# Energy Concepts in the State-Space Theory of Nonlinear $n$-Ports: Part I—Passivity

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*Abstract*—This paper is the first in a two-part sequence which aims to state rigorously the energy-based concepts which are fundamental to nonlinear network theory, passivity and losslessness, and to clarify the way they enter the input–output and the state-space versions of the subject. In this part we examine the conflicting definitions of passivity which exist in the literature and demonstrate the contradictions between them with several examples. We propose a particular definition of passivity which avoids these contradictions by eliminating the dependency on a state of "zero stored energy," and we show that it has the appropriate properties of representation independence and closure. We apply it to several specific classes of $n$-ports and derive equivalent passivity criteria. The exact conditions are given under which this definition is equivalent to one based on an internal energy function, and we use the concept of an internal energy function to provide a canonical network realization for a class of passive systems.

## I. Introduction

The purpose of this paper is to clarify the meaning of passivity and some of its consequences in nonlinear circuit theory. A sequel [1] will deal with losslessness. Due to space limitations we have omitted all proofs. They are given in complete detail in [2], where many topics in this article are discussed more fully than is possible here.

This work deals with *finite dimensional time-invariant systems described by state equations.* Our viewpoint throughout this paper will be that the state equations for an $n$-port are available at the outset. We will not be concerned with the problem of constructing a state representation or describing an $n$-port as active or passive on the basis of input–output measurements alone, although many of our results will have a bearing on the latter problem. We require that inputs be applied and outputs observed over the time interval $\mathbf{R}^+=[0,\infty)$, and "initial state" means the state at $t=0$. Voltages and currents will always have associated reference directions so that the instantaneous power into an $n$-port is

$$\sum_{j=1}^{n} v_j(t)i_j(t)=\langle v(t),i(t)\rangle.$$

Although a large number of slightly different definitions of passivity appear in the literature, three distinct ideas can be isolated if we overlook minor differences. Examples below will show that two of them lead to odd or even nonphysical consequences if examined critically. The first concept considered below, Passivity 1, is taken from [3], [4].

*Passivity 1.* An $n$-port is passive if, whenever the state $x$ at time zero is $\mathbf{0}$,

$$\int_0^T \langle v(t),i(t)\rangle\,dt\geq 0 \tag{1-1}$$

for all admissible pairs $\{v(\cdot),i(\cdot)\}$ and all $T>0$.

According to this concept of passivity, we only need to know the zero-state response of an $n$-port in order to determine if it is active or passive. The following two examples point out some disturbing consequences of this concept of passivity.

*Example 1.* Consider a capacitor characterized by $v(q)=q-1$, as in Fig. 1(a). Its terminal behavior, $i=dv/dt$ or

$$v(t)=v(0)+\int_0^t i(\tau)\,d\tau \tag{1-2}$$

cannot be distinguished from that of a 1-F capacitor by any possible voltage and current measurements.

In terms of Passivity 1, this element must be classified as active, since (1-1) need not be satisfied when the initial state is $q(0)=0$, as we can see by connecting it to a 1-Ω load, for example. But a 1-F capacitor, which has identical input–output behavior, is a passive element according to this (or any other reasonable) concept of passivity. Thus Passivity 1 makes a distinction where none can be observed experimentally.

*Example 2.* Consider the linear 2-port in Fig. 2. Its state and output equations are

$$\dot v_c=-\frac{v_c}{C}\triangleq Av_c+B\begin{bmatrix} i_1\\ i_2\end{bmatrix}$$

$$\begin{bmatrix} v_1\\ v_2\end{bmatrix}=\begin{bmatrix} 0\\ v_c\end{bmatrix}+\begin{bmatrix} i_1\\ 0\end{bmatrix}\triangleq Cv_c+D\begin{bmatrix} i_1\\ i_2\end{bmatrix}.$$

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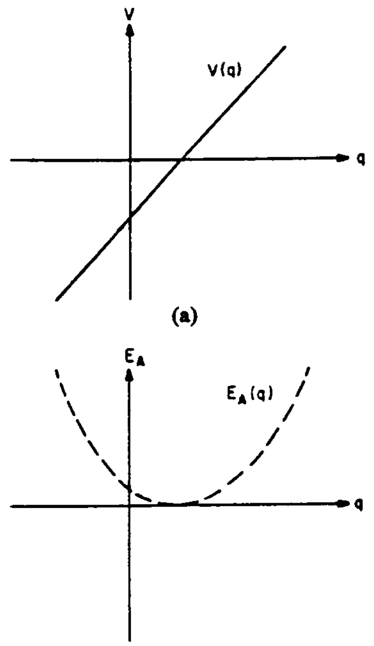


Fig. 1. (a) The capacitive constitutive relation $v(q)=q-1$. (b) The available energy for this system is given by $E_A(q)=(q-1)^2/2$.

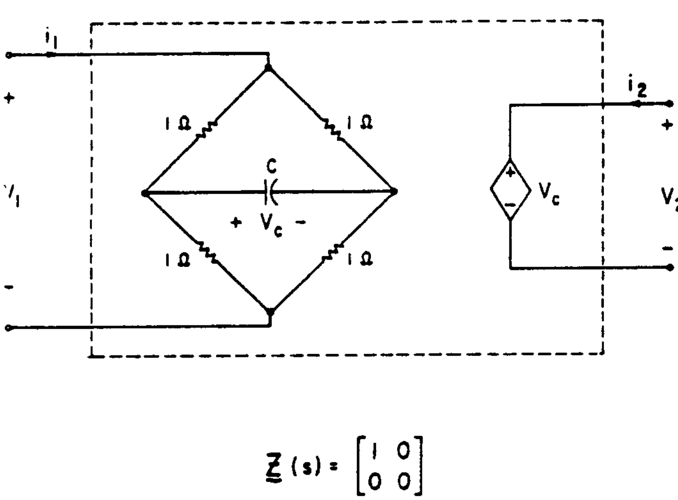


Fig. 2. Judging from its impedance matrix alone, this 2-port would appear to be passive. But in any nonzero initial state it can furnish unlimited energy to the outside world. Furthermore, it is violently unstable if $C<0$.

The impedance matrix is calculated by the usual formula to be

$$Z(s)=C\{sI-A\}^{-1}B+D=\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}.$$

This completely expresses its zero-state response. If $v_c(0)=0$, then port 1 looks like a 1-Ω resistor and port 2 looks like a short circuit. But for any nonzero initial condition $v_{co}\neq 0$, the voltage at port 2 is $v_{co}e^{-t/C}$ regardless of the input at either port. If $v_{co}\neq 0$, then there is no limit to the amount of energy we can extract from this system simply by connecting an arbitrarily small resistor across port 2. (We will show later that the peculiarities of this example arise because it is not completely controllable.)

In terms of Passivity 1, Example 2 must be classified as passive since (1-1) will be satisfied so long as $v_{co}=0$. But this seems a strange classification for a system which can supply unbounded energy if started in any nonzero initial condition. In fact it is passive according to Passivity 1 even if $C<0$, although it is unstable in that case. Thus Passivity 1 is further called into question. (A related but simpler example appears in [5].)

