Structured and simultaneous Lyapunov functions for system stability problems

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It is shown that many system stability and robustness problems can be reduced to the question of when there is a quadratic Lyapunov function of a certain structure which establishes stability of $\dot{x} = Ax$ for some appropriate $A$. The existence of such a Lyapunov function can be determined by solving a convex program. We present several numerical methods for these optimization problems. A simple numerical example is given.

1. Notation and preliminaries

$\mathbb{R} (\mathbb{R}^n)$ will denote the set of real numbers (non-negative real numbers). The set of $m \times n$ matrices will be denoted $\mathbb{R}^{m \times n}$. $I_k$ will denote the $k \times k$ identity matrix (we will sometimes drop the subscript $k$ if it can be determined from context). $\mathbb{R}I_k$ will denote all multiples of $I_k$. If $G_i \in \mathbb{R}^{k \times k}$, $i = 1, \ldots, m$, then $\Theta_{I_k}^{m} = G_1 \oplus \cdots \oplus G_m$ will denote the block diagonal matrix with diagonal blocks $G_1, \ldots, G_m$. We extend this notation to sets of matrices, so that for example $\Theta_{I_k}^{3} \mathbb{R}$ is the set of diagonal $3 \times 3$ matrices, and

$$\mathbb{R}^{2 \times 2} \oplus \mathbb{R}I_2 = \left\{ \begin{bmatrix} a_1 & a_2 & 0 & 0 \\ a_3 & a_4 & 0 & 0 \\ 0 & 0 & a_5 & 0 \\ 0 & 0 & 0 & a_6 \end{bmatrix} \mid a_1, \ldots, a_6 \in \mathbb{R} \right\}$$

Many of our results will pertain to the basic feedback system (shown in Fig. 1),

$$\begin{align*}
\dot{x} &= Ax + Bu \\
y &= Cx + Du \\
u &= \Delta(y)
\end{align*} \tag{1}$$

where $x(t) \in \mathbb{R}^n$, $u(t), y(t) \in \mathbb{R}^k$, and $\Delta$ is a (possibly non-linear) causal operator mapping $[L^\infty(\mathbb{R}_+)]^k$ into itself (see Desoer and Vidyasagar 1975 for a complete definition of causality and more background). Throughout this paper we will assume that the linear system (1) is minimal, i.e. controllable and observable. We will say that the system (1), (2) is stable if for all solutions, $x(t)$ is bounded for $t \geq 0$.

Sometimes the operator $\Delta$ can be decomposed into a number of smaller operators in parallel, as shown in Fig. 2. More precisely, suppose $u$ and $y$ can be partitioned as $u^T = [u_1^T \ldots u_m^T]$, $y^T = [y_1^T \ldots y_m^T]$, $u_i(t), y_i(t) \in \mathbb{R}^k$, such that (2) can be expressed as

$$u_i = \Delta_i(y_i), \quad i = 1, \ldots, m \tag{3}$$

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In this case we say the operator $\Delta$ has the block structure $[k_1, \ldots, k_m]$. If $\Delta$ has block structure $[1, \ldots, 1]$, we say $\Delta$ is a diagonal operator.

The term 'block structure' and the symbol $\Delta$ follow the usage of Doyle (1982).

If $\Delta$ has block structure $[k_1, \ldots, k_m]$ and $a_1, \ldots, a_m$ are non-zero constants, we can perform the block structure preserving scaling transformation $\bar{u} = a_i u_i$, $\bar{y} = a_i y_i$, so that (1), (2) can be expressed as

$$\begin{align*}
\dot{x} &= Ax + BF^{-1}\bar{u} \\
\dot{y} &= FCx + FDF^{-1}\bar{u} \\
\bar{u} &= \bar{\Delta}(\bar{y})
\end{align*}$$

where $F = \bigoplus_{i=1}^m a_i I_{k_i}$ and $\bar{\Delta}$ is the operator defined by $\bar{u}_i = a_i \Delta_i (a_i^{-1} \bar{y}_i)$. A block diagram of this scaling transformation is shown in Fig. 3. We note for future reference
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\[ \dot{x} = Ax + Bu \]
\[ y = Cx + Du \]

\[ \Delta_m \]

\[ \Delta_1 \]

\[ \Delta \]

Figure 3. Basic feedback system with block structured feedback, showing the 'block structure preserving scaling transformation'.

that \( \Delta \) also has block structure \( [k_1, \ldots, k_m] \) and the transfer matrix of the scaled linear system (4) is \( FH(s)F^{-1} \), where \( H(s) = C(sI - A)^{-1}B + D \) is the transfer matrix of the original linear system (1).

2.Structured Lyapunov functions

We say \( A \in \mathbb{R}^{n \times n} \) is stable if all solutions of \( \dot{x} = Ax \) are bounded for \( t > 0 \), or, equivalently, all eigenvalues of \( A \) have non-positive real parts, and the purely imaginary eigenvalues are simple zeros of the minimal polynomial of \( A \). (This is often called marginal stability in the linear systems literature.) A famous result of Lyapunov theory states that \( A \) is stable if and only if there is a \( P = P^T > 0 \) such that \( A^T P + PA \leq 0 \). In this case we say the Lyapunov function \( V(x) = x^T P x \) establishes stability of the differential equation \( \dot{x} = Ax \), since \( V \) is positive definite and \( \dot{V}(x) = -2x^T P A x \) is negative semidefinite.

The topic of this paper is the following question: given \( A \), is there a \( P \) of a certain structure, for example block diagonal, for which the Lyapunov function \( V(x) = x^T P x \) establishes stability of \( \dot{x} = Ax \)? We call this the **structured Lyapunov problem** for \( A \). We will show that

(a) several problems involving stability of the basic feedback system (1), (2) can be answered by solving a structured Lyapunov problem for a certain structure and matrix \( A \), and

(b) practical (numerical) solution of the structured Lyapunov problem involves a convex minimization problem.

The implication of (b) is that there are effective algorithms for solving the structured Lyapunov problem.
Definition

Let $S$ be a subspace of $\mathbb{R}^{n \times n}$. $A \in \mathbb{R}^{n \times n}$ is $S$-structured Lyapunov stable ($S$-SLS or just SLS if $S$ is understood) if there is a $P \in S$ such that $P = P^T > 0$ and $A^T P + PA \leq 0$.

We will refer to $S$ as a "structure" and $V$ an $S$-structured Lyapunov function (S-LF) for $A$. The structures we will encounter will be very simple, usually consisting of block diagonal matrices, perhaps with some blocks repeated.

Note the distinction between a "block structure" $[k_1, ..., k_m]$ (an attribute of an operator), and a "structure" $S$ (a subset of $\mathbb{R}^{n \times n}$).

If $S = \mathbb{R}^{n \times n}$, then by Lyapunov's theorem, $A$ is SLS if and only if $A$ is stable (this could be called unstructured Lyapunov stability); but in general the condition that $A$ is $S$-SLS is stronger than mere stability of $A$. At the other extreme, if $S = \mathbb{R} I_n$, then $A$ is SLS if and only if $A + A^T \leq 0$, which is sometimes referred to as dissipative dynamics. This is precisely the condition under which all solutions of $\dot{x} = Ax$ are not only bounded, but in addition $\|x\| = (x^T x)^{1/2}$ is non-increasing. For intermediate structures, the condition that $A$ be SLS will fall between these two extremes: stability, and dissipative dynamics.

A very important special case of the structured Lyapunov problem is the following:

$$S = \{ P \oplus ... \oplus P | P \in \mathbb{R}^{n \times n} \}$$

$$A = A_1 \oplus ... \oplus A_k, \quad A_i \in \mathbb{R}^{n \times n} \quad (6)$$

In this case, $A$ is $S$-SLS if and only if there is a single Lyapunov function $V(x) = x^T P x$ (no structure requirement on $P$) which establishes the stability of the matrices $A_1, ..., A_k$. If the matrix $A$ in (7) is $S$-SLS, then we say the set of matrices $\{A_1, ..., A_k\}$ is simultaneously Lyapunov stable (SLS), and the Lyapunov function $V$ is a simultaneous Lyapunov function (SILF) for the set.

Several non-trivial cases of the structured Lyapunov problem have been investigated, notably for the case where $P$ is diagonal. This problem is considered by Araki (1976), Barker, Berman, and Plemons (1978), Moylan and Hill (1978), Khalil and Kokotovic (1979), Khalil (1982), and others; their applications range from stability of large interconnected systems to multi-parameter singular perturbations. Matrices which are diagonal-SLS or satisfy a very similar condition (for example, $A^T P + PA > 0$) are sometimes called $D$-stable or diagonally stable. In the papers cited above various relations (often, sufficient conditions for $D$-stability) have been found between such matrices and $M$-matrices, so-called quasi-dominant matrices, and $P_0$-matrices. We refer the reader to the papers mentioned above and the references therein. We mention that a structured Lyapunov stability problem with a block-diagonal structure is briefly mentioned by Khalil and Kokotovic (1979).

The simultaneous Lyapunov problem has also been investigated, more or less directly, by Horisberger and Belanger (1976), and also in connection with the absolute stability problem (see § 4) by, for example, Kamenetskiii (1983) and Pyatnitskii and Skorodinskii (1983).

