Automatica, Vol. 13, pp. 497–505. Pergamon Press, 1977. Printed in Great Britain

Ref: OB 2 [9]
[Garrard Jordan 77

Design of Nonlinear Automatic Flight Control Systems*

1/95

WILLIAM L. GARRARD† and JOHN M. JORDAN‡

A nonlinear aircraft automatic flight control system, developed for use at high angles of attack, reduces altitude loss during stall and increases the magnitude of the angle of attack from which the aircraft can recover from stall.

Key Word Index—Aerospace control; attitude control; closed loop systems; control nonlinearities; nonlinear control systems; perturbation techniques.

Summary—A method for the synthesis of nonlinear automatic flight control systems is developed, and the performance of a control system synthesized by use of this method is compared to the performance of control system designed by use of linear quadratic optimal control theory. Comparisons are made on the basis of aircraft dynamic response at high angles of attack. It is found that the nonlinear controller reduces the altitude loss during stall and increases the magnitude of the angle of attack for which the aircraft can recover from stall.

1. INTRODUCTION

MODERN high-performance aircraft often operate in flight regimes where nonlinearities significantly affect dynamic response. For example, fighter aircraft may operate at high angles of attack where the lift coefficient cannot be accurately represented as a linear function of angle of attack or at high roll rates where nonlinear, inertial cross-coupling may result in instabilities. In such situations, dynamic response may be improved if controller design is based on nonlinear rather than linear models of aircraft dynamics.

A number of investigators have studied the problem of using optimal control theory as the basis for the design of suboptimal, feedback controllers for nonlinear systems and a systematic procedure has been developed for systems in which the nonlinearities can be expressed as a power series in the state vector[1–9]. This procedure has been applied to only a few problems of practical interest and results previously reported[10, 11] do not indicate that nonlinear control produces clear-cut improvements in dynamic response when compared with controllers designed using linear quadratic optimal control theory.

The objective of this paper is to apply nonlinear feedback control theory to the design of a flight control system which can provide acceptable dynamic response over the entire range of angle of attack which a modern high performance aircraft may operate. Control system performance is particularly critical at large angles of attack as the uncompensated dynamic characteristics of the aircraft may result in abnormal and sometimes hazardous flying qualities.

The paper is divided into three major sections. In the first section, the nonlinear equations describing the longitudinal motion of an aircraft are developed. The general equations are derived and are applied to a specific aircraft, the F-8 Crusader. Synthesis of the linear and nonlinear controllers is presented in the second section. The lesser known nonlinear case is given the majority of attention. Evaluation of the linear and nonlinear control systems are presented in the third section. It is found that the nonlinear system results in considerably improved dynamic response when compared with the linear system.

2. NONLINEAR DYNAMICAL MODEL

The forces considered and the coordinate system used are shown in Fig. 1. The drag is small compared with the lift and weight and is neglected in this analysis. The lift is separated into its wing and tail components[12].

The basic equations of longitudinal motion are

$$m\dot{u}+w\dot{\theta}) = -mg\sin\theta + L_w\sin\alpha + L_t\sin\alpha_t \quad (1)$$

$$m(\dot{w}-u\dot{\theta}) = mg\cos\theta - L_w\cos\alpha - L_t\cos\alpha_t \quad (2)$$

$$I_y\ddot{\theta} = M_w + lL_w\cos\alpha - l_tL_t\cos\alpha_t - c\dot{\theta} \quad (3)$$

where

m = mass of aircraft
u = velocity of aircraft in X direction
w = velocity of aircraft in Z direction
θ = angular displacement about Y axis, measured clockwise from the horizon as shown in Fig. 1
I_y = moment of inertia of aircraft about Y axis
L_w = wing lift
L_t = tail lift
α = wing angle of attack
α_t = tail angle of attack
M_w = wing moment

*Received 9 December 1976; revised 21 March 1977. The original version of this paper was not presented at any IFAC meeting. This paper was recommended for publication in revised form by associate editor M. Jamshidi.

†Dept. of Aerospace Engineering and Mechanics, University of Minnesota, Minneapolis, Minnesota 55455, U.S.A.
‡Redstone Arsenal, Huntsville, Alabama 35809, U.S.A.

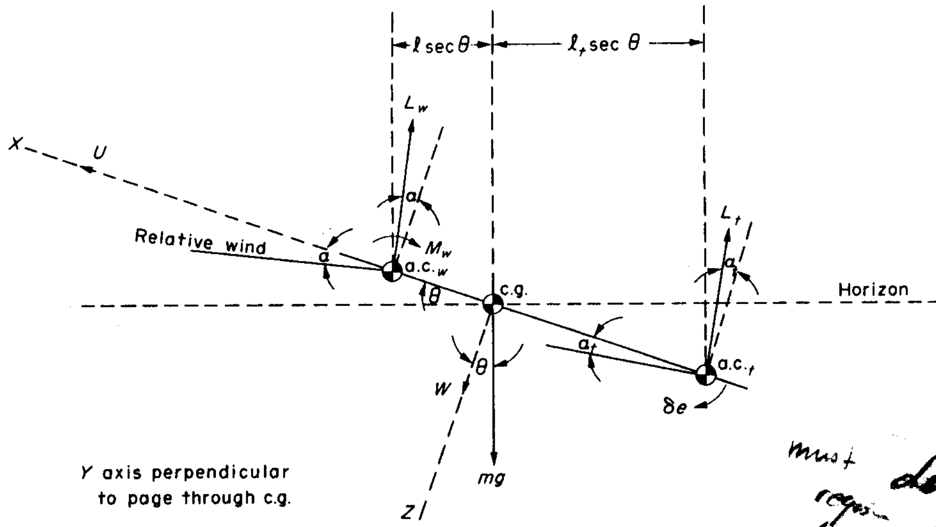

FIG. 1. Aircraft dynamical model.

l = distance between wing aerodynamic center and aircraft center of gravity

l_t = distance between tail aerodynamic center and aircraft center of gravity

$c\dot{\theta}$ = damping moment.

