4

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Some Extensions of Liapunov's Second Method*

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Introduction

Liapunov's second method has long been recognized in the Soviet Union as the most general method for the study of the stability of equilibirum positions of systems described by differential or difference equations. The method was first presented by Liapunov in his now classical memoir,[1] which appeared in Russian in 1892 and was translated into French in 1907. Good sources for the statements and proofs of the mathematical theorems underlying the method can be found in works by Hahn,[2] Antosiewicz,[3] and Cesari.[4] These references also contain extensive bibliographies.

By way of introduction, let us consider first Liapunov's asymptotic stability theorem for autonomous systems. Let the systems of differential equations be ($\dot{x} = dx/dt$)

$$\dot{x}_i = X_i(x_1, \cdots, x_n), \qquad i = 1, \cdots, n. \tag{1}$$

The state of the system at time t is given by n real numbers $x_1(t), x_2(t), \cdots, x_n(t)$. Thus, the state of the system at time t can be represented simply by the n vector $x(t) = (x_1(t), \cdots, x_n(t))$. The phase velocity of the system at the point $x = (x_1, \cdots, x_n)$ is defined by the vector field $X(x) = (X_1(x), \cdots, X_n(x))$. In vector notation, the system of differential equations is simply the vector differential equation

$$\dot{x} = X(x). \tag{2}$$

The objective of this paper is to present methods rather than to obtain general results, and we shall confine our illustrations to simple equations such as Liénard's:

$$\ddot{x} + f(x)\dot{x} + g(x) = 0. \tag{3}$$

Letting $F(x) = \int_0^x f(u)du$ and $y = \dot{x} + F(x)$, we obtain, as a convenient system of first-order equations equivalent to (3), the system

$$\begin{aligned} \dot{x} &= y - F(x), \\ \dot{y} &= -g(x). \end{aligned} \tag{4}$$

For van der Pol's equation $f(x) = \epsilon(x^2 - 1)$, $F(x) = \epsilon(\frac{1}{3}x^3 - x)$, and $g(x) = x$.

Returning to the general system (2), we shall assume that $X(x)$ has continuous first partials for all x. Thus, for any x^0 there is a unique solution $x(t)$ of (2) satisfying $x(0) = x^0$. This assumption on $X(x)$ is much stronger than is required, but this is of little concern to us here, and our attitude throughout is to present the principal features of methods. The *equilibrium states*, sometimes called critical points, are those states where $X(x) = 0$. Thus, if $X(x^0) = 0$, then $x = x^0$ is a solution of (2). Started at x^0, the system remains in this state for all t. This is, of course, a mathematical statement, and the actual behavior of physical systems raises the problem of stability.

It is never possible to start the system exactly in its equilibrium state, and the system is always subject to outside forces not taken into account by the differential equations. The system is disturbed and is displaced slightly from its equilibrium state. What happens? Does it remain near the equilibrium state? This is stability. Does it remain near the equilibrium state and in addition tend to return to the equilibrium? This is asymptotic stability.

Let us make these notions more precise. Assume, as one always can, that the equilibrium state being investigated is located at the origin: $X(0) = 0$. A translation of coordinates accomplishes this. Let $\|x\|$ be the Euclidean length of the vector x: $\|x\|^2 = x_1^2 + \cdots + x_n^2$. Let $S(R)$ be the spherical region of radius $R > 0$ about the origin: $S(R)$ consists of the points x satisfying $\|x\| < R$. The origin is said to be *stable* if corresponding to each $S(R)$ there is an $S(r)$, such that a solution starting in $S(r)$ does not leave $S(R)$: $x(0)$ in $S(r)$ implies $x(t)$ is in $S(R)$ for all $t \geq 0$ (Fig 1). If, in addition, there is a neighborhood $S(R_0)$ such that every solution starting in $S(R_0)$ approaches the origin as $t \to \infty$, the system is said to be *asymptotically stable* (Fig. 2).