A second concept of passivity is taken from [6]–[8].[1]

*Passivity 2.* An *n*-port, storing no energy at $t=0$, is passive if (1-1) holds for all $T>0$ and all admissible pairs $\{v(\cdot), i(\cdot)\}$.

One difficulty with this definition is that it offers no operational rule for determining the "stored energy" at $t=0$. If we mean by the term "*n*-port" a black box which we are not allowed to open, this is not a trivial objection. Considering Example 2 again, it is unclear whether $v_{co}=0$ should be called a state of zero stored energy. If so, then Example 2 is passive in the sense of Passivity 2, a classification which throws considerable doubt on the appropriateness of Passivity 2.

Even if we had such an operational rule, the demand that we begin with a state of zero stored energy is itself unclear. If for a given *n*-port we cannot find such a state, is that *n*-port active or does it fall outside the scope of the definition altogether? The following example illustrates the problem.

*Example 3.* The nonlinear capacitor characterized by $v(q)=e^q$ has no state of zero voltage, as shown in Fig. 3(a). For each initial state $q$ it is possible to extract some energy from it, so no state of zero stored energy exists.

Example 3 cannot be clearly classified in terms of Passivity 2 and thus emphasizes the inadequacy of Passivity 2 for a general theory of nonlinear networks. Example 3 is evidently active according to Passivity 1, another questionable classification since Example 3 is not capable of supplying unlimited energy like the classical unambiguously active elements: ideal voltage and current sources and negative linear resistors, capacitors, and inductors.

The final definition considered below resolves these anomalies. It was given a detailed analysis in [9]. And it is essentially the concept given earlier in [10]–[13], although it has not been widely recognized that this definition does not require the existence of a state of "zero stored energy." A more complete and rigorous statement of it will be given in Section 3.1.

*Passivity 3.* Given an *n*-port $\mathfrak{N}$, let the *available energy* $E_A(x)$ be the maximum energy that can be extracted from $\mathfrak{N}$ when its initial state is $x$, with the convention that $E_A(x)=+\infty$ if the available energy is unbounded. Then $\mathfrak{N}$ is *passive* if $E_A(x)$ is finite for each initial state $x$.

[1] The passivity definition from [6] was given in the context of an arbitrary black box which need *not* have a state representation. It was intended to apply to a larger class of systems than is addressed in this paper. A satisfactory general definition of passivity for an arbitrary black box, which reduces to the definition proposed in this paper for *n*-ports described by state equations, remains an unsolved problem to the best of our knowledge.

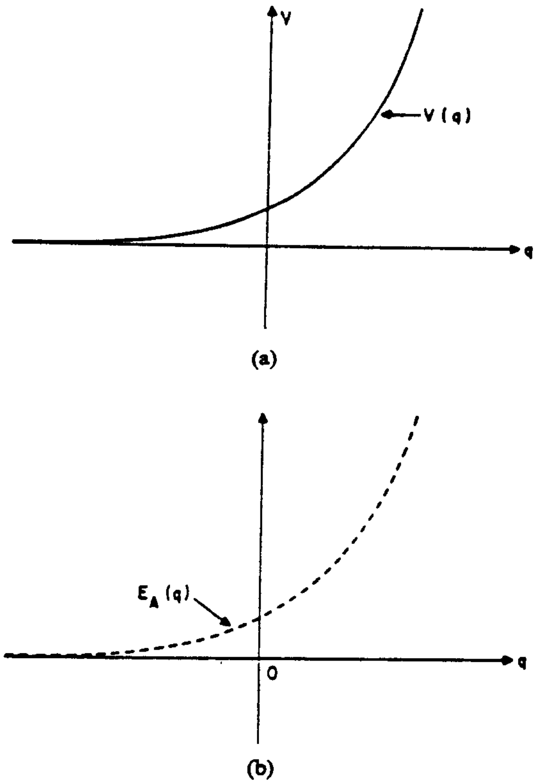


Fig. 3. (a) The constitutive relation $v(q)=e^q$ for a nonlinear capacitor. (b) The available energy for this element is given by $E_A(q)=e^q$.

In other words, an $n$-port is active if for some initial state it can furnish unlimited energy, and otherwise it is passive. It is straightforward to calculate $E_A$ for Examples 1 and 3 (a formal derivation is given in Section 3.1), and the result appears in Figs. 1(b) and 3(b). Since $E_A$ is finite for each state $q$, Examples 1 and 3 are passive according to Passivity 3. For Example 2 we showed previously that $E_A(v_{co})=+\infty$ for all $v_{co}\neq 0$, and so Example 2 is active.

With all the examples and objections that we are presently aware of, Passivity 3 appears to be the most reasonable concept of passivity for a general theory of lumped nonlinear $n$-ports because it does not single out any particular state in $\Sigma$ for special attention, and it does not require that a state of zero stored energy be found.

## II. Definitions and Assumptions

It is probably best to skim this section quickly the first time through and then refer back to it as needed.

**The $n$-ports dealt with in this paper are assumed to possess a state representation**; this is our fundamental assumption. Roughly speaking, a state representation of an $n$-port is a state equation and two readout maps which give the port voltages and port currents as functions of the input and state, together with a set of rules defining the class of inputs which can be applied.

*Definition 1.* A *state representation* $S$ for an $n$-port is a quintuplet $\{U, \mathcal{U}, \Sigma, E, R\}$, where

(1) $U\subset \mathbf{R}^n$ is a nonempty set called the *set of admissible input values*.

(2) $\mathcal{U}$ is a nonempty set of functions mapping $\mathbf{R}^+$ to $U$ called the *set of admissible input waveforms*.

(3) $\Sigma\subset\mathbf{R}^m$ is a nonempty set called the *state space*.

(4) $E$ is a pair of equation

$$\dot{x}=f(x,u) \tag{2-1}$$

$$y=g(x,u) \tag{2-2}$$

where $f(\cdot,\cdot)$ maps $\Sigma\times U\to\mathbf{R}^m$ and $g(\cdot,\cdot)$ maps $\Sigma\times U\to\mathbf{R}^n$. Equation (2-1) is called the *state equation* and (2-2) is called the *output equation*.

(5) $R$ is a pair of readout maps: $V$: $\Sigma\times U\to\mathbf{R}^n$ is called the *port voltage readout map* and $I$: $\Sigma\times U\to\mathbf{R}^n$ is called the *port current readout map*.

*Definition 2.* The *power input function* $p$: $\Sigma\times U\to\mathbf{R}$ is defined by

$$p(x,u)\triangleq\langle V(x,u), I(x,u)\rangle.$$

*Definition 3.* A choice of input and output variables $u$ and $y$ for an $n$-port is called a *hybrid pair* if $u$ and $y$ are $n$-dimensional and for each $k\in\{1,\cdots,n\}$, either $u_k=v_k$ and $y_k=i_k$ or else $u_k=i_k$ and $y_k=v_k$, where $u_k$ and $y_k$ denote the $k$th components of $u$ and $y$, respectively, and $v_k$ and $i_k$ denote the $k$th-port voltage and current, respectively.