Equations (1)–(3) can be refined into three equations of longitudinal motion in which cubic and lower order terms are retained.

The tail and wing lift forces are

$$L_w = C_L \bar{q} S$$
$$L_t = C_{L_t} \bar{q} S_t$$

where

C_L = coefficient of wing lift

C_{L_t} = coefficient of tail lift

$\bar{q}$ = dynamic pressure

S = wing area

S_t = horizontal tail area.

Figure 2 shows the lift coefficient C_L vs α curve for the F-8 wing and linear and cubic approximations of this curve. For large angles of attack, the cubic approximation is more accurate than the linear; therefore, C_{L_w} and C_{L_t} will be approximated as

$$C_{L_w} = C^0_{L_w} + C^1_{L_w}\alpha_w - C^2_{L_w}\alpha^3_w$$
$$C_{L_t} = C^0_{L_t} + C^1_{L_t}\alpha_t - C^2_{L_t}\alpha^3_t$$

where $C^0_{L_w}$, $C^1_{L_w}$, $C^2_{L_w}$, $C^0_{L_t}$, $C^1_{L_t}$ and $C^2_{L_t}$ are constants peculiar to the individual aircraft.

The symbol δ_e represents the tail deflection angle measured clockwise from the x-axis in Fig. 1. If $\partial C_{L_t}/\partial \delta_e$ is written as a_e, then the coefficient of tail lift becomes

$$C_{L_t} = C^0_{L_t} + C^1_{L_t}\alpha_t - C^2_{L_t}\alpha^3_t + a_e\,\delta_e.$$

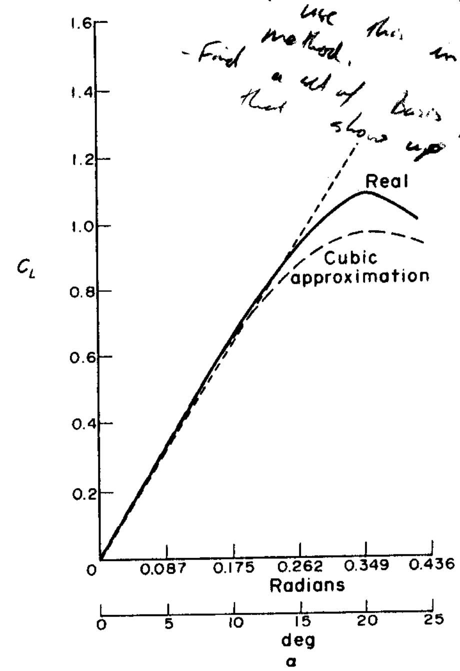

FIG. 2. C_L vs α.

Because the entire tail of the F-8 rotates, the tail angle of attack is:

$$\alpha_t = \alpha - \varepsilon + \delta_e$$

where ε is the downwash angle.

The velocity in the z direction, w, and its rate of change, $\dot{w}$, can be eliminated from (1) and (2) by noting that

$$w = u \tan\alpha \simeq u\left(\alpha + \frac{\alpha^3}{3}\right)$$

and

$$\dot{w} = \dot{u}\tan\alpha + u\dot{\alpha}\sec^2\alpha \simeq u\dot{\alpha}\sec^2\alpha$$

since $\dot u$ is very small compared with u. A nominal level flight condition of $\theta_0=0$, is assumed so that $\theta=\Delta\theta$, and

$$\cos\theta \simeq 1-\frac{\theta^2}{2} \qquad \cos\alpha \simeq 1-\frac{\alpha^2}{2}$$

$$\sin\theta \simeq \theta-\frac{\theta^3}{6} \qquad \sin\alpha \simeq \alpha-\frac{\alpha^3}{6}.$$

The F-8's tail is within the wing wake, and therefore terms including the downwash angle, ε, cannot be ignored. ε may be approximated by a linear function of α.

$$\varepsilon=\varepsilon_0+a_\varepsilon\alpha$$

where ε_0 is a constant and $a_\varepsilon=\partial\varepsilon/\partial\alpha$.

With these substitutions, (1)–(3) combine to give the general nonlinear equations of motion:

$$\dot u = -u\left(\alpha+\frac{\alpha^3}{3}\right)\dot\theta - g\left(\theta-\frac{\theta^3}{6}\right) + \left(C^0_{L_w}\alpha + C^1_{L_w}\alpha^2 - C^2_{L_w}\frac{\alpha^3}{6}\right)\frac{\bar q}{m}S + \{C^0_{L_t}+C^1_{L_t}(\alpha-\varepsilon_0-a_\varepsilon+\delta_e) - C^2_{L_t}(\alpha-\varepsilon_0-a_\varepsilon\alpha+\delta_e)^3 + a_e\delta_e\} \times\left(\alpha-\varepsilon_0-a_\varepsilon\alpha+\delta_e-\frac{(\alpha-\varepsilon_0-a_\varepsilon\alpha+\delta_e)^3}{6}\right)\frac{\bar q}{m}S_t \quad (4)$$

$$\dot\alpha = \dot\theta(1-\alpha^2)+\frac{g}{u}\left(1-\frac{\theta^2}{2}-\alpha^2\right) - \left(C^0_{L_w}+C^1_{L_w}\alpha - C^2_{L_w}\alpha^3 - C^0_{L_w}\frac{3\alpha^2}{2} - C^1_{L_w}\frac{3\alpha^3}{2}\right)\frac{\bar qS}{mu} - \{C^0_{L_t}+C^1_{L_t}(\alpha-\varepsilon_0-a_\varepsilon\alpha+\delta_e) - C^2_{L_t}(\alpha-\varepsilon_0-a_\varepsilon\alpha+\delta_e)^3+a_e\delta_e\}\times\left(1-\frac{(\alpha-\varepsilon_0-a_\varepsilon\alpha+\delta_e)^2}{2}\right)\frac{\bar q S_t}{mu} \quad (5)$$