In §§ 3 and 4 we show how the important system theoretic notions of passivity and non-expansivity are easily characterized in terms of SLS problems. The Kalman–Yacubovich–Popov lemma establishes the equivalence between these important system theoretic notions and the existence of quadratic Lyapunov functions which establish the stability of the basic feedback system for appropriate classes of $\Delta$s (passive and non-expansive, respectively).
More importantly, we show that the questions of whether a linear system can be scaled so as to be passive or non-expansive are also readily cast as SLS problems. These conditions are weaker than passivity or non-expansivity, and we show that the conditions are related to the existence of a quadratic Lyapunov function establishing the stability of the basic feedback system for appropriate classes of block diagonal $\Delta$s.

In § 5 we show how the general (multiple non-linearity, non-zero $D$) absolute stability problem can be attacked as a SLS problem, extending the results of Kamenetskii (1983), Pyatnitskii and Skorodinskii (1983), and Horisberger and Belanger (1976).

In § 6 we show how the results of the previous sections can be combined to yield SLS problems which can determine the existence of a quadratic Lyapunov function establishing the stability of a complex system containing sector-bounded memoryless non-linearities and non-expansive $\Delta$s.

In § 7 we discuss numerical methods for determining whether a given $A$ is $S$-SLS for some given structure $S$. We establish that this question can be cast as a non-differentiable convex programming problem, a fact which has been noted for several special cases by several authors (see § 7). We give some basic results for this optimization problem, such as optimality conditions and descriptions of subgradients and descent directions. We describe several algorithms appropriate for these convex programs.

In § 8 we present a numerical example which demonstrates some of the results of this paper.

3. Passivity and scaled passivity

Recall that the linear system (1) is passive if every solution of (1) with $x(0) = 0$ satisfies

$$\int_0^T u(t)^T y(t) \, dt \geq 0$$

(8)

for all $T \geq 0$. This implies that $A$ is stable. (We remind the reader of the minimality assumption in force, and our use of the term stable.) Passivity is equivalent to the transfer matrix $H(s) = C(sI - A)^{-1} B + D$ being positive real (PR) (see, for example, Desoer and Vidyasagar 1975), meaning

$$H(s) + H(s)^* \geq 0 \quad \text{for all } s > 0$$

(9)

Theorem 3.1

Let $S = \mathbb{R}^{n \times n} \oplus \mathbb{R}I$. Then the matrix

$$\begin{bmatrix} A & B \\ -C & -D \end{bmatrix}$$

(10)

is $S$-stable if and only if the linear system (1) is passive.

Proof

First suppose that the linear system (1) is passive. By the Kalman–Yacubovich–Popov (KYP) lemma (see, for example, Anderson 1967), there is a
symmetric positive definite matrix $P$ and matrices $L$ and $W$ such that

$$A^T P + PA = -LL^T$$  \hspace{1cm} (11)

$$PB = C^T - LW$$  \hspace{1cm} (12)

$$W^T W = D + D^T$$  \hspace{1cm} (13)

The only property of $P$ important for us, and indeed in any application, is that the function $V(x) = x^T Px$ satisfies the inequality (see, for example, Willems 1971 a, b, 1972)

$$\frac{1}{2} \frac{d}{dt} V(x) = u^T y - \frac{1}{2} (L^T x + W u)^T (L^T x + W u) \leq u^T y$$  \hspace{1cm} (14)

for any solution of (1) [equation (14) has the following simple interpretation: the time rate of increase of the Lyapunov function $V$ does not exceed the power input ($u^T y$) to the system] or equivalently

$$x^T P (Ax + Bu) \leq u^T (Cx + Du) \quad \forall x, u$$  \hspace{1cm} (15)

We rewrite (15) as

$$\begin{bmatrix} x^T P 0 \\ u^T 0 \\ 0 1 \end{bmatrix} \begin{bmatrix} A & B \\ -C & -D \end{bmatrix} \begin{bmatrix} x \\ u \end{bmatrix} \leq 0$$  \hspace{1cm} (16)

for all $x, u$. Since the left-hand matrix in (16) is in $S$, we have shown that the matrix (10) is $S$-SLS.

Now we prove the converse. Suppose that (10) is $S$-SLS. Without loss of generality we may assume the SLF has the form of the left-hand matrix in (16), so that (16) holds. If we define $V(x) = x^T Px$, then from (16) we may conclude the inequality (15), and hence (14). This implies the system (1) is passive, since (14) implies for $T \geq 0$

$$\int_0^T u(t)^T y(t) \, dt \geq -x(0)^T Px(0)$$

In the single-input single-output strictly proper case, we can recast the structured Lyapunov stability condition in Theorem 3.1 as a simultaneous Lyapunov stability condition.

**Corollary 3.2**

The single-input, single-output, strictly proper system $\dot{x} = Ax + bu$, $y = cx$, is passive if and only if the matrices $A$ and $-bc$ are simultaneously Lyapunov stable.

**Proof**

First suppose the system is passive. By Theorem 3.1, there is a symmetric positive definite matrix $P$ such that

$$\begin{bmatrix} P & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} A & b \\ -c & 0 \end{bmatrix} + \begin{bmatrix} A & b \\ -c & 0 \end{bmatrix}^T \begin{bmatrix} P & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} A^T P + PA & Pb - c^T \\ b^T P - c & 0 \end{bmatrix} \leq 0$$
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This implies \( Pb = c^T \), \( A^T P + PA \preceq 0 \) (which is the simple form the KYP lemma takes in this case). \( P \) also establishes the stability of \( -bc \), since

\[
(-bc)^T P + P(-bc) = -2c^T c \preceq 0
\]

and thus \( V(x) = x^T P x \) is an SILF for \( \{A, -bc\} \).

For the converse direction, see the proof of Corollary 3.4.

Passivity is an important tool in stability analysis. The passivity theorem (e.g. Desoer and Vidyasagar 1975) can be used to establish the stability of the feedback system (1), (2). It states that if the linear system (1) is passive and \( -\Delta \) is a passive operator, meaning for any signal \( z \) and \( T > 0 \),

\[
\int_0^T z(t)^T (-\Delta z)(t) \, dt \geq 0
\]

then the feedback system (1), (2) is stable. This conclusion is immediate from (14) and (17), since integration yields

\[
V(x(T)) \leq V(x(0)) + \int_0^T u(t)^T y(t) \, dt \leq V(x(0))
\]

and thus \( x \) is bounded for \( t \geq 0 \).

If \( -\Delta \) is not only passive but has block structure \( [k_1, \ldots, k_m] \), then it can be advantageous to apply a block structure preserving transformation to the system before applying the passivity theorem. Such a transformation does not affect the passivity of the feedback, that is, \( -\Delta \) is also passive. This results in the following less conservative condition for stability: if there exists an invertible matrix \( F \in \bigoplus_{i=1}^m \mathbb{R} I_{k_i} \) such that \( FH(s)F^{-1} \) is \( \mathbb{P} \mathbb{R} \), where \( H \) is the transfer matrix of the system (1), then the feedback system (1), (2) is stable. This block-structure preserving scaled passivity condition is also readily cast as an SLS problem.

**Theorem 3.3**

Let \( S = \mathbb{R}^{n 	imes n} \oplus \bigoplus_{i=1}^m \mathbb{R} I_{k_i} \). Then the matrix (10) is \( S \)-stable if and only if there exists an invertible \( F \in \bigoplus_{i=1}^m \mathbb{R} I_{k_i} \) such that \( FH(s)F^{-1} \) is \( \mathbb{P} \mathbb{R} \). Under this condition, the feedback system (1), (2) is stable whenever \( -\Delta \) is passive and \( \Delta \) has block structure \( [k_1, \ldots, k_m] \).

**Proof**

First suppose (10) is \( S \)-stable. We express the \( P \in S \) which establishes stability of (10) as \( P = P_0 \oplus P_1 \), where \( P_0 \in \mathbb{R}^{n \times n} \) and \( P_1 \in \bigoplus_{i=1}^m \mathbb{R} I_{k_i} \). Thus we have

\[
0 \geq \begin{bmatrix} A & B \\ -C & -D \end{bmatrix}^T \begin{bmatrix} P_0 & 0 \\ 0 & P_1 \end{bmatrix} \begin{bmatrix} A & B \\ -C & -D \end{bmatrix} = (I \oplus P_1^{1/2}) \begin{bmatrix} A & \bar{B} \\ -\bar{C} & -\bar{D} \end{bmatrix}^T (P_0 \oplus I) + (P_0 \oplus I) \begin{bmatrix} A & \bar{B} \\ -\bar{C} & -\bar{D} \end{bmatrix} (I \oplus P_1^{1/2})
\]

where \( \bar{B} = B \).
where
\begin{align*}
\bar{B} &= BP_i^{-1/2} \\
\bar{C} &= P_i^{1/2} C \\
\bar{D} &= P_i^{1/2} DP_i^{-1/2}
\end{align*}
(18) (19) (20)

It follows from Theorem 3.1 that \( \bar{C}(sI - A)^{-1} \bar{B} + \bar{D} = P_i^{1/2} H P_i^{1/2} \) is PR. Thus, there is an invertible (indeed, positive definite) matrix \( F = P_i^{1/2} \in \bigoplus_{i=1}^n \mathbb{R} I_k \) such that \( FH(s)F^{-1} \) is PR.