If $u$ and $y$ are a hybrid pair, then $p(x,u)\triangleq\langle V(x,u), I(x,u)\rangle=\langle u, g(x,u)\rangle=\langle u,y\rangle$.

*Definition 4.* Let $D\subset\mathbf{R}^p$ be an open set. A function $h$: $D\to\mathbf{R}^q$ is $C^0$ if it is continuous, and it is $C^k$ for some positive integer $k$ if each of its component functions possesses continuous partial derivatives of all orders up to and including $k$.

*Definition 5.* A function $u(\cdot)$: $\mathbf{R}^+\to\mathbf{R}^n$ is said to be *locally* $L^p$, $1<p<+\infty$, if $u(\cdot)$ is measurable and for every choice of $a,b\in\mathbf{R}^+$,

$$\int_a^b(\|u(t)\|)^p\,dt<+\infty$$

where $\|\cdot\|$ is the Euclidean norm on $\mathbf{R}^n$. We will let $L^p_{loc}(\mathbf{R}^+\to\mathbf{R}^n)$ denote the class of all such functions.[2] A function $u(\cdot)$: $\mathbf{R}^+\to\mathbf{R}^n$ is said to be *locally* $L^\infty$ if $u(\cdot)$ is measurable and for every finite $T>0$ there exists a finite $M_T>0$ such that $\|u(t)\|<M_T$ for almost all $t\in[0,T]$. We will let $L^\infty_{loc}(\mathbf{R}^+\to\mathbf{R}^n)$ denote the class of all such functions.

*Definition 6.* Given a function $u(\cdot)$: $\mathbf{R}^+\to\mathbf{R}^n$ and a real number $\tau>0$, let $u_\tau(\cdot)$: $\mathbf{R}^+\to\mathbf{R}^n$ be obtained from $u(\cdot)$ by translating $u(\cdot)$ $\tau$ units to the left, i.e., $u_\tau(t)=u(t+\tau)$, $\forall t\in\mathbf{R}^+$. We say that $\mathcal{U}$ is *translation invariant* if $u(\cdot)\in\mathcal{U}\Rightarrow u_\tau(\cdot)\in\mathcal{U}$, $\forall\tau>0$.

*Definition 7.* Given two functions $u_1(\cdot), u_2(\cdot)$: $\mathbf{R}^+\to\mathbf{R}^n$ and given a real number $\tau>0$, we define $u_{12\tau}$ and $\hat{u}_{12\tau}$: $\mathbf{R}^+\to\mathbf{R}^n$ by

$$u_{12\tau}(t)=\begin{cases}u_1(t), & 0<t<\tau\\ u_2(t-\tau), & t>\tau\end{cases}$$

$$\hat{u}_{12\tau}(t)=\begin{cases}u_1(t), & 0<t<\tau\\ u_2(t-\tau), & t>\tau.\end{cases}$$

[2]These classes of functions are the same as the *extended* $L^p_n$ *spaces* defined by Desoer and Vidyasagar [14] and denoted by $L^p_{ne}$.

We say that $\mathcal{U}$ is *closed under concatenation* if for every $u_1(\cdot)$, $u_2(\cdot) \in \mathcal{U}$, $u_{12\tau}(\cdot)$ is an element of $\mathcal{U}$ for all $\tau > 0$ and $\hat{u}_{12\tau}(\cdot)$ is an element of $\mathcal{U}$ for all $\tau > 0$.

Given an input waveform $u(\cdot)$, we will say that a function $x(\cdot)$: $\mathbb{R}^+ \to \Sigma$ is a solution of the state equation $\dot{x} = f(x,u)$ if $x(\cdot)$ is absolutely continuous on every bounded interval $[0, T]$ with $T > 0$ [15], and satisfies $\dot{x}(t) = f(x(t), u(t))$ for almost all $t$.

*Standing Assumptions on State Representations*

(1) The functions $f(\cdot,\cdot)$, $g(\cdot,\cdot)$, $V(\cdot,\cdot)$, and $I(\cdot,\cdot)$ are continuous.

(2) For every $x_0 \in \Sigma$ and every $u(\cdot) \in \mathcal{U}$ there exists a unique solution $x(\cdot)$: $\mathbb{R}^+ \to \Sigma$ of the differential equation $\dot{x} = f(x,u)$ such that $x(0) = x_0$.

(3) If $S$ is a state representation for an $n$-port and if the pair $\{u(\cdot), x(\cdot)\}$ is as described in (2), then the port voltage and port current of the $n$-port are, respectively, $v(t) = V(x(t), u(t))$ and $i(t) = I(x(t), u(t))$.

(4) For every pair $\{u(\cdot), x(\cdot)\}$ as described in (2), the function $t \to p(x(t), u(t))$ is locally $L^1$.

(5) The set of admissible input waveforms $\mathcal{U}$ is translation invariant and closed under concatenation, and all functions in $\mathcal{U}$ are measurable.

The second assumption implies that $x(\cdot)$ is defined and continuous on $\mathbb{R}^+$, so systems with finite escape times are ruled out. Since $x(\cdot)$ must take values in $\Sigma$, we assume that no admissible input can drive the state out of the state space.

The fourth assumption implies that the input energy is finite over any finite interval of time.

The assumption that $\mathcal{U}$ is translation invariant is a natural one for time-invariant systems, and closure under concatenation means roughly that any two input waveforms which can be applied separately can be applied in sequence. While we do not require that $\mathcal{U}$ be a vector space, all the $L^p$ and locally $L^p$ spaces of functions mapping $\mathbb{R}^+$ to $\mathbb{R}^n$ will satisfy assumption (5). But the corresponding spaces of continuous and of differentiable functions will not.

Assumptions (1)–(5) will seldom be mentioned again in this paper, but they are essential to the formal proofs, which are given in [2].

*Definition 8.* A *state space trajectory* is a function $x$: $\mathbb{R}^+ \to \Sigma$ which is a solution of $\dot{x} = f(x,u)$ for some $u(\cdot) \in \mathcal{U}$. If $x(\cdot)$ is a state space trajectory with $x(t_1) = x_1$ and $x(t_2) = x_2$, $t_1 < t_2$, we will call the restriction of $x(\cdot)$ to $[t_1, t_2]$ a *trajectory from* $x_1$ *to* $x_2$. The restriction of $x(\cdot)$ to $[t_1, t_2]$ will be denoted by $x(\cdot)|[t_1, t_2]$. An *input-trajectory pair* is a pair of functions $u(\cdot) \in \mathcal{U}$ and $x$: $\mathbb{R}^+ \to \Sigma$ such that $x(\cdot)$ is a solution of $\dot{x} = f(x,u)$. If $\{u(\cdot), x(\cdot)\}$ is an input-trajectory pair with $x(0) = x'$, we call it an *input-trajectory pair with initial state* $x'$. If $\{u(\cdot), x(\cdot)\}$ is an input-trajectory pair with $x(t_1) = x_1$ and $x(t_2) = x_2$, $t_1 < t_2$, we call $\{u(\cdot), x(\cdot)\}|[t_1, t_2]$ an *input-trajectory pair from* $x_1$ *to* $x_2$. The *energy consumed* by $\{u(\cdot), x(\cdot)\}|[t_1, t_2]$ is the quantity

$$\int_{t_1}^{t_2} p(x(t), u(t))\, dt.$$

It follows from standing assumption (4) that this quantity is always finite when $t_1$ and $t_2$ are finite.

