grashof Four-Link Mechanism	Either location opposite the shortest link
Change-Point Four-Link Mechanism	Any location
Non-Grashof Four-Link Mechanism	Any location

Summary of Results

The results of each theorem and its proof are summarized in Table 4 for mechanisms which meet the condition that the links opposite the spring are not collinear in the initial position. An example of a four-link bistable mechanism with a spring at position 4 is shown in Fig. 6(a). For a compliant equivalent, the spring would be replaced by either a small-length flexural pivot or a fixed-pinned segment. Fig. 6(b) shows a bistable compliant mechanism made by replacing the spring and pin joint in Fig. 6(a) with a small-length flexural pivot.

APPLICATION TO BISTABLE MECHANISM DESIGN

The theory presented in the preceding sections greatly simplifies bistable mechanism design. Knowledge of the mechanism configurations which lead to bistable behavior allows a designer to focus on the other constraints of a bistable mechanism design problem, adding springs to guarantee bistable behavior after other considerations, such as mechanism path, are met. Two examples are presented to demonstrate the idea.

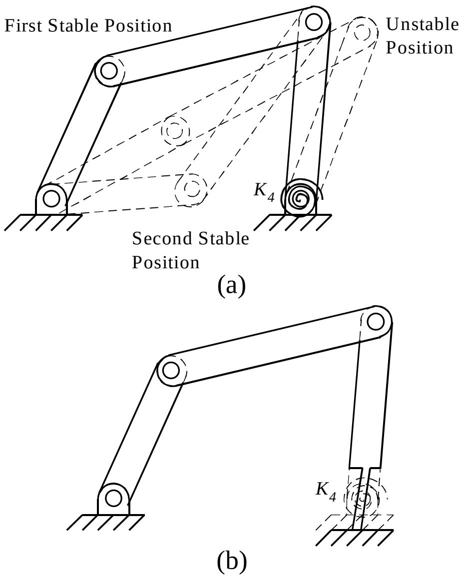
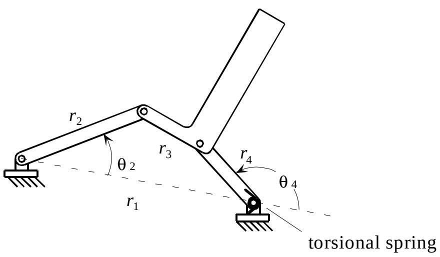


Figure 6: A bistable four-link mechanism showing the two stable positions and one unstable position (a), and a compliant equivalent (b).



r_1	3.61 cm
r_2	2.0 cm
r_3	1.0 cm
r_4	1.23 cm
θ_{40}	142°

Figure 7: A rigid-body four-link mechanism that meets the design requirements

Example: A Fully-Compliant Bistable Switch

A fully-compliant bistable light switch would allow reduced assembly cost and complexity. However, the switch should retain the look and feel of a conventional light switch for the consumer market. The problem can be most easily solved by using well-known mechanism synthesis techniques to design a rigid-body mechanism with a coupler point which moves approximately in a circular arc to mimic the motion of a conventional light switch. The bistable mechanism theory presented above can then be used to choose a joint for spring placement that will guarantee bistable behavior.

The four-link mechanism design shown in Fig. 7 satisfies the motion requirements of the problem. The table next to the drawing gives mechanism dimensions. As it is a non-Grashof mechanism, adding a spring at any joint will guarantee bistable behavior. Here, the spring is placed at position 4. The spring stiffness can be chosen to give the switch a similar force response to a conventional light switch.

The mechanism can be made fully compliant by putting a small-length flexural pivot [20] in place of the spring. The other joints can be made compliant without introducing significant stiffness by using living hinges, which are short, very thin hinges whose behavior closely approximates that of a rigid-body revolute joint. The material should be highly ductile to allow these hinges to pivot without fracture. Polypropylene is commonly used to meet this condition. The completed fully-compliant switch layout in both positions is shown in Fig. 8. It has been fabricated and tested to verify proper function.

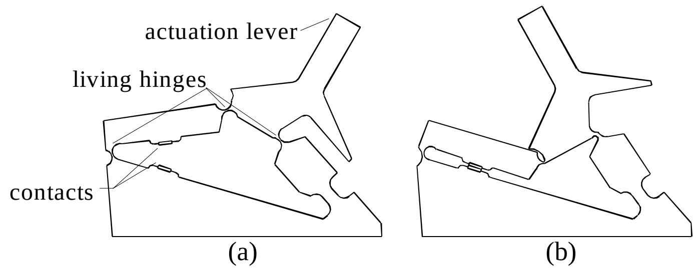


Figure 8: Layout for a fully-compliant bistable switch as-fabricated (a) and closed (b). (Patent pending.)

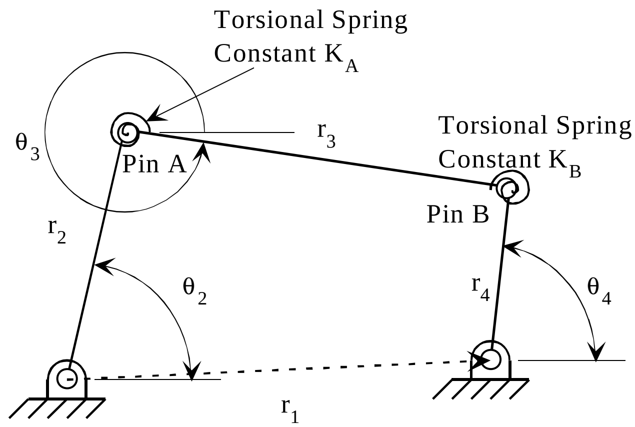


Figure 9: A model of the four-link mechanism class chosen for the bistable micro-mechanism.