To prove the converse, suppose \( F \in \bigoplus_{i=1}^n \mathbb{R} I_k \) and \( FH(s)F^{-1} = (FC)(sI - A)(BF^{-1})^{-1} \) is PR. By Theorem 3.1 there is a symmetric positive definite \( P_0 \in \mathbb{R}^{n \times n} \) and positive \( p_i \in \mathbb{R} \) such that \( P_0 \bigoplus p_i I_k \) establishes stability of
\[
\begin{bmatrix}
A & BF^{-1} \\
-FC & FDF^{-1}
\end{bmatrix}
\]
By a calculation similar to the one above, it follows that \( P = P_0 \bigoplus p_i F^2 \in \mathcal{S} \) and establishes stability of the matrix \( (10) \). We defer the proof of the last assertion in Theorem 3.3.

Just as in Corollary 3.2, we can recast the structured Lyapunov stability condition appearing in Theorem 3.3 as a simultaneous Lyapunov stability condition when the linear system is strictly proper and the structure is diagonal.

**Corollary 3.4**

Suppose that \( D = 0, b_i \neq 0 \) and \( c_i \neq 0, i = 1, \ldots, k \), where \( b_i (c_i) \) is the \( i \)th column (row) of \( B (C) \). Then there exists a diagonal invertible \( F \) such that \( FHF^{-1} \) is PR if and only if \( \{ A, -b_1 c_1, \ldots, -b_k c_k \} \) is simultaneously Lyapunov stable.

**Proof**

First suppose that \( FHF^{-1} \) is PR. Let \( P_0 \bigoplus P_1 \in \mathbb{R}^{n \times n} \bigoplus \bigoplus_{i=1}^k \mathbb{R} \) establish stability of the matrix \( (10) \) in Theorem 3.3, where \( P_0 \in \mathbb{R}^{n \times n} \) and \( P_1 \in \bigoplus_{i=1}^k \mathbb{R} \). Then we have
\[
\begin{bmatrix}
A^T P_0 + P_0 A & P_0 B - C^T P_1 \\
B^T P_0 - P_1 C & 0
\end{bmatrix} \preceq 0
\]
from which we conclude that \( A^T P_0 + P_0 A \preceq 0 \) and \( P_0 B = C^T P_1 \). Thus \( P_0 b_i = \lambda_i c_i^T \) where \( P_1 = \bigoplus_{i=1}^k \lambda_i I_k \). It follows that \( V(x) = x^T P_0 x \) is an SILF for \( \{ A, -b_1 c_1, \ldots, -b_k c_k \} \), since
\[
(-b_i c_i)^T P_0 + P_0 (-b_i c_i) = -\lambda_i c_i^T c_i \leq 0
\]

Conversely suppose that \( x^T P_0 x \) is an SILF for the set \( \{ A, -b_1 c_1, \ldots, -b_k c_k \} \), so that for each \( i \)
\[
(P_0 b_i c_i + c_i^T (P_0 b_i))^T \geq 0
\]
Quite generally if \( u^T + v^T \geq 0 \) for two non-zero vectors \( u \) and \( v \), then \( u = \lambda v \) for some \( \lambda > 0 \). Thus we conclude that \( P_0 b_i = \lambda_i c_i^T \) for some positive constants \( \lambda_i \) (here we use the
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We can give a more intuitive statement of Theorem 3.3, which moreover provides an interpretation of the SLF of Theorem 3.3.

Theorem 3.5
There exists an invertible \( F \in \Theta_{m-1}^{m} \oplus I_{k} \) such that \( FH(s)F^{-1} \) is PR if and only if there is a symmetric positive definite \( P_0 \in \mathbb{R}^{n \times n} \) and positive constants \( \lambda_1, \ldots, \lambda_m \) such that for all solutions of (1), with \( V(x) = x^{T}P_0x \), we have

\[
\frac{d}{dt} V(x(t)) \leq \sum_{i=1}^{m} \lambda_i u_i(t)^{T}y_i(t)
\]  

(21)

Like (14), (21) has a simple and obvious interpretation. We note that this theorem provides an immediate proof of the last assertion of Theorem 3.3, since if (21) and (17) hold, integration yields

\[
V(x(T)) \leq V(x(0)) + \sum_{i=1}^{m} \lambda_i \int_{0}^{T} u_i(t)^{T}y_i(t) \, dt \leq V(x(0))
\]

(22)

and hence the stability of the feedback system.

We also note that Theorem 3.5 shows that the structured Lyapunov condition of Theorem 3.3 is very nearly the most general condition under which a quadratic Lyapunov function exists which establishes stability of the feedback system (1), (2) for arbitrary \( \Delta \) with block structure \([k_1, \ldots, k_m]\) and \(- \Delta \) passive. The gap is simply this: the quadratic Lyapunov function \( V \) would still establish stability if it satisfied (22), but with some of the \( \lambda_i \) zero, as opposed to positive.

Proof
Suppose \( F \in \Theta_{m-1}^{m} \oplus I_{k} \) is invertible and \( FH(s)F^{-1} \) is PR. Let the matrix of the SLF of Theorem 3.3 be

\[
P_0 \oplus \lambda_1 I_{k} \oplus \ldots \oplus \lambda_m I_{k_m}
\]

(23)

Routine computations establish (21).

Conversely if (21) holds, let us define \( F = \oplus_{i=1}^{m} \sqrt{\lambda_i} I_{k_i} \). Then the calculation (22) implies that \( FH(s)F^{-1} \) is PR.

\[\Box\]

4. Non-expansivity and scaled non-expansivity

We now turn to the important notion of non-expansivity. The linear system (1) is non-expansive if every solution with \( x(0) = 0 \) satisfies

\[
\int_{0}^{T} y(t)^{T}y(t) \, dt \leq \int_{0}^{T} u(t)^{T}u(t) \, dt, \quad \forall T > 0
\]

(24)

Non-expansivity also implies that \( A \) is stable. In terms of the transfer matrix \( H(s) = C(sI - A)^{-1}B + D \), non-expansivity is equivalent to

\[
\|H\|_{\infty} = \sup_{s \in \mathbb{C}} \sigma_{\text{max}}(H(s)) \leq 1
\]

(25)

where \( \sigma_{\text{max}}( \cdot ) \) denotes the maximum singular value.
If the linear system (1) is non-expansive, then the feedback system (1), (2) is stable for any non-expansive $\Delta$, meaning
\[
\int_0^T \Delta(z)^T \Delta(z) \, dt \leq \int_0^T z^T z \, dt \quad \forall z, T \geq 0
\tag{26}
\]
A simple proof of this follows from a non-expansivity form of the KYP lemma which states that the linear system (1) is non-expansive if and only if there exists a symmetric positive definite $P \in \mathbb{R}^{n \times n}$ such that with $V(x) = x^T P x$, we have
\[
\frac{d}{dt} V(x(t)) \leq u(t)^T u(t) - y(t)^T y(t)
\tag{27}
\]
for any solution of (1). (Equation (27) has exactly the same interpretation as (14), if we think of $u$ and $y$ as scattering variables, since then $u^T u - y^T y$ represents the power input to the system (1)). Integration of (27), along with (26) yields $V(x(T)) \leq V(x(0))$ for all $T \geq 0$.

By means of the Cayley transformation, the results on passivity in the previous section can be made to apply to non-expansivity. If $S$ is a complex $k \times k$ matrix with $\det(I + S) \neq 0$, we define its Cayley transform to be $Z = (I - S)(I + S)^{-1}$. It can be shown that $\sigma_{\text{max}}(S) \leq 1$ if and only if $Z + Z^* \geq 0$. Let us now apply the Cayley transform to the transfer matrix $H = C(sI - A)^{-1} B + D$. If $\det(I + D) \neq 0$, the transfer matrix $H = C(sI - A)^{-1} B + D$ satisfies $\|H\|_{\infty} \leq 1$ if and only if $G = (I - H)(I + H)^{-1}$ is PR. A state space realization of $G$ can be derived: $G = C_c(sI - A_c)^{-1} B_c + D_c$, where
\[
A_c = A - B(I + D)^{-1} C
B_c = B(I + D)^{-1}
C_c = -2(I + D)^{-1} C
D_c = (I - D)(I + D)^{-1}
\tag{28}
\]

**Theorem 4.1**

Let $S = \mathbb{R}^{n \times n} \oplus \mathbb{R}^{n \times 1} \oplus \mathbb{R}^{1 \times n}$, and suppose that $\det(I + D) \neq 0$. Then the matrix
\[
\begin{bmatrix}
A_c & B_c \\
-C_c & -D_c
\end{bmatrix}
\tag{29}
\]
is $S$-stable if and only if there exists an invertible $F \in \mathbb{R}^{n \times n}$ such that $\|FHF^{-1}\|_{\infty} \leq 1$. If this condition holds, then the feedback system (1), (2) is stable for any non-expansive $\Delta$ with block structure $[k_1, \ldots, k_m]$.