$$\ddot\theta = M_w I_y^{-1} + \left(C^0_{L_w}+C^1_{L_w}\alpha - C^2_{L_w}\alpha^3 - C^0_{L_w}\frac{\alpha^2}{2} - C^2_{L_w}\frac{\alpha^3}{2}\right)\bar q S I_y^{-1} - \{C^0_{L_t}+C^1_{L_t}(\alpha-\varepsilon_0-a_\varepsilon\alpha+\delta_e) - C^2_{L_t}(\alpha-\varepsilon_0-a_\varepsilon+\delta_e)^3 + a_e\delta_e\} \times\left(1-\frac{(\alpha-\varepsilon_0-a_\varepsilon\alpha+\delta_e)^2}{2}\right)\bar q S_t l_t I_y^{-1} - c\dot\theta I_y^{-1} \quad (6)$$

If we consider disturbances of an F-8 in level,

"TABLE 1. F-8 AIRCRAFT DATA

$C^0_{L_w}$	$=C^0_{L_t}=0$
$C^1_{L_w}$	$=C^1_{L_t}=4.0$
$C^2_{L_w}$	$=C^2_{L_t}=12.0$
a_e	$=0.1$
S	$=375$ ft^2 (33.75 m^2)
S_t	$=93.4$ ft^2 (8.41 m^2)
m	$=667.7$ slugs (9773 kg)
a_ε	$=0.75$
ε_0	$=0$
$C_{m_{a.c.}}$	$=0$
$\bar c$	$=11.78$ ft (3.53 m)
I_y	$=96{,}800$ slug ft^2 (127,512 kg-m^2)
l	$=0.189$ ft (0.06 m)
l_t	$=16.7$ ft (5.01 m)

unaccelerated flight at Mach $=0.85$ and an altitude of 30,000 ft (9000 m) using the data given in Table 1 equations (4), (5) and (6) become

$$\dot u = 845\alpha\dot\theta - 281.67\alpha^3\dot\theta - 32.2\theta + 5.37\theta^3 + 724\alpha^2 + 0.25\delta_e\alpha \quad (7)$$

$$\dot\alpha = 0.038\dot\theta - \alpha^2\dot\theta - 0.019\theta^2 - 0.038\alpha^2 - 0.896\alpha + 3.486\alpha^3 - 0.215\delta_e + 0.28\delta_e\alpha^2 + 0.47\delta_e^2\alpha + 0.63\delta_e^3 \quad (8)$$

$$\ddot\theta = -0.396\dot\theta - 4.187\alpha - 3.564\alpha^3 - 20.967\delta_e + 6.265\delta_e\alpha^3 + 46\delta_e^2\alpha + 61.4\delta_e^3 \quad (9)$$

The trim conditions are

$$\alpha = 0.044 \text{ radians} \qquad \delta_e = -0.009 \text{ radians}.$$

Substituting the new variables, $\bar\alpha=\alpha+0.044$ and $\bar\delta_e=\delta_e-0.009$ into (8) and (9), and changing the $\bar\alpha$, $\bar\delta_e$ notation back to α, δ_e, we obtain

$$\dot\alpha = \dot\theta - \alpha^2\dot\theta - 0.088\alpha\dot\theta - 0.877\alpha + 0.47\alpha^2 + 3.846\alpha^3 - 0.215\delta_e + 0.28\delta_e\alpha^2 + 0.47\delta_e^2\alpha + 0.63\delta_e^3 - 0.019\theta^2 \quad (10)$$

$$\ddot\theta = -0.396\dot\theta - 4.208\alpha - 0.47\alpha^2 - 3.564\alpha^3 - 20.967\delta_e + 6.265\delta_e\alpha^2 + 46\delta_e^2 + 61.4\delta_e^3. \quad (11)$$

3. CONTROL SYNTHESIS

We will first synthesize a linear controller for (10) and (11) to provide a basis for comparison with the nonlinear controller.

Linearizing (10) and (11) yields

$$\dot{\alpha}=\dot{\theta}-0.877\alpha-0.215\delta_e \qquad (12)$$

$$\ddot{\theta}=-0.396\dot{\theta}-4.208\alpha-20.967\delta_e \qquad (13)$$

Letting $\alpha=x_1, \theta=x_2, \dot{\theta}=x_3, \delta_e=\mu$

$$\begin{bmatrix}\dot{x}_1\\ \dot{x}_2\\ \dot{x}_3\end{bmatrix}=\begin{bmatrix}-0.877 & 0 & 1\\ 0 & 0 & 1\\ -4.208 & 0 & -0.396\end{bmatrix}\begin{bmatrix}x_1\\ x_2\\ x_3\end{bmatrix}$$

$$+\begin{bmatrix}-0.215\\ 0\\ -20.976\end{bmatrix}\mu \qquad (14)$$

$$\dot{\mathbf{X}}=A\mathbf{X}+\mathbf{b}\mu.$$

The control is selected to minimize the quadratic performance index

$$J=\tfrac{1}{2}\int_0^\infty [\mathbf{X}^{\mathrm{T}}Q\mathbf{X}+r\mu^2]\mathrm{d}t \qquad (15)$$

It is well known that the optimal linear control is

$$\mu=-r^{-1}\mathbf{b}^{\mathrm{T}}P\mathbf{X} \qquad (16)$$

where P is the positive definite solution of the matrix Riccati equation

$$A^{\mathrm{T}}P+PA-P\mathbf{b}r^{-1}\mathbf{b}^{\mathrm{T}}P+Q=0. \qquad (17)$$

A choice of

$$Q=\begin{bmatrix}0.25 & 0 & 0\\ 0 & 0.25 & 0\\ 0 & 0 & 0.25\end{bmatrix} \text{ and } r=1$$

yielded the control law

$$\mu=-0.053x_1+0.5x_2+0.521x_3 \qquad (18)$$

which was found to give good response, without exceeding a maximum tail deflection of 25° and a tail deflection rate of 60°/sec.