For instance, if within a neighborhood of the equilibrium state the total energy of the system is always decreasing, we expect the equilibrium state to be asymptotically stable. Liapunov's second method generalizes this idea. Suppose that within some neighborhood $S(R)$ of the origin one can construct a scalar function $V(x)$ having continuous first partials and such that $V(0) = 0$ and $V(x) > 0$ for $x \neq 0$. Define, and this is with reference to the system (2) being investigated,

$$\dot{V}(x) = \frac{\partial V}{\partial x_1} X_1 + \cdots + \frac{\partial V}{\partial x_n} X_n = (\text{grad } V) \cdot X. \tag{5}$$

Then, if $x(t)$ is a solution of (2), the rate of change of the

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[1] A. Liapounov, "Problème général de la stabilité du mouvement," in "Annals of Mathematical Studies No. 17," Princeton University Press, Princeton, N. J.; 1949.

[2] W. Hahn, "Theorie und Anwendung der Direckten Methode von Ljapunov," Springer-Verlag, Berlin, Germany; 1959.

[3] H. A. Antosiewicz, "A survey of Lyapunov's second method contributions to the theory of nonlinear oscillations IV," in "Annals of Mathematical Studies No. 41," Princeton University Press, Princeton, N. J.; 1958.

[4] L. Cesari, "Asymptotic Behavior and Stability Problems in Ordinary Differential Equations," Springer-Verlag, Berlin, Ger.; 1959.

the system is stable, whereas stability with a voltage variation of 1/4 volt may be quite acceptable. Ideally, we might like to have the system return to equilibrium regardless of the size of the perturbation. This is "asymptotic stability in the large."

In this section, we develop and illustrate a number of theorems that relate to the question of determining the extent to which a system is asymptotically stable. Nothing can be said about the extent of asymptotic stability by examining only the linear approximation. The effect of the nonlinearities must be taken into account, and the Liapunov method gives us a means of doing this.

Before we begin the development of these theorems, there are a few more preliminaries. We consider again the autonomous system

$$\dot{x} = X(x), \tag{7}$$

assuming that the function $X(x)$ has continuous first partials, or any other conditions that guarantee the existence and uniqueness of solutions and the continuity of the solutions relative to the initial conditions. We assume also that $X(0) = 0$—there is an equilibrium at the origin.

We need first to familiarize ourselves with Birkhoff's concept of a limiting set. Let $x(t)$ be a solution of (7). A point p is said to be in the *positive limiting* set Γ^+ of $x(t)$, if corresponding to each $\epsilon > 0$ and each $T > 0$ there is a $t > T$ with the property that $\|x(t) - p\| < \epsilon$. This is equivalent to saying that there is a sequence t_n approaching infinity with n and such that $x(t_n) \to p$ as $n \to \infty$. One of the fundamental properties of limiting sets is the following:

If $x(t)$ is bounded for $t \geq 0$, then its positive limiting set Γ^+ is a nonempty, compact, invariant set.

A set M is said to be invariant if each solution starting in M remains in M for all t. The proof of this property of limiting sets is given in Lefschetz's book.[4]

We say also that $x(t)$ *approaches a set M* as t approaches infinity, if each $\epsilon > 0$ there is a $T > 0$ with the property that for each $t > T$ there is a p in M with $\|x(t) - p\| < \epsilon$; that is, for all $t > T$ the points $x(t)$ are within a distance ϵ of M. For instance, if $x(t)$ is bounded for $t \geq 0$, then $x(t)$ approaches its positive limiting set Γ^+ as $t \to \infty$. If this were not so, there would be an $\epsilon > 0$ with the property that for each $T > 0$ there is a $t > T$, such that $\|x(t) - p\| \geq \epsilon$ for all p in Γ^+. Hence, there would be a sequence t_n tending to infinity with n and such that $\|x(t_n) - p\| \geq \epsilon$ for all p in Γ^+. But since $x(t)$ is bounded for $t \geq 0$, the sequence $x(t_n)$ has a limit point which is in Γ^+, which is a contradiction. This proves the proposition. The way in which we shall use this proposition is as follows:

If $x(t)$ is bounded for $t \geq 0$ and if M contains the positive limiting set Γ^+ of $x(t)$, then $x(t) \to M$ as $t \to \infty$.

With these preliminaries behind us, we can now establish a theorem which leads to a number of possible methods for determining the extent of asymptotic stability.

[4] S. Lefschetz, "Differential Equations: Geometric Theory," Interscience Publishers, New York, N. Y.; 1957.