**Proof**

The proof follows from Theorem 3.3, the observations above, and the facts that $\det(I + D) = \det(I + FDF^{-1})$ and
\[
F(I - H)(I + H)^{-1} F^{-1} = (I - FHF^{-1})(I + FHF^{-1})^{-1}
\]

**Remark**

To apply Theorem 4.1 when $\det(I + D) = 0$, we simply pick a sign matrix $S$ such that $\det(I + DS) \neq 0$ and apply the theorem to the modified linear system.
Lyapunov functions for system stability

\{ A, BS, C, DS \}. Let us justify this. The modified linear system has transfer matrix \( HS \).

Since \( F \) above is diagonal, it commutes with any sign matrix, so that \( F(HS)F^{-1} = FHF^{-1}S \), and thus \( \| F(HS)F^{-1} \|_\infty = \| FH \|_\infty = \| FS \|_\infty \).

Now let us show that we can always pick a sign matrix \( S \) such that \( \det (I + DS) \neq 0 \). Let \( S = s_1 \oplus \ldots \oplus s_k \), where \( s_i \in \{ -1, 1 \} \). By elementary properties of determinants, we have e.g.

\[
\det (I + D(1 \oplus s_2 \oplus \ldots \oplus s_k)) + \det (I + D(-1 \oplus s_2 \oplus \ldots \oplus s_k)) = 2 \det (I + D(0 \oplus s_2 \oplus \ldots \oplus s_k))
\]

and thus we have

\[
\sum_{s_i \in \{-1, 1\}} \det (I + DS) = 2^k
\]

Since the sum of these \( 2^k \) numbers is \( 2^k \), at least one of them is non-zero, and that is precisely what we wanted to show.

Block diagonal scaled non-expansivity can be restated in a 'KYP' form, that is, in terms of the existence of a quadratic Lyapunov function with certain properties.

**Theorem 4.2**

There exists an invertible \( F \in \bigotimes_{k=1}^m \mathbb{F}I_{k_k} \) such that \( \| FHF^{-1} \|_\infty \leq 1 \) if and only if there is a symmetric positive definite \( P \in \mathbb{R}^{n \times n} \) and positive constants \( \lambda_1, ..., \lambda_m \) such that for all solutions of (1), with \( V(x) = x^T P x \), we have

\[
\frac{d}{dt} V(x(t)) \leq \sum_{i=1}^m \lambda_i (u_i(t)^T u_i(t) - y_i(t)^T y_i(t))
\]

This can be proved by applying a Cayley transform and Theorem 3.5.

**Remark**

Doyle (1982) has studied the feedback system (1), (2) for the case when \( \Delta \) is non-expansive, has block structure \([k_1, ..., k_m] \), and in addition \( \Delta \) is a linear time-invariant system. He shows that necessary and sufficient conditions for the stability of the feedback system for all such \( \Delta \) are that

\[
\mu(H(j \omega)) < 1 \quad \forall \omega \in \mathbb{R}
\]

where \( \mu \) denotes the \(([k_1, ..., k_m] - )\) structured singular value of a matrix.

Doyle demonstrates that for any matrix \( G \) and any invertible \( F \in \bigotimes_{k=1}^m \mathbb{F}I_{k_k} \), we have \( \mu(G) \leq \sigma_{min}(FGF^{-1}) \) (and indeed if the right-hand side is minimized over \( F \), the result is thought to be an excellent approximation to \( \mu(G) \)). Thus the condition \( \| FHF^{-1} \|_\infty \leq 1 \) appearing in Theorem 4.1 immediately implies Doyle’s condition (31). Alternatively, we may note that \( \| FHF^{-1} \|_\infty \leq 1 \) is sufficient to guarantee stability of the feedback system for all (non-linear, time-varying) non-expansive \( \Delta \) with block structure \([k_1, ..., k_m] \), and hence in particular for those \( \Delta \) which are in addition linear and time-invariant. Since Doyle’s condition (31) is necessary for stability of the feedback system for all non-expansive linear time-invariant block structured \( \Delta \), it must be implied by \( \| FHF^{-1} \|_\infty \leq 1 \).
Remark

Theorem 4.1 yields an effective algorithm for computing

$$\hat{\mu}^{-1} = \inf \left\{ \| FHF^{-1} \|_\infty \left| F \in \bigoplus_{i=1}^m \mathbb{R} I_{k_i}, \det F \neq 0 \right. \right\}$$

$$\hat{\mu}$$ has the following interpretation: the feedback system (1), (2) is stable for all $\Delta$ with block structure $[k_1, \ldots, k_m]$ and $L^2$-gain at most $\hat{\mu}$, that is

$$\int_0^T \Delta(z)^T \Delta(z) \, dt \leq \hat{\mu}^2 \int_0^T z^T z \, dt \quad \forall z, T \geq 0$$

Thus $\hat{\mu}$ could be considered an upper bound on a non-linear version of Doyle's structured singular value.

5. Absolute stability problem

We now consider the system (1), (2) with $\Delta$ diagonal and memoryless, meaning

$$u_i(t) = \Delta_i(y_i)(t) = f_i(y_i(t), t)$$

where the $f_i$ are functions from $\mathbb{R} \times \mathbb{R}_+$ into $\mathbb{R}$, in sector $[\alpha_i, \beta_i]$, meaning, for all $a \in \mathbb{R}$ and $t \geq 0$

$$u_i a^2 \leq f_i(u, t) \leq \beta_i a^2$$

The absolute stability problem is to find conditions under which all trajectories of the system (1), (34) are bounded for $t \geq 0$, for all $f_i$ satisfying (35). It is well known that we do not change the absolute stability problem by restricting the $f_i$ to be time-varying linear gains, since the set of trajectories $x(t)$ satisfying (1), (34) for some $f_i$ satisfying (35) is identical with the set of trajectories satisfying the equations (1) and

$$u_i(t) = k_i(t) y_i(t)$$

for some $k_i(t)$ which satisfy

$$\alpha_i \leq k_i(t) \leq \beta_i$$

Since the time-varying linear gains (36) satisfy the sector conditions (35), it is clear that if $x$ satisfies (1) and (36) for some $k_i$ satisfying (37), then $x$ satisfies (1) and (34) for some $f_i$ satisfying the sector conditions (35). Conversely, suppose $x$ is a trajectory of (1), (34). Then $x(t)$ is also a trajectory of the linear time-varying system (1), (36), where

$$k_i(t) = \begin{cases} f_i(y_i(t), t) & y_i(t) \neq 0 \\ \alpha_i & y_i(t) = 0 \end{cases}$$

(note that the $k_i$ defined in (38) depends on the particular trajectory $x(t)$). Of course, the $k_i$ defined in (38) satisfy $\alpha_i \leq k_i(t) \leq \beta_i$.

A Lyapunov method can be used to establish absolute stability of the feedback system. The feedback system is absolutely stable if there is a symmetric positive definite $P \in \mathbb{R}^{n \times n}$ such that for any trajectory $x(t)$ satisfying (1) and (34), or, equivalently, (1) and (36), $x(t)^T P x(t)$ is non-increasing. In this case we say that the (quadratic) Lyapunov function $V(x) = x^T P x$ establishes the absolute stability of the feedback system (1), (34).
Lyapunov functions for system stability

In this section we will show that the quadratic Lyapunov function $V$ establishes absolute stability of the feedback system if and only if $V$ is a simultaneous Lyapunov function for the $2^n$ linear systems resulting when the linear time-varying gains $k_1$ are constant and set to every combination of their extreme values. For the $D = 0$ case, which is considerably simpler, this result appears in Kamenetski (1983) and is implicit in the work of Pyatnitskii and Skorodinskii (1983). We note that sufficiency of the simultaneous Lyapunov stability condition for the $D = 0$ case follows from Theorem 1 of Horisberger and Belanger (1976).

We will prove this main result after examining a more fundamental question.

5.1. Well-posedness

We first consider the question of when the feedback system is well posed for any non-linearities satisfying the sector condition (35). By this we mean simply that equations (34) and

$$y = Cx + Du$$

should determine $u$ as a function of $x$, for any $f_i$ satisfying (35). Of course, if $D = 0$ the system is well posed, since then $u_i(t) = f_i(c_i x(t), t)$, where $c_i$ is the $i$th row of $C$.

In view of the equivalence discussed above, the system will be well posed if and only if (36) and (39) determine $u$ as a function of $x$ whenever (37) holds. This is the case if and only if

$$\det \left( I - D(k_1 \oplus \cdots \oplus k_m) \right) \neq 0 \quad \forall k_i \in [a_i, b_i]$$

(40)

Let $\phi(k_1, \ldots, k_m)$ denote the left-hand side of (40).

**Theorem 5.1**

Necessary and sufficient conditions for (40) are that the $2^n$ numbers

$$\phi(k_1, \ldots, k_m), \quad k_i \in [a_i, b_i]$$

all have the same non-zero sign.