Example: A Bistable Micro-Device

Applications such as micro-switching would benefit from a bistable micro-mechanism. Because of fabrication constraints in three-layer surface micromachining, pin joints are most easily constructed when they are fixed to the substrate. The requirements can be met using a four-link mechanism with two fixed pin joints. The resulting mechanism has springs placed at positions 2 and 3. A model for this mechanism is shown in Fig. 9. Because of the usefulness of this general mechanism model, it has been more rigorously defined and classified [5]. General design observations are made here.

Because of the two torsional springs in the pseudo-rigid-body model, Theorems 1 through 3 do not guarantee bistable behavior. However, by choosing one spring to be stiff compared to the other, the mechanism's behavior may be approximated by a mechanism with only one spring. Note that the other, weaker

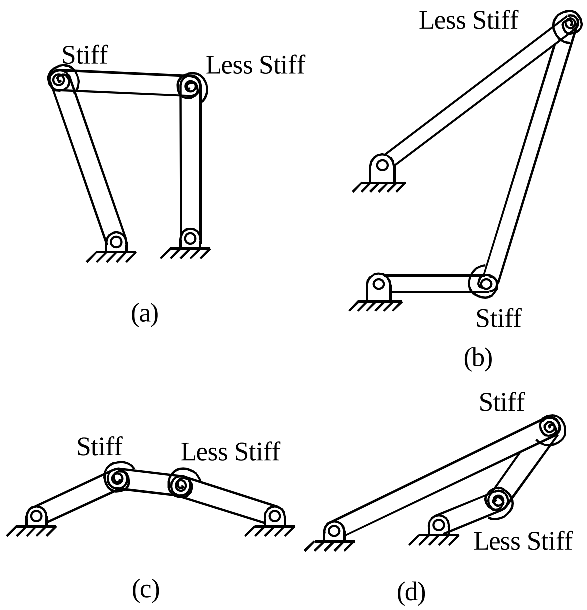


Figure 10: Four possible mechanism configurations for bistable micro-mechanisms. (a) is Grashof, and (b), (c), and (d) are non-Grashof.

spring will change the location of the second stable position, as well as the energy (and force) required to reach the unstable position. In fact, such behavior may be desired, as in a bistable switch which requires only a small force to move it out of its second energy well. Simple calculations may be used to verify bistable behavior and calculate the new stable position and force required to reach it [1]. Further generalization cannot be made because each case involving two or more springs will involve trade-offs between energy storage in each spring. However, Theorems 1 through 3 give a designer knowledge of which spring locations work toward bistable behavior and which work against it.

Fig. 10 shows mechanism designs which meet the design criteria. Fig. 10(a) is a Grashof mechanism with the shortest link as ground, and (b), (c), and (d) are non-Grashof mechanisms with the longest link as coupler, ground, and side link, respectively. Fig. 10(b) is chosen for further development. Fig. 11(a) shows how this mechanism design could be implemented as a compliant mechanism using the pseudo-rigid-body model. Because both springs are adjacent to the longest link, each requires the two links opposite it to be at the same angle in its unstable position. Thus, if each spring were considered separately, each one would require motion in opposite directions to result in bistable behavior. However, the spring on the shorter link has a much higher spring stiffness, causing its potential energy curve to dominate in the mechanism's total potential energy curve. For this reason, the mechanism is stable in the two positions shown in Fig. 11(b). Note that in the second position, the short compliant link is nearly undeflected. This example micro-mechanism has been fabricated and tested in a separate study [5].

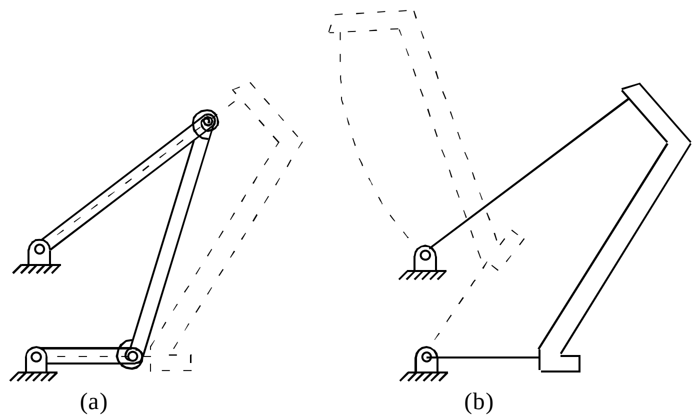


Figure 11: An example of a bistable compliant micro-mechanism whose pseudo-rigid-body model is a non-Grashof four-link mechanism. (a) shows the mechanism and its pseudo-rigid-body model, and (b) shows the two stable positions.

CONCLUSION

Four-link mechanisms have been studied to determine compliant mechanism configurations that result in bistable behavior. The analysis has shown that Grashof mechanisms will be bistable if a torsional spring is placed at either joint opposite the shortest link, provided that the two links opposite the spring are not collinear in the initial position. Similarly, change-point and non-Grashof mechanisms will be bistable if a spring is placed at any joint, subject to the same condition. This knowledge simplifies bistable mechanism design in many cases by allowing a designer to consider motion and stability requirements of a design problem separately. The two example designs have demonstrated the added design flexibility possible using the theory presented.

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