*Definition 9.* Let $\{u(\cdot), x(\cdot)\}$ be an input-trajectory pair. If $y(t) = g(x(t), u(t))$ for all $t \geq 0$, then $\{u(\cdot), y(\cdot)\}$ is called an *input–output pair*. If $v(t) = V(x(t), u(t))$ and $i(t) = I(x(t), u(t))$ for all $t \geq 0$, then $\{v(\cdot), i(\cdot)\}$ is called an *admissible pair*. If $x(0) = x'$, then $\{u(\cdot), y(\cdot)\}$ is called an *input–output pair with initial state* $x'$ and $\{v(\cdot), i(\cdot)\}$ is called an *admissible pair with initial state* $x'$. We will adopt the notation $\{v(\cdot), i(\cdot)\}|[0, T]$ for the restriction of $\{v(\cdot), i(\cdot)\}$ to $[0, T]$.

Finally, we shall often loosely use the term "system" to denote an $n$-port $\mathcal{N}$ along with a given state representation.

## III. The Proposed Definition of Passivity and Some of its Consequences

### 3.1 Available Energy and Passivity

The following definition formalizes the concept of available energy introduced in Section I.

*Definition 10.* Given an $n$-port $\mathcal{N}$ with a state representation $S$, we define the *available energy* $E_A$: $\Sigma \to \mathbb{R}^+ \cup \{+\infty\}$ by

$$E_A(x) \triangleq \sup_{\substack{x\to \\ T\geq 0}} \left\{ \int_0^T -\langle v(t), i(t)\rangle\, dt \right\} \tag{3-1}$$

where the notation $\sup_{\substack{x\to \\ T\geq 0}}$ indicates that the supremum is taken over all $T \geq 0$ and all admissible pairs $\{v(\cdot), i(\cdot)\}$ with the fixed initial state $x$ (Definition 9).

Since we have assumed that $t \to \langle v(t), i(t)\rangle$ is locally $L^1$ (standing assumption (4), Section II), the integral in (3-1) always exists and is finite; however, it is possible for $E_A(x)$ to be infinite for certain values of $x$. **Roughly speaking, the available energy at a particular state $x$ is the maximum energy that can be extracted from the system when its initial state is $x$.** Note that the above expression defines $E_A(\cdot)$ exactly, not merely to within an additive constant. Sine the value $T = 0$ is allowed as an upper limit of the integral in (3-1), $E_A(x)$ is the supremum of a set of numbers which includes zero. Therefore, $E_A(\cdot)$ is a nonnegative function, as claimed in the definition.

Note that the concept we have called Passivity 1 in Section I is equivalent to the condition $E_A(\mathbf{0}) = 0$.

*Example 4.* A 2-terminal capacitor characterized by a continuous function $v = v(q)$ has the natural state representation

$$\dot{q} = i$$
$$v = v(q) \tag{3-2}$$

where $\Sigma = U = \mathbb{R}^1$ and $\mathcal{U}$ is the class of all locally $L^1$ waveforms $i(\cdot)$: $\mathbb{R}^+ \to \mathbb{R}$.

It is well known that the energy extracted in driving a 1-port capacitor from any initial state $q_1$ to any final state $q_2$ depends only on the endpoints $q_1$ and $q_2$ and is given by

$$E(q_1, q_2) = -\int_{q_1}^{q_2} v(q)\, dq = \int_{q_2}^{q_1} v(q)\, dq. \tag{3-3}$$

Therefore, when $S$ is in the form of (3-2), Definition 10 reduces to

$$E_A(q_1) = \sup_{q_2 \in \mathbf{R}} \{E(q_1, q_2)\}. \tag{3-4}$$

Let's briefly reconsider the constitutive relations in Figs. 1 and 3. For the constitutive relation in Fig. 1(a) we have $E(q_1, q_2) = (q_1-1)^2/2 - (q_2-1)^2/2$. Taking the supremum over $q_2$ we have $E_A(q_1) = (q_1-1)/2$ or $E_A(q) = (q-1)^2/2$, as drawn in dotted lines in Fig. 1(b). Clearly an energy-optimal control exists. It just drives $q$ to the point $q=1$. For the constitutive relation in Fig. 3(a), we have $E(q_1, q_2) = e^{q_1} - e^{q_2}$. Taking the supremum over $q_2$ we have $E_A(q_1) = e^{q_1}$, or $E_A(q) = e^q$, as shown in Fig. 3(b).

The way to extract the maximum energy from this last element is to drive the charge as far negative as possible. While there is no trajectory which succeeds in reaching $q = -\infty$ in finite time and extracting *all* the energy possible, the supremum in (3-1) includes admissible pairs and values of $T$ for which the extracted energy approaches the value $e^q$. In fact Definition 10 was stated in terms of a supremum in order to handle precisely this type of situation—one in which no finite-time energy-optimal control exists.

*Definition 11.* An $n$-port $\mathfrak{N}$ with a state representation $S$ is *passive* if for each $x \in \Sigma$, $E_A(x) < +\infty$. Otherwise $\mathfrak{N}$ is *active*.

Definition 11 is just a formal restatement of Passivity 3 from Section I. If for some initial state $x_0$ there is no upper bound on the amount of energy which can be extracted, then $E_A(x_0) = +\infty$ and so $\mathfrak{N}$ is active; but if no such state exists, then $\mathfrak{N}$ is passive. Observe that passivity requires only that $E_A(x)$ be finite for each $x \in \Sigma$, and we do not consider infinity or any point with one or more coordinates equal to $\pm\infty$ to be an element of $\Sigma$. In particular, passivity does *not* require that $E_A(\cdot)$ be a bounded function on $\Sigma$ (cf. Figs. 1(b) and 3(b)).

The nonlinear capacitor in Fig. 3 violates intuitive notions of passivity because it has no state where $E_A(q) = 0$; nevertheless, it is passive by Definition 11. The capacitor in Fig. 3 has no state where $v(q) = 0$, but it is important to realize that the condition, $E_A(q) > 0$ for all $q$, is not limited to systems with no state of zero voltage, as we see below.

*Example 5.* Consider the nonlinear capacitor with constitutive relation $\hat{v}(q) = q(1-q^2)e^{-q^2/2}$, shown in Fig. 4.

This capacitor is unbiased, in the sense that if $q(0) = 0$ and $i(t) = 0$ for $t > 0$, then $v(t) = 0$ for $t > 0$. A straightforward calculation shows that $E_A(q) = (1+q^2)e^{-q^2/2}$, as shown in the figure. Therefore, it is passive by Definition 11, and $E_A(q) > 0$ for all $q$. This example is interesting because if $|q(t)|$ remains sufficiently small for all $t$, then $v(t) = \hat{v}(q(t)) \approx q(t)$; hence, this capacitor behaves locally at $q=0$ as though it were a 1-F linear capacitor, yet it has no state of zero available energy.