**Remark**

When the intervals $[a_i, b_i]$ are replaced by $(0, \infty)$, the condition (40) is the definition of $D$ being a 'P_0-matrix', and there is a similar necessary and sufficient condition for $D$ to be a 'P_0-matrix' (Fiedler and Ptak 1962, 1966).

**Proof**

It is clear that this condition is necessary, since the image of

$$\mathcal{X} = [a_1, b_1] \times \cdots \times [a_m, b_m]$$

under $\phi$ is connected, and therefore an interval, so if it contains numbers of different signs, it contains zero.

To prove sufficiency we will show that the maximum and minimum of $\phi$ over $\mathcal{X}$ are achieved at its vertices (it is generally false that the determinant of a polytope of matrices achieves its maximum or minimum at a vertex). Suppose that the maximum is achieved at $k^* \in \mathcal{X}$ ($\phi$ is continuous and $\mathcal{X}$ compact). We will find $k \in \mathcal{X}$ which is
an extreme point of $\mathcal{K}$ and also achieves the maximum, that is, $\phi(\tilde{k}) = \phi(k^*)$. By elementary properties of determinants

$$
\phi(k_1, k_2^*, \ldots, k_m^*) = \phi(0, k_2^*, \ldots, k_m^*) + k_1 [\phi(1, k_2^*, \ldots, k_m^*) - \phi(0, k_2^*, \ldots, k_m^*)] \tag{43}
$$

so $\phi$ is a polynomial of degree one in $k_1$. Hence the maximum of (43) over $k_1$, $\alpha_1 \leq k_1 \leq \beta_1$, must occur at an endpoint unless (43) is in fact independent of $k_1$. In the first case, $k_1^*$ is extreme (i.e. $\alpha_1$ or $\beta_1$), and we set $\tilde{k}_1 = k_1^*$; in the second case we may set $\tilde{k}_1 = \alpha_1$ without affecting the value of $\phi$. We now apply the same argument to $k_2$, and so on. We have then found a $\tilde{k}$ which achieves the maximum of $\phi$ on $\mathcal{K}$, and for which each $\tilde{k}_i$ is extreme.

A similar argument establishes that the minimum is achieved at a vertex. \hfill \Box

5.2. Existence of quadratic Lyapunov function

We suppose now that the well-posedness condition is satisfied, so that $V(x) = x^T P x$, $P = P^T > 0$, establishes absolute stability of the feedback system if and only if every trajectory $x$ of the linear time-varying system

$$
\dot{x} = (A + BK(t)(I - DK(t))^{-1} C)x \tag{44}
$$

has $V(x(t))$ non-increasing for arbitrary $K(t) \in \mathcal{K}_1 \in [\alpha_t, \beta_t]$. Of course this is equivalent to

$$(A + BK(I - DK)^{-1} C)^T P + P(A + BK(I - DK)^{-1} C) \leq 0 \tag{45}
$$

for all $K \in \mathcal{K}$.

We note that the set of matrices

$$\mathcal{A} = \{ A + BK(I - DK)^{-1} C | K \in \mathcal{X} \} \tag{46}
$$

is not in general a polytope of matrices, although in fact $\mathcal{A}$ is contained in the convex hull of the images of the vertices of $\mathcal{X}$, that is

$$\mathcal{A} \subseteq C_0 \{ A + B\tilde{K}(I - D\tilde{K})^{-1} C | \tilde{K} a \text{ vertex of } \mathcal{X} \} \tag{47}
$$

We now state the main result of this section.

**Theorem 5.2**

There exists a positive definite quadratic Lyapunov function which establishes absolute stability of the system (1), (34) if and only if the set of $2^n$ matrices

$$\{ A + BK(I - DK)^{-1} C | K \text{ a vertex of } \mathcal{X} \} \tag{48}
$$

is simultaneously Lyapunov stable.

**Proof**

If $P$ satisfies (45) for all $K \in \mathcal{X}$, then in particular it satisfies (45) for $K$ a vertex of $\mathcal{X}$. This means that $V(x) = x^T P x$ is a simultaneous Lyapunov function for the $2^n$ matrices in (48).

To prove the converse, suppose that $V = x^T P x$ is a simultaneous Lyapunov function for the matrices (48). We must show that (45) holds for all $K \in \mathcal{X}$. This follows from the fact (47) noted above, since if $V$ is an SILF for $A_1, \ldots, A_4$, then $V$ establishes stability of any matrix $A$ in their convex hull. We will give a direct proof instead.
Lyapunov functions for system stability

Let \( z \in \mathbb{R}^n \) and consider the quadratic form on the left-hand side of (45) evaluated at \( z \), that is,

\[
\psi(k_1, \ldots, k_n) = 2z^T P(A + BK(I - DK)^{-1} C)z
\]

\( \psi \) is non-positive at the vertices of \( \mathcal{K} \), and we will show that \( \psi \) achieves its maximum at a vertex of \( \mathcal{K} \), so that \( \psi \) is actually non-positive for all \( K \in \mathcal{K} \). Since this holds for all \( z \), this will show that (45) holds for all \( K \in \mathcal{K} \), and hence that \( V \) establishes absolute stability of the feedback system (1), (34).

It remains only to show that \( \psi \) achieves its maximum on \( \mathcal{K} \) at a vertex. Suppose that \( k^* \) maximizes \( \psi \) over \( \mathcal{K} \). As in our proof of the condition for well-posedness, we will show that if \( k^*_\ell \) is not extreme, then in fact \( \psi \) does not depend on \( k_\ell \) at all, and we may then set \( k^*_\ell = z_\ell \), without affecting the value of \( \psi \). We then apply the same argument to \( k_\ell \), and so on. Thus we construct a vertex of \( \mathcal{K} \) at which the maximum of \( \psi \) is achieved.

By the elementary properties of determinants, \( \psi \) is a linear fractional function of \( k_1 \) with denominator \( \phi \) defined in (40):

\[
\psi(k_1, k^*_2, \ldots, k^*_n) = \frac{\phi_0(k^*_1, \ldots, k^*_n) + k_1 \phi_1(k^*_1, \ldots, k^*_n)}{\phi(k_1, k^*_2, \ldots, k^*_n)}
\]  

(49)

(\( \phi_0 \) and \( \phi_1 \) are readily determined, but not important to our argument). By the well-posedness condition, the denominator of (49) does not vanish for \( k_1 \in [z_1, \beta_1] \).

Now if we consider a linear fractional function on an interval not containing its pole, then either it achieves its maximum at one of the endpoints only, or else is constant, and hence achieves its maximum at, say, the left endpoint (the derivative of such a function is either never zero or always zero). Thus if \( k^*_1 \) is not extreme, then in fact \( \psi \) does not depend on \( k_1 \) at all, and we may set \( k^*_1 = z_1 \) without affecting the value of \( \psi \).

A generalization of this argument may be used to prove (47).

5.3. Brayton-Tong and Safonov results

For the absolute stability problem there are two very nice results available. Brayton and Tong (1979, 1980) have derived necessary and sufficient conditions for absolute stability: simply, the existence of a convex Lyapunov function which (simultaneously) establishes stability of the \( 2^n \) matrices (48). They give an effective algorithm for constructing such a Lyapunov function or determining that none exists (in which case the system is not absolutely stable). Note that Theorem 5.2 only determines conditions for the existence of a quadratic Lyapunov function establishing absolute stability.

Safonov and Wyetzner (1987) have studied a variation on the absolute stability problem: \( A \) is single-input, single-output, memoryless, time-invariant, and incrementally sector bounded. They have shown that the stability of the system for all such \( A \) can be determined by solving an (infinite dimensional) convex program, and gives a simple algorithm for solving it. Thus for this variation of the absolute stability problem, Safonov and Wyetzner have developed an effective algorithm for determining absolute stability.
6. Comparison and hybrid results

Let us compare Theorem 4.1, which pertains to the feedback system with $\Delta$ diagonal and non-expansive, with Theorem 5.2 with the sector conditions $\alpha_i = -1$, $\beta_i = 1$, which pertains to the feedback system with $\Delta$ diagonal, non-expansive and memoryless. As mentioned above, Theorem 4.1 essentially determines the conditions under which a quadratic Lyapunov function establishes stability of the feedback system for all diagonal non-expansive $\Delta$, whereas Theorem 5.2 determines the conditions under which a quadratic Lyapunov function establishes stability of the feedback system for all diagonal non-expansive memoryless $\Delta$, a weaker condition (on $A$, $B$, $C$, $D$). Doyle's condition (31) is also weaker (as a condition on $A$, $B$, $C$, $D$) than that of Theorem 3.3; it determines the precise conditions under which the feedback system is stable for all diagonal non-expansive linear time-invariant $\Delta$. Doyle's condition (31) and the absolute stability condition of Theorem 5.2 are not comparable, that is, neither is a weaker condition on $A$, $B$, $C$, $D$.