In the nonlinear equations (10) and (11) terms involving δ_e^n, $n=2,3,4,\ldots$, and $\alpha^n\delta_e^m$, $n,m=1,2,3,\ldots$, are eliminated. These terms are small and the controller synthesis technique to be used cannot account for nonlinear control terms.

The resulting nonlinear equations of motion are

$$\begin{bmatrix}\dot{x}_1\\ \dot{x}_2\\ \dot{x}_3\end{bmatrix}=\begin{bmatrix}-0.877 & 0 & 1\\ 0 & 0 & 1\\ -4.208 & 0 & -0.396\end{bmatrix}\begin{bmatrix}x_1\\ x_2\\ x_3\end{bmatrix}$$

$$+\begin{bmatrix}-x_1^2x_3-0.088x_1x_3-0.019x_2^2+0.47x_1^2+3.846x_1^3\\ 0\\ -0.47x_1^2 \quad -3.564x_1^3\end{bmatrix}$$

$$+\begin{bmatrix}-0.215\\ 0\\ -20.967\end{bmatrix}\mu \qquad (19)$$

This is of the form

$$\dot{\mathbf{X}}=A\mathbf{X}+\boldsymbol{\phi}(\mathbf{X})+\mathbf{b}\mu \qquad (20)$$

where $\mathbf{X}$, A, $\mathbf{b}$, and μ are the same as for the linear case, and $\boldsymbol{\phi}(\mathbf{X})$, is an analytic vector function representing the system nonlinearities. The optimization problem is, as in the linear case, to determine the feedback control which transfers any initial state to the origin and which minimizes the quadratic index of performance (16).

As demonstrated by Lee and Markus[13], the unique optimal feedback control is

$$\mu=-\tfrac{1}{2}r^{-1}\mathbf{b}^{\mathrm{T}}\frac{\partial V}{\partial \mathbf{X}}$$

where $V(x)$ satisfies the Hamilton–Jacobi partial differential equation:

$$\frac{\partial V^{\mathrm{T}}}{\partial \mathbf{X}}A\mathbf{X}+\frac{\partial V^{\mathrm{T}}}{\partial \mathbf{X}}\boldsymbol{\phi}-\tfrac{1}{4}\frac{\partial V^{\mathrm{T}}}{\partial \mathbf{X}}\mathbf{b}r^{-1}\mathbf{b}^{\mathrm{T}}\frac{\partial V}{\partial \mathbf{X}}$$

$$+\mathbf{X}^{\mathrm{T}}Q\mathbf{X}=0, \quad V(0)=0 \qquad (21)$$

Since (21) usually cannot be solved analytically, perturbational procedures are used to obtain approximate solutions[1, 5, 6, 8, 10]. The solution of (21) can be represented in series form as

$$V(\mathbf{X})=\sum_{n=0}^{\infty}V_n(\mathbf{X}). \qquad (22)$$

If

$$\boldsymbol{\phi}(\mathbf{X})=\sum_{n=1}^{N}\mathbf{f}_{n+1}(\mathbf{X}) \qquad (23)$$

where $\mathbf{f}_{n+1}$ is of order $n+1$ in $\mathbf{X}$ then the V_n's are given by the following equations

$$\frac{\partial V_0^{\mathrm{T}}}{\partial \mathbf{X}}A\mathbf{X}-\tfrac{1}{4}\frac{\partial V_0^{\mathrm{T}}}{\partial \mathbf{X}}\mathbf{b}r^{-1}\mathbf{b}^{\mathrm{T}}\frac{\partial V_0}{\partial \mathbf{X}}+\mathbf{X}^{\mathrm{T}}Q\mathbf{X}=0$$

$$\frac{\partial V_1^{\mathrm{T}}}{\partial \mathbf{X}}A\mathbf{X}-\tfrac{1}{4}\frac{\partial V_1^{\mathrm{T}}}{\partial \mathbf{X}}\mathbf{b}r^{-1}\mathbf{b}^{\mathrm{T}}\frac{\partial V_0}{\partial \mathbf{X}}$$

$$-\tfrac{1}{4}\frac{\partial V_0^{\mathrm{T}}}{\partial \mathbf{X}}\mathbf{b}r^{-1}\mathbf{b}^{\mathrm{T}}\frac{\partial V_1}{\partial \mathbf{X}}+\frac{\partial V_0^{\mathrm{T}}}{\partial \mathbf{X}}f_2=0$$

$$\frac{\partial V_n^{\mathrm{T}}}{\partial \mathbf{X}}A\mathbf{X}-\tfrac{1}{4}\frac{\partial V_n^{\mathrm{T}}}{\partial \mathbf{X}}\mathbf{b}r^{-1}\mathbf{b}^{\mathrm{T}}\frac{\partial V_0}{\partial \mathbf{X}}$$

$$-\tfrac{1}{4}\frac{\partial V_0^{\mathrm{T}}}{\partial \mathbf{X}}\mathbf{b}r^{-1}\mathbf{b}^{\mathrm{T}}\frac{\partial V_n}{\partial \mathbf{X}}$$

$$+\frac{\partial V_0^{\mathrm{T}}}{\partial \mathbf{X}}\mathbf{f}_{n+1}+\sum_{K=1}^{n-1}\frac{\partial V_K^{\mathrm{T}}}{\partial \mathbf{X}}\mathbf{f}_{n+1-K}$$

$$-\tfrac{1}{4}\sum_{K=1}^{n-1}\frac{\partial V_K^{\mathrm{T}}}{\partial \mathbf{X}}\mathbf{b}r^{-1}\mathbf{b}^{\mathrm{T}}\frac{\partial V_{n-K}}{\partial X}=0. \qquad (24)$$