Examples 3 and 5 may seem counterintuitive since experience with common linear elements suggests that a passive system ought to possess a state of "zero stored energy," or more precisely, a state of zero *available* energy. For this reason, we will define in Section 3.3 a narrower concept of passivity called *strong passivity* which is closer to engineering intuition.

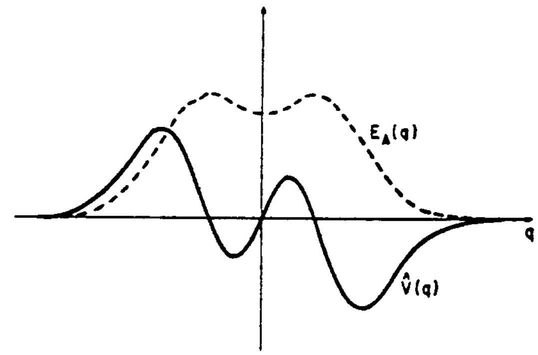


Fig. 4. The nonlinear capacitor with constitutive relation $\hat{v}(q) = q(1-q^2)e^{-q^2/2}$. Note that $E_A(q)$ is never exactly zero.

### 3.2 Controllable Systems

Passivity requires that $E_A(x) < +\infty$ for *each* $x \in \Sigma$. Suppose we know that $E_A(x_0) < +\infty$ for a *particular* $x_0 \in \Sigma$. When can we conclude that $E_A(x) < +\infty$ for *all* $x \in \Sigma$? The following theorem and its corollaries answer this question, but first we must define reachability and complete controllability.

*Definition 12.* Given a state representation $S$, let $x_0, x_1 \in \Sigma$. The *state* $x_1$ is said to be *reachable from* $x_0$ if there exists a finite $T > 0$ and an input/trajectory pair $\{u(\cdot), x(\cdot)\}|[0,T]$ from $x_0$ to $x_1$ (Definition 8). The *state space* $\Sigma$ is said to be *reachable from* $x_0$ if every $x \in \Sigma$ is reachable from $x_0$.

Because of our standing assumption that $t \to \langle v(t), i(t) \rangle$ is locally $L^1$, it follows that the transfer from $x_0$ to $x_1$ can always be effected with a finite (positive or negative) amount of energy.

*Definition 13.* A state representation $S$ is said to be *completely controllable* if $\Sigma$ is reachable from $x$ for every $x \in \Sigma$.

*Theorem 1.* Given an $n$-port $\mathfrak{N}$ with a completely controllable state representation $S$, let $x_0$ be any state in $\Sigma$. Then $\mathfrak{N}$ is passive if and only if $E_A(x_0) < +\infty$.

Necessity follows directly from Definition 11. Sufficiency follows from Definitions 11 and 13, and the fact that the transfer from $x_0$ to any state $x_1$ requires only finite energy. A formal proof is given in [2].

### 3.3 Relaxed States and Strong Passivity

Definition 11 is definitely weaker than the concept of passivity that one would gain from experience with common circuit elements. In practice it is natural to associate with a passive element some sort of "rest state" or "relaxed state" or state of "zero stored energy." While we do not wish to *found* a general nonlinear theory on the existence of such states, it is reasonable to try to *incorporate* relaxed states into our approach.

*Definition 14.* Given an $n$-port $\mathfrak{N}$ with a state representation $S$, a point $x \in \Sigma$ is said to be a *relaxed state* if $E_A(x) = 0$.

In Fig. 2 the only relaxed state is $v_c = 0$, while in Fig. 1 the state $q = 1$ is relaxed. The systems in Figs. 3 and 4 do not have any relaxed states.

*Definition 15.* An *n*-port $\mathfrak{N}$ with a state representation $S$ is *strongly passive* iff

(1) $\mathfrak{N}$ is passive by Definition 11, and

(2) there exists a relaxed state $x^* \in \Sigma$.

The constitutive relation in Fig. 1 defines a strongly passive[3] 1-port, for example. The constitutive relations in Figs. 3 and 4 define systems which are passive but not strongly passive. We will see in the next section, however, that passivity and strong passivity are equivalent concepts for linear, resistive, and memristive *n*-ports. Fig. 2 has a relaxed state but is neither passive nor strongly passive. Theorem 2 shows that such a situation can never arise in completely controllable systems.

*Theorem 2.* Suppose an *n*-port $\mathfrak{N}$ has a completely controllable state representation $S$. If there exists a relaxed state $x^* \in \Sigma$, then $\mathfrak{N}$ is passive (and hence strongly passive as well).

This is an immediate consequence of Theorem 1.

### 3.4 Equivalent State Representations

The definition of passivity in Section 3.1 is not based directly on the physical properties of an *n*-port $\mathfrak{N}$, but rather on a certain function $E_A(\cdot)$ which depends on the particular state representation we have chosen for $\mathfrak{N}$. The following example will help make this point clear.

*Example 6.* When viewed as a 1-port, a 1-F capacitor is completely characterized by the relation $i = dv/dt$ or (1-2). Let's consider the following three state representations for such an element. (In all three cases we let $\mathfrak{U} = L^1_{\text{loc}}(\mathbf{R}^+ \to \mathbf{R})$.)

| $S_1$ | $S_2$ | $S_3$ |
|---|---|---|
| $\dot{x}_1 = i$ | $\dot{x}_2 = i$ | $\dot{x}_3 = (\cos^2 x_3)i$ |
| $v = x_1$ | $v = x_2 - 1$ | $v = \tan x_3$ |
| $\Sigma_1 = \mathbf{R}$ | $\Sigma_2 = \mathbf{R}$ | $\Sigma_3 = (-\pi/2, \pi/2)$. |

Representation $S_2$ is just a restatement of Example 1, with $x_2$ substituted for $q$. To see that $S_3$ represents the same 1-port as $S_1$ and $S_2$, we calculate

$$\frac{dv}{dt} = \frac{dv}{dx_3}\frac{dx_3}{dt} = \left(\frac{1}{\cos^2 x_3}\right)(\cos^2 x_3)(i) = i.$$

Since $S_1$, $S_2$, and $S_3$ all represent a 1-F capacitor, we certainly hope that $E_{A_1}(x_1)$, $E_{A_2}(x_2)$, and $E_{A_3}(x_3)$ will all be finite for each value of their arguments. Otherwise our definition of passivity will be dependent on the particular state representation we select. For $S_1$ and $S_2$ we know from Section 3.1 that $E_{A_1}(x_1) = x_1^2/2$ and $E_{A_2}(x_2) = (x_2 - 1)^2/2$, so $E_{A_1}(x_1) < +\infty$ for all $x_1 \in \Sigma_1$ and $E_{A_2}(x_2) < +\infty$ for all $x_2 \in \Sigma_2$. To calculate $E_{A_3}(x_3)$, note that if $v(0) = v_0$, then for any $T > 0$ and any input $i(\cdot) \in \mathfrak{U}$, the energy extracted over the interval $[0, T]$ is

$$\int_0^T -v(t)i(t)\,dt = \int_0^T -v(t)\dot{v}(t)\,dt = \int_0^T -\frac{d}{dt}\left(\frac{v^2(t)}{2}\right)dt$$
$$= \frac{1}{2}\left(v_0^2 - v^2(T)\right) < \frac{1}{2}v_0^2.$$

Therefore, the available energy as a function of $v_0$ is $v_0^2/2$. Writing it as a function of the initial state $x_3$ we have $E_{A_3}(x_3) = (\tan x_3)^2/2$, which is finite for each $x_3 \in \Sigma_3$.