In terms of Lyapunov functions, the difference between Theorem 4.1 and Theorem 5.2 with sector $[-1,1]$ non-linearities can be stated as follows. These theorems determine conditions under which there exists a symmetric positive definite $P \in \mathbb{R}^{n \times n}$ such that $V(x) = x^T Px$ satisfies:

(Theorem 4.1; $\Delta$ diagonal and non-expansive):

$$\frac{d}{dt} V(x(t)) \leq \sum \lambda_i (u_i(t)^2 - y_i(t)^2), \quad \lambda_i > 0$$  \hfill (50)

(Theorem 5.2; $\Delta$ diagonal, non-expansive, and memoryless):

$$\frac{d}{dt} V(x(t)) \leq 0 \quad \text{whenever} \quad \|u_i\| \leq \|y_i\|$$  \hfill (51)

It is clear that (50) implies (51).

This last observation suggests that the two theorems can be combined. Consider the case where $\Delta$ is non-expansive with block structure $[k_1, \ldots, k_m]$, with $k_1 = \ldots = k_s = 1$ and $\Delta_1, \ldots, \Delta_m$ memoryless (Fig. 4).

We will assume that this system is well posed, meaning that the absolute stability problem resulting by considering only the memoryless operators $\Delta_1, \ldots, \Delta_m$ is well posed.

A quadratic Lyapunov function $V = x^T Px$ would establish stability of the feedback system (1), (2) for all such $\Delta$ if there are positive $\lambda_{s+1}, \ldots, \lambda_m$ such that

$$\frac{d}{dt} V(x(t)) \leq \sum_{i=s+1}^m \lambda_i (u_i^T u_i - y_i^T y_i) \quad \text{whenever} \quad \|u_i\| \leq \|y_i\|, \quad i = 1, \ldots, s$$  \hfill (52)

Note that this combines the conditions (50) and (51).

We will see that the condition (52) can also be cast as a structured Lyapunov stability question. As in the absolute stability problem, we may consider the case where the memoryless non-linearities are linear time-varying gains, that is, $u_i = k_i(t) y_i$, $i = 1, \ldots, s$. Let us eliminate $u_1, \ldots, u_s$ from (1) to yield:

$$\dot{x} = A^k x + B^k \begin{bmatrix} u_{s+1} \\ \vdots \\ u_m \end{bmatrix}$$  \hfill (53)
Lyapunov functions for system stability

\[
\begin{bmatrix}
  y_{i+1} \\
  \vdots \\
  y_m
\end{bmatrix}
= \begin{bmatrix} C^{(k)} & D^{(k)} \end{bmatrix}
\begin{bmatrix} x \\
  \vdots \\
  u
\end{bmatrix}
\]

(54)

We will spare the reader the formulae for \( A^{(k)}, B^{(k)}, C^{(k)}, D^{(k)} \), only noting that they are linear fractional in the \( k_i \). Let \( A^{(k)}, B^{(k)}, C^{(k)}, D^{(k)} \) denote the state-space Cayley transform of the system (53), (54) (formulae (28); we assume that \( \det (I + D^{(k)}) \neq 0 \).

We can now state our result.

\[
\begin{array}{c}
\dot{x} = Ax + Bu \\
y = Cx + Du
\end{array}
\]

Figure 4. Basic feedback system with block structured non-expansive \( \Delta \), with \( \Delta_1, \ldots, \Delta_m \) memoryless.

\textbf{Theorem 6.1}

There exists a symmetric positive definite \( P \in \mathbb{R}^{n \times n} \) and positive \( \lambda_1, \ldots, \lambda_m \) such that for all solutions of (1), with \( V(x) = x^T P x \), (52) holds, if and only if there is a symmetric positive definite matrix in \( \mathbb{R}^{n \times n} \otimes \bigoplus_{\Delta_{s+1}, \ldots, \Delta_m \text{ in sector } [-1, 1]} \), which establishes the stability of the \( 2^n \) matrices

\[
\begin{bmatrix}
  A^{(k)} & B^{(k)} \\
  -C^{(k)} & -D^{(k)}
\end{bmatrix}
\]

(55)
where the linear gains \( k_i(t) \) are constant and set to their extreme values, \( \pm 1 \). In this case, the feedback system (1), (2) is stable for all non-expansive block structured \( A \) with \( \Delta_1, \ldots, \Delta_s \) memoryless.

The proof simply combines several of the arguments used above, and is left to the reader.

7. Numerical methods for the SLS problem

In this section we consider the problem of actually determining whether \( A \) is \( \mathbf{S} \)-SLS, given \( A \) and \( \mathbf{S} \). Our first observation is that \( A \) is \( \mathbf{S} \)-SLS if and only if the set

\[
\mathcal{P} = \{ P | P = P^T > 0, P \in \mathbf{S}, A^T P + P A \leq 0 \}
\]

is non-empty. Since \( \mathcal{P} \) is convex, we see that the question of whether \( A \) is \( \mathbf{S} \)-SLS is really a convex feasibility program, that is, a (non-differentiable) convex optimization problem. Similar observations can be found in Horisberger and Belanger (1976), Khalil (1982), Kamenetskii and Pyatnitskii (1987), and Pyatnitskii and Skorodinskii (1983). Of course, this means that there are effective algorithms for determining whether a given \( A \) is \( \mathbf{S} \)-SLS.

Although it is possible to use numerical methods to determine whether \( \mathcal{P} \) is empty, there are several reasons for in practice preferring to check the slightly stronger condition that \( \mathcal{P} \) has a non-empty interior. In terms of \( A \), this stronger condition, which we will call strict SLS or SSLS, is the existence of a symmetric positive definite \( P \in \mathbf{S} \) such that \( A^T P + P A < 0 \) (note the strict inequality here).

First, small perturbations in \( A \) (e.g. round-off error) do not destroy the SSLS property; the same cannot be said of the SLS property. In other words, the set of \( A \) which are \( \mathbf{S} \)-SSLs for some \( \mathbf{S} \) is always open, whereas the set of \( A \) which are \( \mathbf{S} \)-SLS need not be. Secondly, when the strict SSLS property is determined, it allows us to conclude asymptotic stability \((x(t) \to 0)\) of the system under study, as opposed to mere stability \((x(t) \text{ bounded})\). In the remainder of this section we will consider numerical methods for determining whether \( A \) is SSLS.

Let \( Z_1, \ldots, Z_r \) be a basis for the subspace \( \{ Z | Z = Z^T \in \mathbf{S} \} \). Then \( A \) is SSLS if and only if there exist \( a_1, \ldots, a_r \) such that

\[
\mathcal{P} = \sum_{i=1}^r a_i Q_i < 0
\]

where

\[
Q_i = -Z_i \oplus (A^T Z_i + Z_i A)
\]

Let us define

\[
\Phi(a) = \Phi(a_1, \ldots, a_r) = \max_{i=1}^r \left( \sum_{i=1}^r a_i Q_i \right)
\]

Since \( \Phi \) is positive homogeneous \((\Phi(\lambda a) = \lambda \Phi(a) \text{ for all positive } \lambda)\), \( A \) is SSLS if and only if \( \Phi^* < 0 \), where

\[
\Phi^* = \min_{|a| \leq 1} \Phi(a)
\]

If \( A \) is SSLS, then from the positive homogeneity of \( \Phi \) we conclude that the optimum \( a^* \) always occurs on a boundary of the constraint set, that is, there is at least one \( i \) with \( |a_i^*| = 1 \). Note that \( A \) is not SSLS if and only if \( a = 0 \) is optimal for (57).
Lyapunov functions for system stability

Before turning to numerical algorithms appropriate for the convex program (57), let us comment on the significance of the bounds |a_i| ≤ 1. Suppose the matrices Z_i and A are scaled so that their largest elements are of the order of one, Φ* = Φ(a*) < 0, and
\[ P = \sum_{i=1}^{r} a_i^* Z_i. \]
Then not only is \( P \odot - A^T P - PA \) positive definite, but its condition number is at most of the order of \( 1/|Φ^*| \). Thus if we test whether the minimum value \( Φ^* \) of (57) is less than, say, \( -10^{-4} \), we are really testing whether there is a \( P \) such that \( P \odot - A^T P - PA \) is positive definite and has condition number under approximately \( 10^4 \).

7.1. Descent directions, subgradients, and optimality conditions

A vector \( δa ∈ R^r \) is said to be a descent direction for \( Φ \) at \( a \) if for small positive \( h \),
\[ Φ(a + hδa) < Φ(a). \]
Note that the existence of a descent direction at \( a = 0 \) is equivalent to \( Φ \) being SSLS; it will be useful for us also to consider descent directions at other, non-zero \( a \). The conditions for \( δa \) to be a descent direction at \( a \) are readily determined from perturbation theory for symmetric matrices (Kato 1984). Let \( Φ(a) = λ \), and let \( t \) denote the multiplicity of the eigenvalue \( λ \) of \( \sum_{i=1}^{r} a_i Q_i \). Let the columns of \( U ∈ R^{n × t} \) be an orthonormal basis for the nullspace of
\[ λI - \sum_{i=1}^{r} a_i Q_i. \]
Then \( δa \) is a descent direction if and only if
\[ δa_i U^T Q_i U + ... + δa_r U^T Q_r U = G < 0 \]
and in fact
\[ \lim_{h → 0} \frac{Φ(a + hδa) - Φ(a)}{h} = λ_{max}(G). \]
Thus if the eigenvalue \( λ \) has multiplicity one, so that \( U \) is a single column, one choice of descent direction is \( δa_i = -U_i^T Q_i U \). Indeed this precisely the condition (i.e. \( t = 1 \)) under which \( Φ \) is differentiable at \( a \), and this \( δa_i \) is simply \( -∂Φ(a) \).