The resulting optimal control is

$$\mu=-r^{-1}\mathbf{b}^{\mathrm{T}}\sum_{n=0}^{\infty}\frac{\partial V_n}{\partial \mathbf{X}}. \qquad (25)$$

It should be noted that V_0 is quadratic function in $\mathbf{X}$, V_1 is cubic in $\mathbf{X}$, and in general V_n is of order $n+2$ in $\mathbf{X}$. Also, solution of V_n leads to the $n+1$ order control term.

For the nonlinear aircraft model

$$\phi(\mathbf{X})=\sum_{n=1}^{2}\mathbf{f}_{n+1}(\mathbf{X})$$

$$=\begin{bmatrix}0.47x_1^2-0.088x_1x_3-0.019x_2^2\\0\\-0.47x_1^2\end{bmatrix}+\begin{bmatrix}3.846x_1^3-x_1^2x_3\\0\\-3.564x_1^3\end{bmatrix} \qquad (26)$$

The solution of the first equation of (22) is

$$V_0=\mathbf{X}^{\mathrm{T}}P\mathbf{X} \qquad (27)$$

where P is the positive definite solution of (17).

In order to compare the nonlinear control with the linear control determined previously, the values of Q and r used in the derivation of the linear control are also used in the derivation of the nonlinear control. Thus, the solution for P is the same as in the linear case and the linear terms of the control are the same as the linear controller developed earlier.

Determination of the nonlinear control is very laborious. The general procedure for solving for V_n, $n=1, 2, 3,\ldots$ is as follows

1. Assume $V_n=\sum_{k=0}^{n+2-k}\sum_{j=0}^{n+2}a_{n+2-j-k,j,k}^{n}x_1^{(n+2-j-k)}x_2^jx_3^k$
2. Calculate $\frac{\partial V_n}{\partial \mathbf{X}}$.
3. Substitute $\frac{\partial V_n}{\partial \mathbf{X}}$ into (24).
4. Set the sum of coefficients of like terms equal to zero.
5. Solve the resulting simultaneous linear algebraic equations for $a_{n+2-j-k,j,k}^{n}$.

After V_n is obtained; $\partial V_n/\partial \mathbf{X}$ can be calculated and substituted into (25) to obtain μ_{n+1}. The algebra encountered is simple but tedious.

For the second-order control, ten unknown coefficients must be found. Four of these coefficients are nearly zero and the resulting expression for V_1, $\partial V_1/\partial \mathbf{X}$ and μ_2 are

$$V_1=0.058x_1^3-0.077x_1^2x_2+0.002x_1^2x_3+0.045x_1x_2^2-0.015x_2^3-0.003x_1x_2x_3$$

$$\frac{\partial V_1}{\partial \mathbf{X}}=\begin{bmatrix}0.174x_1^2-0.154x_1x_2+0.004x_1x_3+0.045x_2^2-0.003x_2x_3\\-0.077x_1^2+0.09x_1x_2-0.045x_2^2-0.003x_1x_3\\0.002x_1^2-0.003x_1x_2\end{bmatrix}$$

$$\mu_2=0.04x_1^2-0.048x_1x_2+0.0004x_1x_3+0.005x_2^2-0.0003x_2x_3.$$

The only significant terms in μ_2 are the first two, so that

$$\mu_2\simeq 0.04x_1^2-0.048x_1x_2.$$

For the cubic terms, the assumed form of V_2 has 15 unknown coefficients; however, only two are significant.

The cubic control is

$$\mu_3=0.374x_1^3-0.312x_1^2x_2.$$

The nonlinear control including up to third order terms is

$$\mu=-0.053x_1+0.5x_2+0.521x_3+0.04x_1^2-0.048x_1x_2+0.374x_1^3-0.312x_1^2x_2.$$

The increasing complexity of implementing higher order control terms and the increasing effort needed to derive them make the practicality of including higher order terms questionable. Thus we will stop with cubic control terms. The procedure for generating control laws discussed above can be shown to converge under certain circumstances[6, 14]. It can also be shown that if the vector $\phi(\mathbf{X})$ can be written in terms of a Taylor series in a small parameter ε, truncation of the procedure described above with terms of order ε^k provides a $(2k+1)$ order approximation of the optimal control[8, 15, 16].

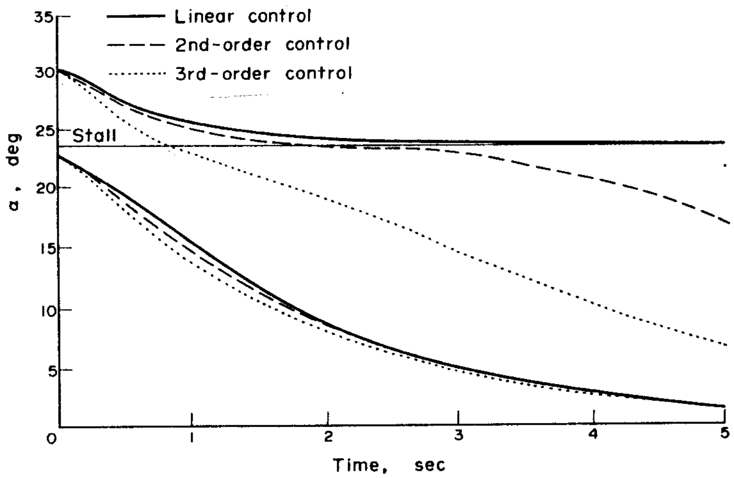

FIG. 3. Angle of attack response for $\alpha(0)=22.9°$ and $\alpha(0)=30.1°$.