So in this example the classification of the *n*-port as active or passive does not depend on which of the state representations we choose. We now want to show that this result holds in general. The following definition is essentially that given in [17].

*Definition 16.* Given two state representations $S_1$ and $S_2$, $x_1 \in \Sigma_1$ and $x_2 \in \Sigma_2$ are said to be *equivalent states* if the class of admissible pairs $\{v(\cdot), i(\cdot)\}$ of $S_1$ with initial state $x_1$ is identical to the class of admissible pairs of $S_2$ with initial state $x_2$. And $S_1$ and $S_2$ are *equivalent state representations* if for each $x_1 \in \Sigma_1$ there exists an equivalent $x_2 \in \Sigma_2$ and for each $x_2 \in \Sigma_2$ there exists an equivalent $x_1 \in \Sigma_1$.

In Example 6, $S_1$, $S_2$, and $S_3$ are all equivalent state representations. Given $x_1 \in \Sigma_1$, the equivalent states are $x_2 = x_1 + 1$ and $x_3 = \tan^{-1} x_1$. Given $x_2 \in \Sigma_2$, the equivalent states are $x_1 = x_2 - 1$ and $x_3 = \tan^{-1}(x_2 - 1)$. And given $x_3 \in \Sigma_3$, the equivalent states are $x_1 = \tan x_3$ and $x_2 = 1 + \tan x_3$.

*Lemma 1.* Let $S_1$ and $S_2$ be two state representations with $x_1 \in \Sigma_1$ and $x_2 \in \Sigma_2$. If $x_1$ and $x_2$ are equivalent states, then $E_{A_1}(x_1) = E_{A_2}(x_2)$.

The proof is immediate from Definitions 10 and 16.

*Theorem 3.* Suppose an *n*-port $\mathfrak{N}$ has two equivalent state representations $S_1$ and $S_2$. Then Definition 11 applied to $S_1$ classifies $\mathfrak{N}$ as passive $\Leftrightarrow$ it classifies $\mathfrak{N}$ as passive when applied to $S_2$. And Definition 15 applied to $S_1$ classifies $\mathfrak{N}$ as strongly passive $\Leftrightarrow$ it classifies $\mathfrak{N}$ as strongly passive when applied to $S_2$.

The proof is immediate from Lemma 1.

### 3.5 Interconnections of Passive N-Ports

*Definition 17.* We say that an attribute of *n*-ports has the property of *closure* if it is preserved under finite interconnections, i.e., if whenever $\mathfrak{N}_1, \cdots, \mathfrak{N}_k$ have the attribute and $\mathfrak{N}$ is obtained by interconnecting $\mathfrak{N}_1, \cdots, \mathfrak{N}_k$ with ideal lossless connecting wires, then $\mathfrak{N}$ must have the attribute as well.

Linearity and time-invariance, for example, possess closure. Observability and controllability do not. Does passivity have the closure property? In other words, will a finite interconnection of passive *n*-ports always be passive?

The purpose of this subsection is to show that passivity as defined in this paper does have the closure property, at least under certain assumptions. We let $\mathfrak{N}_1, \cdots, \mathfrak{N}_k$ have state representations $S_1, \cdots, S_k$ with state spaces $\Sigma_1, \cdots, \Sigma_k$. We will consider only the simplest case, in

[3]Our use of the term "strongly passive" should not be confused with similar terminology which has appeared in the systems literature (e.g., [16]).

which $\mathfrak{N}$ has a state representation $S$ which satisfies our assumptions in Section II and has the state space $\Sigma = \Sigma_1 \times \cdots \times \Sigma_k$, the Cartesian product of the individual state spaces. We will call such an interconnection *admissible*.

*Lemma 2.* Let $\mathfrak{N}$ with state representation $S$ be an admissible interconnection of $\mathfrak{N}_1, \cdots, \mathfrak{N}_k$ as defined above. Let $E_{A_j}: \Sigma_j \to \mathbf{R}^+$ be the available energy for $\mathfrak{N}_j$, $1 < j < k$, and $E_A: \Sigma = \Sigma_1 \times \cdots \times \Sigma_k \to \mathbf{R}^+$ be the available energy for $\mathfrak{N}$. Then if $x = (x_1, \cdots, x_k) \in \Sigma$, we have $E_A(x) < E_{A_1}(x_1) + \cdots + E_{A_k}(x_k)$.

The proof is immediate, since when $\mathfrak{N}_j$ is in initial state $x_j$, the total energy leaving its ports is bounded above by $E_{A_j}(x_j)$, and by Tellegen's theorem the power leaving the ports of $\mathfrak{N}$ at any instant is the sum of the powers leaving the ports of $\mathfrak{N}_1, \cdots, \mathfrak{N}_k$.

*Theorem 4.* Let $\mathfrak{N}$ with state representation $S$ be an admissible interconnection of $\mathfrak{N}_1, \cdots, \mathfrak{N}_k$.

(1) If $\mathfrak{N}_1, \cdots, \mathfrak{N}_k$ are passive, then $\mathfrak{N}$ is passive.

(2) If $\mathfrak{N}_1, \cdots, \mathfrak{N}_k$ are strongly passive, then $\mathfrak{N}$ is strongly passive.

Statement (1) follows immediately from Lemma 2. If $\mathfrak{N}_1, \cdots, \mathfrak{N}_k$ are strongly passive, then there exist states $x_j^* \in \Sigma_j$ with $E_{A_j}(x_j^*) = 0$. Since $x^* = (x_1^*, \cdots, x_k^*)$ is in $\Sigma$, we have from Lemma 2 that $E_A(x^*) = 0$, and statement (2) follows.

## IV. Necessary and Sufficient Conditions for Passivity of Several Classes of $n$-Ports

For certain special classes of $n$-ports[4] it is possible to find necessary and sufficient conditions for passivity which can be verified directly by inspection of the state equations.

### 4.1 Resistive n-Ports

The resistive $n$-ports considered here are completely characterized by the relation $y = g(u)$, where $u$ and $y$ are a hybrid pair (Definition 3). Let $U$ be any nonempty subset of $\mathbf{R}^n$; $g: U \to \mathbf{R}^n$ be any function, and $\mathfrak{U}$ be the class of all functions $u(\cdot)$: $\mathbf{R}^+ \to U$ such that $t \to \langle u(t), g(u(t)) \rangle$ is locally $L^1$.