Whenever \( t > 1 \), (e.g. when \( a = 0 \), we have \( t = n \)) determining a descent direction (or that none exists) is much harder. One general method uses the notion of the subgradient \( ∂Φ(a) \) of a convex function \( Φ \) at \( a ∈ P^r \), defined in Rockefellar (1972) and Clarke (1983)
\[ ∂Φ(a) = \{ g ∈ R^r | Φ(\tilde{a}) - Φ(a) ≥ g^T (\tilde{a} - a), \forall \tilde{a} ∈ P^r \}. \]
\( ∂Φ(a) \) can be shown to be non-empty, compact, and convex, and moreover \( δa \) is a descent direction at \( a \) if and only if
\[ δa^T g < 0 \quad ∀ g ∈ ∂Φ(a). \]
so that descent directions correspond precisely to hyperplanes through the origin with the subgradient in the negative half-space. Thus we have the standard conclusion that there exists a descent direction at \( a \) if and only if \( 0 ∉ ∂Φ(a) \), and indeed in this case we may take as 'explicit' descent direction the negative of the element of \( ∂Φ(a) \) of least norm. In particular we have: \( A \) is SSLS if and only if \( 0 ∉ ∂Φ(0) \).

Polak and Wardi (1982) have shown that for our particular \( Φ \),
\[ ∂Φ(a) = C_0 \{ g ∈ R^r | g_i = z^T U_i^T Q_i U z, z ∈ P^r, z^T z = 1 \}. \]
In particular if \( u \) is any unit eigenvector of \((58)\) corresponding to the maximum eigenvalue \( \Phi(a) \) of the matrix \((58)\), then \( g_i = u^T Q_i u \) yields \( g \in \partial \Phi(a) \). So it is very easy to find elements of the set \( \partial \Phi(a) \).

From Polak and Wardi's characterization of the subgradient we can readily derive conditions for \( a = 0 \) to be optimal. These conditions can be found in Overton (1987) and Overton and Wormley (1987), but with a completely different proof.

**Theorem 7.1**

\( A \) is not SSLS, or, equivalently, \( 0 \) is a global minimizer of \( \Phi \), if and only if there is a non-zero \( R = R^T \geq 0 \) such that \( \text{Tr} \ Q_i R = 0, \ i = 1, \ldots, r \).

**Proof**

First suppose that \( A \) is not SSLS, so that \( 0 \in \partial \Phi(0) \). By Polak and Wardi's characterization of \( \partial \Phi(0) \), there are \( \lambda_1, \ldots, \lambda_d \), with \( \lambda_j > 0 \), \( \sum_{j=1}^d \lambda_j = 1 \), and unit vectors \( \vec{z}_1, \ldots, \vec{z}_d \), such that

\[
\sum_{j=1}^d \lambda_j z_j^T Q_i z_j = 0, \ i = 1, \ldots, r
\]

We rewrite this as \( \text{Tr} \ Q_i R = 0, \ i = 1, \ldots, r \), where we define \( R = \sum_{j=1}^d \lambda_j z_j z_j^T \). Of course \( R = R^T \geq 0 \); \( R \) is non-zero since \( \text{Tr} \ R = 1 \). This establishes one direction of Theorem 7.1.

To prove the converse, suppose that \( A \) is SSLS, say, \( Q = \sum_{j=1}^d a_j^T Q_i < 0 \). We must show that there is no non-zero \( R \) such that \( R = R^T \geq 0 \) and \( \text{Tr} \ Q_i R = 0, \ i = 1, \ldots, r \). Suppose that \( R = R^T \geq 0 \) and \( \text{Tr} \ Q_i R = 0, \ i = 1, \ldots, r \); we will show that \( R = 0 \). \( Q < 0 \), so we may write it as \( Q = -GG^T \) for some non-singular \( G \). Thus

\[
0 = \text{Tr} \left( \sum_{j=1}^d a_j^T Q_i \right) R = \text{Tr} \ Q R = -\text{Tr} \ G^T R G
\]

Since \( G^T R G \) is positive semidefinite and has trace zero, it must be the zero matrix, and thus \( R = 0 \).

### 7.2. Cutting-plane method

The algorithm we have found most effective for solving \((57)\) is Kelley's cutting-plane algorithm (Kelley 1960). The algorithm requires only the ability to evaluate the function (i.e., compute \( \Phi(a) \)) and find an element in the subgradient at a point (i.e., compute an \( e \in \partial \Phi(a) \)), which we have already explained how to do. Suppose that \( d^{(1)}, \ldots, d^{(s)} \) are the first \( s \) iterates with \( e^{(i)} \in \partial \Phi(d^{(i)}) \). Then from the definition of subgradient we have

\[
\Phi(z) \geq \max_{i=1,\ldots,s} \Phi(d^{(i)}) + e^{(i)^T}(z - d^{(i)})
\]

for all \( z \) and thus

\[
\Phi^* \geq \Phi^*_{LB} = \min_{||z|| \leq 1} \max_{i=1,\ldots,s} \Phi(d^{(i)}) + e^{(i)^T}(z - d^{(i)})
\]

(63)
The right-hand side of (63) is readily solved via linear programming, and we take \( d^{(s+1)} \) to be the argument which minimizes the right-hand side of (63), that is, \( d^{(s+1)} \) is chosen such that
\[
\Phi_{LB}^{(s)} = \max_{i=1, \ldots, r} \Phi(d^{(i)}) + g^{(i)T}(d^{(s+1)} - d^{(i)})
\]
Of course \( \Phi_{LB}^{(s)} \) is a lower bound for \( \Phi^* \), which is extremely useful in devising stopping criteria—for example, we may stop when \( \Phi_{LB}^{(s)} \) exceeds some threshold, say, \(-10^{-4}\), or when the difference \( \Phi(d^{(s)}) - \Phi_{LB}^{(s)} \) is smaller than some specified tolerance.

Although the number of constraints in the linear program which must be solved at each iteration grows with iteration number, if these linear programs are initialized at the last iterate it usually takes only a very few iterations to converge.

The great advantage of the cutting-plane algorithm is that at all times a lower bound \( \Phi_{LB}^{(s)} \) and upper bound \( \Phi_{UB}^{(s)} \) (the minimal \( \Phi(d^{(i)}) \)) on \( \Phi^* \) are maintained. Of course it is readily shown that \( \Phi_{LB}^{(s)} - \Phi_{UB}^{(s)} \to 0 \) as \( s \to \infty \), so the cutting-plane algorithm is therefore completely effective—it cannot fail unambiguously to determine in a finite number of steps whether or not \( \Phi^* < -\epsilon \) (as mentioned above, \( 1/\epsilon \) can be interpreted as a maximum allowable condition number for \( P \odot -A^T P - PA \)). The disadvantage is that the computation per iteration can be prohibitive for very large systems. In the next two subsections we describe two other algorithms for (57) which involve less computation per iteration, and thus may be appropriate for large systems.

7.3. Subgradient methods

Shor (1985) has introduced a method for solving non-differentiable convex programs such as (57), the subgradient algorithm. In appearance it is quite similar to a descent method for a differentiable convex function. Shor’s algorithm generates \( d^{(s+1)} \) as
\[
d^{(s+1)} = d^{(s)} + h_s \delta d^{(s)}
\]
where \( \delta d^{(s)} \) is the direction and \( h_s \) the step-size of the \( s \)th iteration. In a descent method, \( \delta d^{(s)} \) would be a descent direction for \( \Phi \) at \( d^{(s)} \), and then \( h_s \) might be chosen to minimize or approximately minimize \( \Phi(d^{(s)} + h_s \delta d^{(s)}) \). In Shor’s subgradient methods, the direction \( \delta d^{(s)} \) is allowed to be the negative of any element of the subgradient \( \partial \Phi(d^{(s)}) \), and usually the step size \( h_s \) depends only on the iteration number \( s \). One possible choice is:
\[
-\delta d^{(s)} \in \partial \Phi(d^{(s)}), \quad h_s = \frac{x}{s \| \delta d^{(s)} \|}
\]
where \( x \) is the largest number under one which ensures \( |d^{(s+1)}| \leq 1 \).

Thus the subgradient method requires at each iteration the computation of any element of the subgradient, as opposed to a descent direction. As we have already noted, finding an element of the subgradient \( \partial \Phi(d) \) is straightforward, essentially involving the computation of the largest eigenvalue of the symmetric matrix \( Q \) and a vector in its associated eigenspace. This computation can be very efficiently done, even for large systems (Parlett 1980).