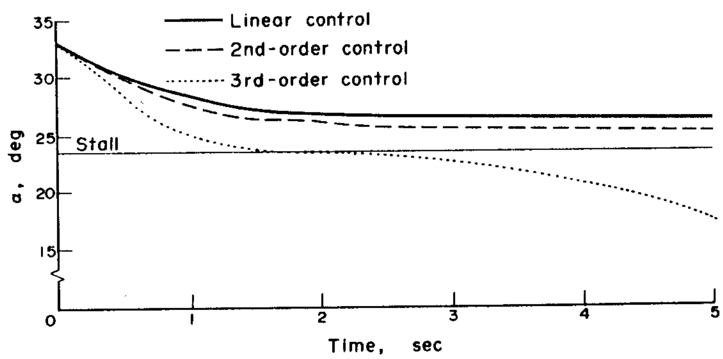

FIG. 4. Angle of attack response for $\alpha(0)=33°$.

4. RESULTS

The response of the aircraft to the three different controllers derived in the previous sections was tested. These controls expressed in terms of the state variables are

1. $\delta_e = -0.053\alpha + 0.5\theta + 0.521\dot{\theta}$,
   linear control

2. $\delta_e = -0.053\alpha + 0.5\theta + 0.521\dot{\theta} + 0.04\alpha^2 - 0.048\alpha\theta$,
   second order control

3. $\delta_e = -0.053\alpha + 0.5\theta + 0.521\dot{\theta} + 0.04\alpha^2 - 0.048\alpha\theta + 0.374\alpha^3 - 0.31\alpha^2\theta$,
   third order control.

At the flight conditions considered, Mach = 0.85 and at 30,000 feet (9000 m), the F-8 stalls when the angle of attack is 23.5°. The time response for several initial values of angle of attack are shown in graphical form in Figs. 3 and 4. In Fig. 3, it can be clearly seen that if the initial angle of attack is below the stall angle, the responses for all three controllers are very similar but as the initial value of angle of attack increases, the beneficial effects of the third order control grow. In all cases the second order controller produced a response closer to that of the linear controller than that of the third order controller. This illustrates the importance of the cubic terms for an effective nonlinear control. Figure 3 shows that when $\alpha(0)=30.1°$ the linear control cannot recover from the stall. The second-order control recovers after 2 sec while the third-order control recovers in less than 1 sec. It was found that as $\alpha(0)$ approaches 34.5°, the third-order control loses effectiveness and for $\alpha(0)>34.5°$, it cannot recover from stall. Close study of the data indicates that the key to the effectiveness of the third order control is its ability to reduce α below 20° more quickly than the linear control. Once α has reached about 20°, its rate of decrease is approximately equal for all three control systems.

Figures 5–7 show the θ, $\dot{\theta}$ and δ_e responses for the case $\alpha(0)=30.1°$. It is evident that the third-order control causes a larger error in θ for the first 3 sec. But the error for all three controls is small; the maximum being about 11°. For $t>3$ sec, the third-order control returns to zero more quickly than the linear control. The $\dot{\theta}$ response shown in Fig. 6 explains the θ response just mentioned. The third-order control produces the largest negative value for $\dot{\theta}$ early in the response, but also switches to a

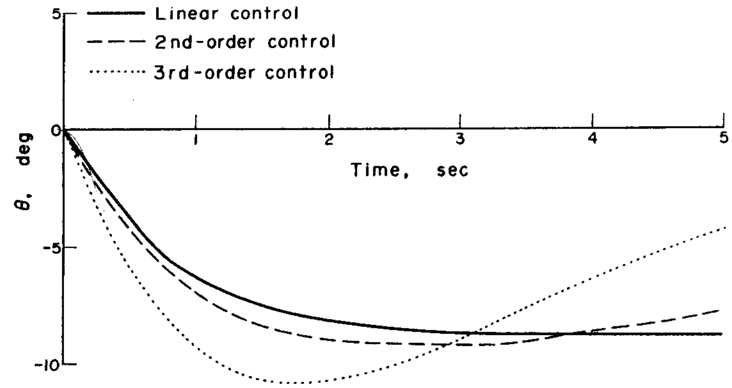

FIG. 5. θ response for $\alpha(0)=30.1°$.

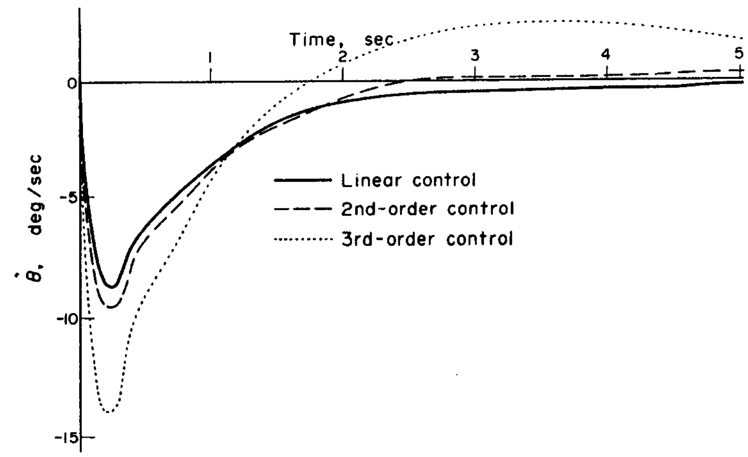

FIG. 6. $\dot{\theta}$ response for $\alpha(0)=30.1°$.