It is unnatural to construct a state representation for a resistive element, but in order to include such elements in our theory we will given them representations of the form $\dot{x} = 0$, $y = g(u)$, with $\Sigma$ taken to be any nonempty subset of $\mathbf{R}^m$.

*Theorem 5.* Let $\mathfrak{N}$ be a resistive $n$-port with a state representation $S$ as described above. Then the following three statements are equivalent:

(i) $\langle u, g(u) \rangle > 0$, $\quad \forall u \in U$.

(ii) $E_A(x) = 0$, $\quad \forall x \in \Sigma$.

(iii) $\mathfrak{N}$ is passive according to Definition 1.

It follows immediately that passivity and strong passivity are equivalent for resistive $n$-ports. The proof, which is trivial, is given in [2].

[4]The resistive, generalized capacitive/inductive, and generalized memristive $n$-ports considered in this section are special cases of the *algebraic n-ports* treated by Chua [18].

### 4.2 Generalized Capacitive/Inductive n-Ports

A generalized capacitive/inductive $n$-port is an $n$-port with state and output equations of the form

$$\dot{x} = u$$
$$y = g(x) \tag{4-1}$$

where $u$ and $y$ form a hybrid pair. We call these generalized capacitive/inductive $n$-ports since they reduce to $n$-port capacitors if $u = i$ and $y = v$ and to $n$-port inductors if $u = v$ and $y = i$. (The relevant technical assumptions are that $\Sigma = U = \mathbf{R}^n$, $\mathfrak{U} = L^1_{loc}(\mathbf{R}^+ \to \mathbf{R}^n)$, and that $g$: $\mathbf{R}^n \to \mathbf{R}^n$ is continuous.)

*Theorem 6.* Let $\mathfrak{N}$ be a generalized capacitive/inductive $n$-port with a state representation $S$ as described above. Then

(1) $\mathfrak{N}$ is passive $\Leftrightarrow g = \nabla\psi$, where $\psi$: $\Sigma \to \mathbf{R}$ is a $C^1$ scalar function which is bounded from below.

(2) $\mathfrak{N}$ is strongly passive $\Leftrightarrow$ the above conditions hold and in addition $\psi(\cdot)$ attains its lower bound, i.e., $\exists x^* \in \Sigma$ such that $\psi(x^*) < \psi(x)$, $\forall x \in \Sigma$.

The proof is given in [2]. The capacitor in Fig. 1 satisfies both conditions (1) and (2) while the capacitors in Figs. 3 and 4 satisfy only (1).

It is easy to see that if $\mathfrak{N}$ is passive and $l$ is the greatest lower bound of $\psi(\cdot)$, then $E_A(x) = \psi(x) - l$, $\forall x \in \Sigma$. An immediate consequence of Theorem 6 is that if $\mathfrak{N}$ is passive and $g(\cdot)$ is $C^1$, then the Jacobian matrix, $[Dg](x)$ is symmetric at each point $x \in \Sigma$. Linearizing (4-1) about any state $x$, it follows that this symmetry condition is equivalent to reciprocity if $u = i$ or if $u = v$. But if $u \neq i$ and $u \neq v$, i.e., if $u$ contains both voltage and currents, then symmetry of $[Dg]$ is an entirely different condition from reciprocity ([18], but contrary to [6]).

*Corollary to Theorem 6.* A passive $n$-port inductor or capacitor with a $C^1$ function $g(\cdot)$ is reciprocal.

### 4.3 Generalized n-Port Memristors

By a generalized $n$-port memristor we mean an $n$-port with state and output equations of the form

$$\dot{x} = u$$
$$y = [R(x)]u \tag{4-2}$$

where $u$ and $y$ form a hybrid pair and where $R(x)$ is an $n \times n$ real matrix which varies with $x$. A system of this sort is, roughly speaking, a state-dependent linear resistor [19]. The relevant technical requirements in this case are that $\Sigma = \mathbf{R}^n$, the entries of $R(x)$ are continuous functions on $\Sigma$, and that $\mathfrak{U} = L^2_{loc}(\mathbf{R}^+ \to \mathbf{R}^n)$.

*Theorem 7.* A generalized $n$-port memristor with a state representation as described above is passive $\Leftrightarrow R(x)$ is positive semidefinite at each point $x \in \Sigma$.

The proof, which is straightforward, is in [2].

*Corollary to Theorem 7.* A generalized $n$-port memristor is passive if and only if it is strongly passive.

### 4.4 Linear n-Ports

We have shown in Example 2 and Fig. 2 that passivity is not exactly equivalent in general to the traditional positive real criterion. They are equivalent, however, *if the n-port is completely controllable*. The definition of a positive real transfer function and its special interpretation in the case of a rational function are well known [2], [7].

We consider the linear time-invariant finite dimensional state representation

$$\dot{x}=Ax+Bu$$
$$y=Cx+Du \tag{4-3}$$

where $u$ and $y$ form a hybrid pair; $U=\mathbf{R}^n$ and $\Sigma=\mathbf{R}^m$; $A$, $B$, $C$, and $D$ are real constant matrices of appropriate dimension; and $\mathfrak{U}$ is taken to be the class of all locally $L^2$ functions $u$: $\mathbf{R}^+\to\mathbf{R}^n$. It follows immediately that the output $y(\cdot)$ is defined for all positive time and is itself locally $L^2$. The matrix condition for complete controllability is also well known [17].

*Theorem 8.* Suppose $\mathfrak{N}$ has a linear, time-invariant, finite-dimensional state representation $S$ as in (4-3). If $S$ is completely controllable, then the following three conditions are equivalent:

(i) $\mathfrak{N}$ is passive by Definition 11;

(ii) $E_A(0)=0$;

(iii) The matrix transfer function

$$H(s)=C[sI-A]^{-1}B+D$$

is positive real.

Slight variations on this theorem are well known [7], [9] and a complete elementary proof is given by Gannett and Chua [20]. Example 2 in Fig. 2 shows why the assumption of complete controllability is essential: for that 2-port $Z(s)$ is positive real and $E_A(0)=0$, but it is active nevertheless.

### *4.5 First-Order Systems*

The systems considered in this subsection are those for which the state space $\Sigma$ is contained in the real line. As far as we know, the results given here are entirely new to the literature. We consider systems with a state equation of the form

$$\dot{x}=f(x,u). \tag{4-4}$$

The technical requirements are that $\Sigma\subset\mathbf{R}^1$, $U$ is a closed subset of $\mathbf{R}^n$, and $\mathfrak{U}$ is the set of all locally $L^\infty$ functions mapping $\mathbf{R}^+$ to $U$. In most examples, $\Sigma$ will be $\mathbf{R}^1$ or an interval in $\mathbf{R}^1$.

*Definition 18.* For each point $x\in\Sigma$, let $U_x^+$ be the set of all input values $u\in U$ such that $f(x,u)>0$. Similarly, let $U_x^-$ be all values of $u$ such that $f(x,u)<0$.