Theorem 1

Let Ω be a bounded closed (compact) set with the property that every solution of (7) which begins in Ω remains for all future time in Ω. Suppose there is a scalar function $V(x)$ which has continuous first partials in Ω and is such that $\dot{V}(x) \leq 0$ in Ω. Let E be the set of all points in Ω where $\dot{V}(x) = 0$. Let M be the largest invariant set in E. Then every solution starting in Ω approaches M as $t \to \infty$.

Proof. Let $x(t)$ be a solution initially in Ω. Since $\dot{V}(x) \leq 0$ in Ω, $V(x(t))$ is a nonincreasing function of t. $V(x)$, being continuous on the compact set Ω, is bounded from below on Ω. Therefore, $V(x(t))$ has a limit c as $t \to \infty$. Note also that the positive limiting set Γ^+ is in Ω (because Ω is a closed set), and since V is continuous on Ω, $V(x) = c$ on Γ^+. Γ^+ is an invariant set, and hence $\dot{V}(x) = 0$ on Γ^+. Thus, Γ^+ is in M. This implies, as was pointed out above, that $x(t) \to M$ as $t \to \infty$. All solutions starting in Ω approach M as t approaches infinity.

In some applications the construction of the Liapunov function $V(x)$ in Theorem I will itself guarantee the existence of a set Ω. It may be that the set Ω defined by $V(x) \leq l$ is a bounded set. If $\dot{V}(x) \leq 0$ in Ω, then $V(x(t))$ is nonincreasing for any solution $x(t)$ which starts in Ω. Therefore, $x(t)$ must remain in Ω for all future time. We then have as a direct consequence of Theorem I:

Theorem 2

Let Ω denote the closed region defined by $V(x) \leq l$, and suppose that $V(x)$ has continuous first partials in Ω. If, in addition, Ω is bounded and $\dot{V}(x) \leq 0$ in Ω, then every solution starting in Ω approaches M as $t \to \infty$. (The set M is as defined in Theorem I.)

Note with regard to Theorem 2 that if $V(x) \to \infty$ as $\|x\| \to \infty$, then the set Ω defined by $V(x) \leq l$ is bounded for all values of l. If $\lim_{\|x\| \to \infty} \inf V(x) = l_0$, then Ω is bounded for all $l < l_0$.

Liénard's equation,

$$\ddot{x} + f(x)\dot{x} + g(x) = 0, \tag{8}$$

provides a good example of the application of Theorem 2. An equivalent system is

$$\dot{x} = y - F(x), \qquad \dot{y} = -g(x), \tag{9}$$

where $F(x) = \int_0^x f(u)du$. We assume that f and g are continuous and that $g(x)$ is locally Lipschitzian. Define $G(x) = \int_0^x g(u)du$. Here a suitable Liapunov function is

$$V = \tfrac{1}{2}y^2 + G(x).$$

V is the total energy $\frac{1}{2}\dot{x}^2 + G(x)$ plus the term $\dot{x}F(x) + \frac{1}{2}F^2(x)$. It is a modified energy function. Now

$$\dot{V} = -g(x)F(x).$$

Suppose that

1) $\quad g(x)F(x) > 0 \quad \text{for} \quad 0 < |x| < a,$

A solution leaving Ω must cross either $V = \frac{1}{2}a^2\beta^2$, $W_1 = -\beta$, or $W_2 = -a\beta$. We already know that $\dot{V} \leq 0$, and hence no solution can cross $V = \frac{1}{2}a^2\beta^2$. Now

$$\dot{W}_1 = y \quad \text{and} \quad \dot{W}_2 = -x(b + x).$$

It is easily seen that along that part of W_1 which is the boundary of Ω, $y \geq 0$, and therefore $\dot{W}_1 \geq 0$. Similarly, along the part of W_2 that is part of the boundary, $x \leq 0$. Hence, for $0 < \beta < b$, $\dot{W}_2 \geq 0$ and the solutions cannot cross W_2. Solutions starting inside Ω remain inside for all future time. It is also evident that the set M is the origin. Hence, by Theorem 1, every solution inside Ω, $0 < \beta < b$, approaches the origin as $t \to \infty$. The origin is asymptotically stable, and the region defined by

$$\frac{1}{2}y^2 + \frac{1}{2}bx^2 + \frac{x^3}{3} < \frac{1}{2}a^2b^2$$

$$x > -b$$

$$y + ax > -ab$$

gives us an estimate of the size of the region of asymptotic stability.