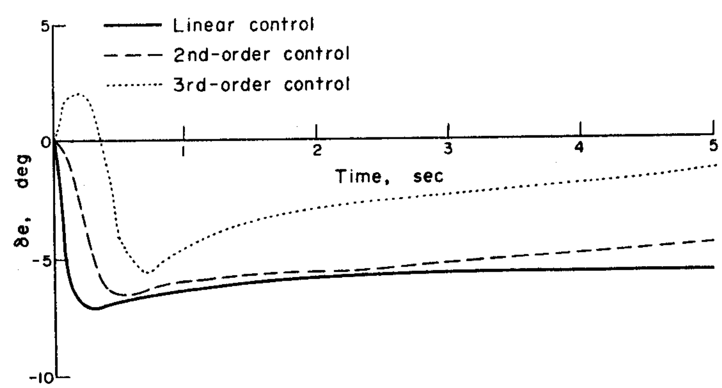

FIG. 7. δ_e vs time for $\alpha(0)=30.1°$.

positive value before the others. Figure 7 shows the difference in control signals produced by the three feedback systems. While the linear controller initially orders a rapid negative tail deflection, the third-order controller initially produces a positive tail deflection. Basically the third-order control is acting to produce a larger pitch error in order to more rapidly reduce the angle of attack. As time passes and α decreases, all of the feedback systems produce similar control signals. The importance of the initial control signal differences is emphasized by the fact that the linear system fails to recover from stall in this case, while the third-order system recovers in less than 1 sec.

A better insight into the initial control signal differences and their effects can be found by examining the control equations and the equations of motion. For all the cases examined, the initial values of $\dot{\theta}$ are negative. Negative values of θ correspond to a nose down effect, resulting in

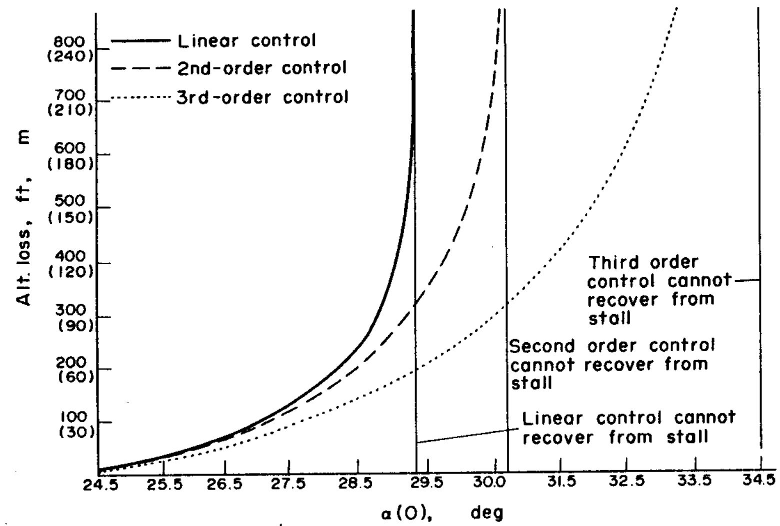

FIG. 8. Altitude loss due to stall vs α(0).

decreasing angle of attack. Since θ and $\dot{\theta}$ are initially zero, the linear control produces a negative value of δ_e. This in turn gives the major control term, $-20.967\delta_e$, a positive value, thereby reducing the nose down effect. For the same initial conditions, the third-order control produces a positive value of δ_e. This is due to the terms $0.04\alpha^2$ and $0.374\alpha^3$. The positive value of δ_e created by the third-order control gives the major control term, $-20.967\delta_e$, a negative value. This increases the nose down effect, resulting in recovery from stall. Once the angle of attack decreases to a non-stall condition, the terms $0.04\alpha^2$ and $0.373\alpha^3$ become less significant and the third-order control behaves very similarly to the linear control.

One example of improved aircraft performance resulting from non-linear controls is in the area of altitude loss due to stall. Figure 8 shows altitude loss as a function of $\alpha(0)$. Figure 9 charts the percent decrease in altitude loss due to nonlinear controls as a function of $\alpha(0)$. The improved performance shown in Fig. 8 and Fig. 9 is of great interest since minimization of height loss due to stall is of major importance.

The recoverable stall range is significantly widened by use of nonlinear controls. The ranges are

1. $23.5° < \alpha < 29.3°$ for the linear-control.
2. $23.5° < \alpha < 30.7°$ for the second-order control.
3. $23.5° < \alpha < 34.5°$ for the third-order control.

The range for the third-order control is almost twice as large as the range for the linear control.

The maximum tail deflection rate required during individual responses averaged about 30% less for the linear control than for the third-order control. These maxima always occurred at the beginning of the response. Within 2 sec the difference in deflection rates was negligible. After two seconds, the linear rate usually became and remained higher.

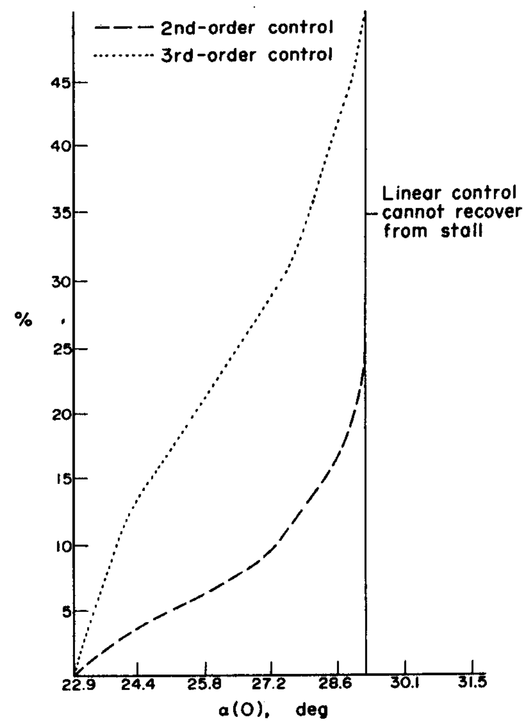

FIG. 9. Percent decrease in altitude loss vs α(0).