If the subgradient \( \partial \Phi(d^{(s)}) \) subtends an angle exceeding \( \pi/2 \) from the origin, then it is possible that \( \delta d^{(s)} \) is not a descent direction, and indeed it (often) occurs that
\[ \Phi(a^{(n+1)}) > \Phi(a^{(n)}) \]. Nevertheless it can be proved that the algorithm (64), (65) has guaranteed global convergence, that is
\[
\lim_{n \to \infty} \Phi(a^{(n)}) = \Phi^* \tag{66}
\]
Thus if \( A \) is SSLS, so that \( \Phi^* < 0 \), then the subgradient algorithm (64), (65) will find an \( a \) with \( \Phi(a) < 0 \) in a finite number of iterations. These assertions follow immediately from the results in Shor (1985) or Demyanov and Vasilyev (1985).

This algorithm involves much less computation per iteration than the cutting-plane method described above, especially for large systems. It has two disadvantages. First, if \( A \) is SSLS, it may take a large number of subgradient iterations to produce an SLF. Secondly, and more important, if \( A \) is not SSLS, there is no good method to know when to stop—no good lower bounds on \( \Phi^* \) are available. In other words, the subgradient method cannot unambiguously determine that \( A \) is not SSLS—it will simply fail to produce an SLF in a large number of iterations. Even if it appears that the \( a \) are converging to zero, as they must if \( A \) is not SSLS, there is no way to be certain of this after only a finite number of iterations.

7.4. Kamenetskii–Pyatnitskii saddle point method

Kamenetskii and Pyatnitskii (1987) have developed an algorithm for (57) which involves even less computation per iteration than the subgradient method, and thus may be useful for very large systems. Kamenetskii and Pyatnitskii consider the function
\[
F(a, x) = x^\top \left( \sum_{i=1}^{s} a_i Q_i \right) x \tag{67}
\]
Recall that \( \bar{a}, \bar{x} \) is said to be a saddle point of \( F \) (Arrow et al. 1958, Rockafellar 1972) if
\[
F(\bar{a}, x) \leq F(\bar{a}, \bar{x}) \leq F(\bar{a}, x) \quad \forall x, a
\]
It is easy to see that \( \bar{a}, \bar{x} \) is a saddle point of \( F \) if and only if \( \Phi(\bar{a}) \leq 0 \) and \( Q_i \bar{x} = 0 \), \( i = 0, ..., r \), which we assume without loss of generality occurs only if \( \bar{x} = 0 \) (otherwise \( A \) is clearly not SSLS). This is Theorem 2 of Kamenetskii and Pyatnitskii (1987).

The Kamenetskii–Pyatnitskii algorithm is just the gradient method for finding saddle points of functions, most simply expressed as a differential equation for \( a \) and \( x \):
\[
\begin{align*}
\dot{x} &\equiv \frac{\partial F}{\partial x} = 2 \sum_{i=1}^{s} a_i Q_i x \\
\dot{a}_i &\equiv -\frac{\partial F}{\partial a_i} = -x^\top Q_i x
\end{align*} \tag{68}
\]
It can be shown that if \( F \) were strictly concave in \( x \) for each \( a \) (which is not true for our \( F \) (67)) and convex in \( a \) for each \( x \), then all solutions of the differential equation (68) would converge to saddle points of \( F \) (Arrow et al. 1958). Despite the fact that (67) is not concave in \( x \) for each \( a \), Kamenetskii and Pyatnitskii prove the remarkable fact that if \( A \) is SSLS, then for arbitrary initial conditions the solutions of differential equation (68) converge to saddle points of \( F \) as \( t \to \infty \). Thus \( x \to 0 \) and \( a \to \bar{a} \), where \( \Phi(\bar{a}) \leq 0 \). They show moreover that for almost all initial conditions (zero is one of the exceptions), \( \Phi(\bar{a}) < 0 \). Thus if \( A \) is SSLS, then the gradient method will find an SLF (for almost all initial \( x \) and \( a \)).
that the algorithm (64), (65) has

\text{(66)}

will find an

Of course, in practice a suitable discretization of the differential equation (68) is

(see Kamenetskii and Pyatnitskii 1987).

Compared to the cutting-plane or subgradient method, this algorithm is extremely

In other words, if \( A \) is not SSLS, there is no good bounds on \( \Phi^* \) are available. In other

\text{(67)}

\text{(68)}

\begin{align*}
\dot{x} &= \Phi(a) + Q_1 \xi + Q_2 \xi^2 \\
\text{and only if } &\Phi(\hat{a}) \leq 0 \text{ and } Q_1 \hat{x} = 0, \\
\text{otherwise occurs only if } &\hat{x} = 0 \text{ (otherwise } \\
\text{Pyatnitskii and Pyatnitskii (1987).}
\end{align*}

\text{for each } a \text{ (which is not true for } \dot{x} = \Phi(a) \text{) of the differential equation}

\text{(Kamenetskii et al. 1958). Despite the fact that Pyatnitskii prove the remarkable

\text{Thus } x \to 0 \text{ and } a \to \hat{a}, \text{ where zero is one of the
}

\text{and an output 'multiplicative' perturbation which consists of a non-expansive but

\begin{align*}
W(s) &= 1.5 \frac{s + 1}{s + 10} I_2
\end{align*}

\text{Very roughly speaking, this means that our nominal plant is moderately accurate

\text{(about 15\%)} \text{ at low frequencies (say, } \omega < 0.5 \text{), less accurate in the range } 0.5 < \omega < 3, \text{ and quite inaccurate for } \omega > 3.}
The two memoryless non-linearities \( f_1 \) and \( f_2 \) represent actuator non-linearities, and are assumed to be in sector \([0, 1/3]\).

The controller is a simple proportional plus integral (PI) controller with transfer matrix

\[
C(s) = -\left( K_P + \frac{K_I}{s} \right) \begin{bmatrix} 0.7 & -1.4 \\ 0 & 0.7 \end{bmatrix}
\]

We set \( K_P = 2\alpha - 1 \) and \( K_I = 2\alpha^2 \), which with the nominal plant without actuator non-linearity would yield closed-loop eigenvalues at approximately \(-\alpha \pm j\alpha\). Thus the parameter \( \alpha \) approximately determines the closed-loop system bandwidth.

The system can be put into the hybrid system form of Fig. 4 and thus the question of whether there exists a quadratic Lyapunov function which establishes stability of this control system can be formulated as an SLS problem (see Theorem 6.1). For this example, the appropriate structure is \( \mathcal{S} = \mathbb{R}^{6 \times 6} \oplus \mathbb{R}^J \). When formulated as a convex optimization problem as described in §7, we find \( r = 22 \) (i.e. there are 22 variables in the optimization problem), and the matrices \( Q \) are \( 40 \times 40 \), in fact block diagonal, with five \( 8 \times 8 \) blocks. Thus to generate a subgradient involves roughly five \( 8 \times 8 \) symmetric matrix eigenvalue computations.

Taking \( \varepsilon = 10^{-4} \) (see §7), which corresponds to an approximate limit of \( 10^4 \) on the condition number of acceptable \( P \oplus -A^TP - PA \), we find that when \( 0.5 \leq \alpha < 1/2 \), \( A \) is \( \mathcal{S} \)-SSLS, and when \( \alpha > 1/2 \), \( A \) is not \( \mathcal{S} \)-SSLS. Thus for \( \alpha < 1/2 \), we can find a quadratic Lyapunov function which establishes (asymptotic) stability of the control system, and for \( \alpha > 1/2 \), there exists no quadratic Lyapunov function establishing stability of our control system.

To give some idea of the performance of the algorithms discussed in §7 we will consider two cases: \( \alpha = 1 \) (\( A \) is \( \mathcal{S} \)-SSLS in this case), and \( \alpha = 2 \) (\( A \) is not \( \mathcal{S} \)-SSLS in this case).

For \( \alpha = 1 \), the cutting-plane method takes 109 iterations to find out that the system is \( \mathcal{S} \)-SSLS (\( \Phi_{\min} < -\varepsilon \)), and another 133 iterations to determine that \( \Phi^* = -0.0031 \) within at most 10% (stopping criterion: \( \Phi_{\text{UB}} - \Phi_{\text{LB}} < 0.1|\Phi_{\text{LB}}| \)). The condition number of the corresponding \( P \oplus -A^TP - PA \) is 382.

For \( \alpha = 1 \), the subgradient method takes 5507 iterations to determine that the system is \( \mathcal{S} \)-SSLS (\( \Phi_{\text{UB}} < 0 \)).

When \( \alpha = 2 \), the cutting-plane method takes 113 iterations to determine that \( A \) is not \( \mathcal{S} \)-SSLS (\( \Phi_{\text{UB}} > -\varepsilon \)). Of course, the subgradient method simply fails to find an SLF for \( A \), after a very large number of iterations we may suspect that \( A \) is not \( \mathcal{S} \)-SSLS, but we cannot be sure.

9. Conclusions

We have introduced the simple notion of structured Lyapunov stability, and shown how several important system theoretic problems involving block diagonally scaled passivity and non-expansivity can be recast as SLS problems. We have shown (Theorem 6.1) how it can be used to determine conditions which guarantee stability of the feedback system (1), (2) for all \( \Delta \) of a specified class, for example of a certain block structure, with some blocks memoryless with sector constraints and other blocks non-expansive (but possibly non-linear and dynamic).

A very important fact is that the structured Lyapunov stability problem is
Equivalent to a convex optimization problem, and can therefore be effectively solved, for example using the methods described in § 7.

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