The curve

$$y^2 = H(x) = a^2b^2 - bx^2 - \tfrac{2}{3}x^3$$

is part of the boundary of the region Ω defined above. Now $H'(x) = -2x(b + x)$, and we see that $H(x)$ has a minimum at $x = -b$. At $x = -b$ the value of $H(x)$ is

$$H(-b) = b^2(a^2 - \tfrac{1}{3}b).$$

Hence, if $a^2 \leq 1/3\ b$, a better estimate of the region of asymptotic stability is obtained by taking Ω to be the region defined by

$$\tfrac{1}{2}y^2 + \tfrac{1}{2}bx^2 + \tfrac{1}{3}x^3 < \tfrac{1}{6}b^3, \quad \text{and} \quad x < -b.$$

This estimate of the region of asymptotic stability is illustrated in Fig. 5 when $a = 1/4$ and $b = 2$.

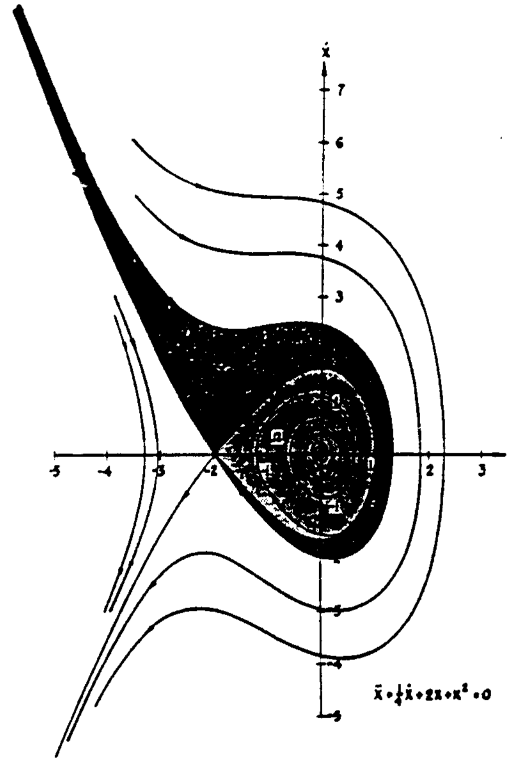

Fig. 5.

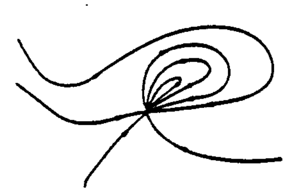

Fig. 6.

Complete Stability (Asymptotic Stability in the Large)

For many systems it may be important to assure that no matter how large the perturbation, or in a feedback control system, regardless of the size of the error, the system tends to return to its equilibrium state. This is asymptotic stability in the large. In place of this awkward expression we shall say completely stable. The system (2) will be said to be *completely stable* if the origin is stable and if every solution tends to the origin as t tends to infinity.

Fig. 6 illustrates the reason for assuming both that the origin is stable and that all solutions tend to the origin. We see that all solutions can tend to the origin and yet the origin need not be stable.

The basic theorem is:

Theorem 3

Let $V(x)$ be a scalar function with continuous first partials for all x. Assume that

1) $V(x) > 0$ for all $x \neq 0$,

2) $\dot{V}(x) \leq 0$ for all x.

Let E be the set of all points where $\dot{V}(x) = 0$, and let M be the largest invariant set contained in E. Then every solution of (2) bounded for $t \geq 0$ approaches M as $t \to \infty$.

The proof is so close to that of Theorem 1 that we omit it.

contains no solution other than the origin, the set M is the origin, and the system is completely stable.

It does occur in some applications that one can construct a Liapunov function V satisfying Theorem 5 and, in addition, $\dot{V}$ may be negative definite; that is, $\dot{V}(x) < 0$ for all $x \neq 0$. Then, of course, M is the origin and the system is completely stable. However, it is often easier to find a Liapunov function whose time derivative is only non-negative, and then to conclude from the differential equations that M is the origin. The next example is a simple instance of this.