A final simulation was run for the simplified third-order control

$$\delta_e = -0.053\alpha + 0.5\theta + 0.521\dot{\theta} + 0.04\alpha^2 + 0.374\alpha^3.$$

The terms $-0.48\alpha\theta$ and $-0.312\alpha^2\theta$ have been eliminated for ease of implementation. The response of this simplified control closely paralleled that of

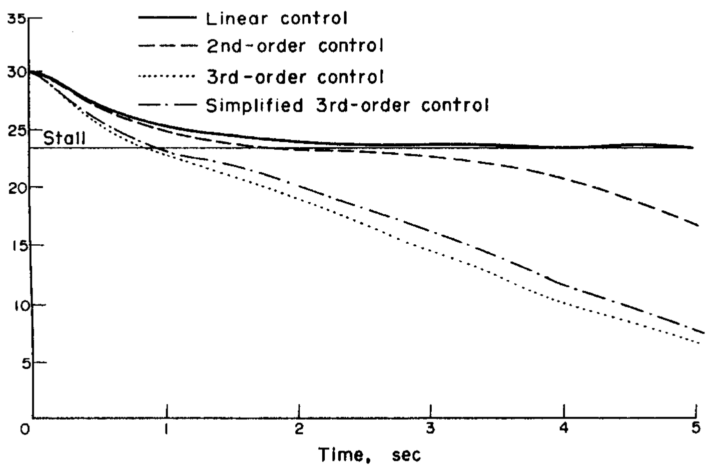

FIG. 10. Angle of attack response for $\alpha(0)=30.1°$, simplified control.

the original third-order control. An example is shown in Fig. 10.

5. CONCLUSIONS

The results presented in this paper indicate that nonlinear controls can lead to significant improvements in aircraft performance. When the aircraft is near the stall condition, the ability of the nonlinear control to recover from stall and reduce the angle of attack below 20° faster than the linear control makes it a superior system. This superiority is evidenced by substantial decreases in altitude loss during stall. Another dividend of nonlinear control is the postponement of non-recoverable stall. For the case presented, the range of recovery from stall was almost doubled through the use of third-order control.

Future research should include development of computer routines capable of eliminating the laborious algebraic calculations used to derive the nonlinear control terms. Currently considerable effort is required to determine even the second order terms and effort required grows quickly as higher order terms are sought. The interaction between the longitudinal and lateral dynamic response of the controlled aircraft should be investigated for high angles of attack. The nonlinear feedback control law would probably be realized using digital methods. Simulations to demonstrate the effects of quantization, computation, and truncation errors on stability and dynamic response should be carried out.

REFERENCES

[1] E. G. AL'BREKHT: On the optimal stabilization of nonlinear systems. *PMM-J, Appl. Math. Mech.* **25**, 1254–1266 (1961).

[2] E. G. AL'BREKHT: The existence of an optimal Lyapunov function and of a continuous optimal controller for one problem in the analytical design of controllers. *Differential'nye Uravneniya*, **1**, (10) 1301–1311 (1965).

[3] Z. V. REKASIUS: Suboptimal design of intentionally nonlinear controllers. *IEEE Trans. Aut. Control* **AC-9**, 380–386 (1964).

[4] Z. V. REKASIUS and R. L. HAUSSLER: Uber die Sub-Optimale Regelung von Nichtlinearen Systemen. *Regelundstechnik* **7**, 290–296 (1964).

[5] W. L. GARRARD, N. H. MCCLAMROCH and L. G. CLARK: An approach to sub-optimal feedback control of nonlinear system. *Int. J. Control* **5**, 425–435 (1967).

[6] D. L. LUKES: Optimal regulation of nonlinear dynamical systems. *SIAM J. Control* **7**, 75–100 (1969).

[7] P. KOKOTOVIC and G. SINGH: Optimization of coupled systems. *Int. J. Control* **XIV**, (1) 51–64, July (1971).

[8] W. L. GARRARD: Suboptimal feedback control for nonlinear systems. *Automatica* **8**, 219–221 (1972).

[9] Y. NISHIKAWA, N. SANNOMIYA and H. ITAKVRA: A method for suboptimal design of nonlinear feedback systems. *Automatica* **7**, 703–713 (1971).

[10] W. L. GARRARD and L. G. CLARK: On the synthesis of suboptimal, inertia-wheel altitude control systems. *Automatica* **5**, 781–789 (1969).

[11] C. K. LING: Quasi-optimum design of an aircraft landing control system. *J. Aircraft* 38–43, Jan.–Feb. (1970).

[12] B. ETKIN: *Dynamics of Atmospheric Flight.* John Wiley, New York (1972).

[13] E. B. LEE and L. MARKUS: *Foundations of Optimal Control Theory.* John Wiley, New York (1967).

[14] W. L. GARRARD: Additional results on sub-optimal feedback control of nonlinear systems. *Int. J. Control* **10**, 657–663 (1969).

[15] R. A. WERNER and J. B. CRUZ: Feedback control which preserves optimality for systems with unknown parameters. *IEEE Trans Aut. Control* **AC-12**, 59–66 (1967).

[16] M. JAMSHIDI: A feedback near optimum control for nonlinear systems. *Inform. Control* **32**, 75–84 (1976).