For Liénard's equation

$$\ddot{x} + f(x)\dot{x} + g(x) = 0,$$

we assume this time that

1) $G(x) = \int_0^x g(\xi)\, d\xi > 0$ for all $x \neq 0$.

2) $G(x) \to \infty$ as $|x| \to \infty$

3) $f(x) > 0$ for all $x \neq 0$.

Thus we assume that: 1) the potential energy $G(x)$ is positive definite ($x = 0$ is its minimum), 2) the potential energy approaches infinity with $|x|$, and 3) the damping is always positive.

An equivalent system is

$$\dot{x} = y,$$

$$\dot{y} = -g(x) - f(x)y.$$

We take the Liapunov function to be the total energy:

$$V(x, y) = \tfrac{1}{2}y^2 + G(x).$$

Then,

$$\dot{V}(x, y) = -f(x)y^2 \leq 0.$$

Now $\dot{V}$ vanishes only on the axes $x = 0$ and $y = 0$, and it is clear that excluding the origin, no solution remains on these axes. Hence, all the conditions of Theorem 5 are satisfied, and the system is completely stable.

A difficulty in using Theorem 5 often is that one can construct a Liapunov function satisfying conditions 1) and 2), but not 3). We illustrate in the next example that it may then be easier to establish the boundedness of the solutions as a separate problem.

We continue our investigation of the complete stability of Liénard's equation with a weaker assumption on $g(x)$ and a stronger assumption on the damping. Assume that

1) $xg(x) > 0$ for $x \neq 0$.

2) $f(x) > 0$ for all $x \neq 0$.

3) $|F(x)| = \left| \int_0^x f(\xi)\, d\xi \right| \to \infty$ as $|x| \to \infty$.

We use the same Liapunov function $V(x, y) = \tfrac{1}{2}y^2 + G(x)$ as before. However, since it may not be true that $G(x) \to \infty$ as $|x| \to \infty$, we can conclude only that every solution bounded for $t \geq 0$ approaches the origin as $t \to \infty$ (Theorem 3). Thus, to establish complete stability, we need to show that all solutions are bounded for $t \geq 0$. To do this, consider the region Ω (Fig. 7) defined by

$$V(x, y) = \tfrac{1}{2}y^2 + G(x) < l,$$

and

$$(y + F(x))^2 < a^2.$$

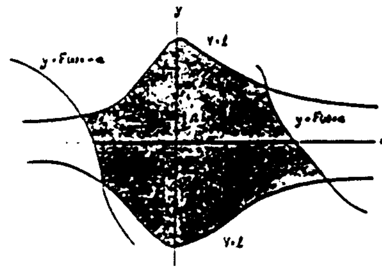

Fig. 7.

For any l and a, this is a bounded region. Let $(x(t), y(t))$ be any solution, and select l and a so large that the solution starts in Ω. Then the solution cannot leave without crossing the boundary of Ω. It must cross either $V = l$ or $y + F(x) = -a$ or $y + F(x) = a$. We can select a sufficiently large that the part of $y + F(x) = a$ which is the boundary of Ω corresponds to $x > 0$ and the part of $y + F(x) = -a$ corresponds to $x < 0$. Since $\dot{V} \leq 0$, a solution starting inside cannot cross $V = l$.

Now,

$$\frac{d}{dt}(y + F(x))^2 = -2(y + F(x))g(x).$$

Along that part of $y + F(x) = -a$ or $y + F(x) = a$ which makes up the boundary of Ω, we have

$$\frac{d}{dt}(y + F(x))^2 = -2a\,|g(x)| < 0.$$

Hence, $(x(t), y(t))$ cannot leave Ω, and every solution is bounded for $t \geq 0$. Thus, under somewhat different conditions we have again shown that Liénard's equation is completely stable.

Usually it is not difficult to study the stability of second-order systems. There are two reasons for this. The phase space is a plane, and we have no trouble visualizing the qualitative behavior of the system. We are also accustomed to identifying the damping in systems with one degree of freedom. It is worthwhile then to illustrate the method for a third-order system. Equations of the