We will let $\mathbf{R}^e$ denote the set of extended real numbers, i.e., $\mathbf{R}^e=\{-\infty\}\cup\mathbf{R}\cup\{+\infty\}$.

*Definition 19.* We define $\underline{h}$ and $\bar{h}$: $\Sigma\to\mathbf{R}^e$ by

$$\underline{h}(x)\triangleq\begin{cases}\sup\limits_{u\in U_x^-}\dfrac{p(x,u)}{f(x,u)}, & \text{if } U_x^-\neq\varnothing\\ -\infty, & \text{if } U_x^-=\varnothing\end{cases} \tag{4-5}$$

$$\bar{h}(x)\triangleq\begin{cases}\inf\limits_{u\in U_x^+}\dfrac{p(x,u)}{f(x,u)}, & \text{if } U_x^+\neq\varnothing\\ +\infty, & \text{if } U_x^+=\varnothing.\end{cases} \tag{4-6}$$

Consider the first line in (4.6). Before taking the infimum, the numerator is energy input per unit time and the denominator is the distance the state moves to the right per unit time; so the quotient is the input energy per unit distance $\Delta x$, in the limit as $\Delta x\to 0$ from above. Taking the infimum, we see that $\bar{h}(x)$ is the minimum energy cost per unit displacement of $x$ to the right, with the convention that $\bar{h}(x)=+\infty$ if it is impossible to drive the state to the right from the point $x$. Similarly, $\underline{h}(x)$ is the maximum energy we can extract per unit displacement of the state to the left from the point $x$, this time as $\Delta x\to 0$ from below: the convention here is that $\underline{h}(x)=-\infty$ if we cannot move to the left from $x$. In general, neither $\underline{h}(\cdot)$ nor $\bar{h}(\cdot)$ will be continuous; however, $\underline{h}(\cdot)$ is lower semicontinuous and $\bar{h}(\cdot)$ is upper semicontinuous (see [2]).

*Example 7.* Consider the following memristive 1-port [19]:

$$\dot{x}=|i|^\alpha$$
$$v=r(x)i \tag{4-7}$$

where $\Sigma=U=\mathbb{R}$, $\mathfrak{U}$ consist of all locally $L^\infty$ functions mapping $\mathbf{R}^+$ to $\mathbb{R}$, $r(\cdot)$ is a continuous function which is negative on the interval $(0,1)$ and zero elsewhere, and $\alpha$ is some real positive number.

It is clear that $\underline{h}(x)=-\infty$ for all $x$, since $f(x,i)\triangleq|i|^\alpha$ is never negative; moreover, $\bar{h}(x)=0$ for all $x$ outside of $(0,1)$, since $p(x,i)\triangleq r(x)i^2=0$ for $x\notin(0,1)$. When $x\in(0,1)$, $\bar{h}(x)$ is the infimum over all $i\neq 0$ of $i^2r(x)/|i|^\alpha$, i.e., the infimum of $r(x)|i|^{(2-\alpha)}$: if $\alpha=2$, this is just $r(x)$; if $\alpha\neq 2$, the infimum is $-\infty$ since $r(x)<0$.

For any state representation $S$ and for any $x\in\Sigma$, let $R(x)$ denote the set of states reachable from $x$ (Definition 12). For first-order systems it will be useful to define, for each $x_0\in\Sigma$, $R^-(x_0)\triangleq\{x\in R(x_0)\colon\ x<x_0\}$ and $R^+(x_0)\triangleq\{x\in R(x_0)\colon\ x>x_0\}$. In Example 7, for instance, $R^-(x_0)=\varnothing$ and $R^+(x_0)=(x_0,+\infty)$, for any $x_0\in\Sigma$.

*Theorem 9.* Let $\mathfrak{N}$ be an $n$-port with a state representation $S$ as given in (4-4) and the paragraph following (4-4) and suppose that for each $x\in\Sigma$, $R^+(x)$ and $R^-(x)$ are open in $\mathbf{R}^1$. Then $\mathfrak{N}$ is passive if and only if all three of the following conditions are satisfied:

(i) $p(x,u)>0$ at every point $(x,u)\in\Sigma\times U$ such that $f(x,u)=0$;

(ii) $\underline{h}(x)<\bar{h}(x)$, $\forall x\in\Sigma$;

(iii) there exists a function $W$: $\Sigma\to\mathbf{R}^+$ such that, for every $x_0\in\Sigma$,

$$\int_{x_0}^{x_1}\underline{h}(x)\,dx+W(x_0)>0,\qquad\forall x_1\in R^-(x_0) \tag{4-8}$$

$$\int_{x_0}^{x_2}\bar{h}(x)\,dx+W(x_0)>0,\qquad\forall x_2\in R^+(x_0). \tag{4-9}$$

The proof is given in [2]. Note that there is no need to actually calculate $W(x_0)$. Its existence is just another way of saying that the integrals in (4-8) and (4-9) remain bounded from below as their upper limits are allowed to vary in $R^-(x_0)$ and $R^+(x_0)$, respectively. Since $x_1 < x_0$, the integral in (4-8) will be positive if the integrand is everywhere negative. No such problem occurs in (4-9). If $R^-(x_0)$ is empty then (4-8) is satisfied automatically for that value of $x_0$; (4-9) is similar.

The physical interpretation of the three conditions in Theorem 9 is straightforward and quite interesting. The first condition says that it is impossible to extract power from $\mathfrak{N}$ while $x$ stands still, i.e., while $\mathfrak{N}$ remains in equilibrium. The second condition says that the maximum energy payoff per unit displacement of $x$ to the left is less than or equal to the minimum energy cost per unit displacement to the right. This means that it is impossible to extract energy by driving the state around a closed path. The integral in (4-8) represents the minimum energy consumed while driving the system from $x_0$ to $x_1$, and the integral in (4-9) is the minimum energy consumed while driving from $x_0$ to $x_2$. If the system is to be passive, then it is clear that these quantities must be bounded from below as $x_1$ and $x_2$ range over all states reachable from $x_0$. If (i) and (ii) are both satisfied, then the least function $W(\cdot)$ which satisfies (iii) is in fact the available energy $E_A(\cdot)$.

Let's reconsider Example 7. Condition (i) is always satisfied because $|i|^\alpha = 0 \Rightarrow i = 0 \Rightarrow r(x)i^2 = 0$. Since $x$ can only move to the right, $\underline{h}(x) = -\infty$ everywhere and $R^-(x_0) = \varnothing$ for all $x_0$; hence, condition (ii) is trivially satisfied and (4-8) always holds by default. If $\alpha = 2$, then $\bar{h}(x) = r(x)$ and (4-9) is satisfied by choosing for $W(\cdot)$ the constant function

$$W \triangleq \int_0^1 -r(x)\,dx.$$

If $\alpha \neq 2$, then $\bar{h}(x) = -\infty$ for all $x$ in (0,1), (4-9) cannot be satisfied, and the system is active.

## V. Internal Energy Functions and Passive $n$-Ports

Many of the results in this section have been stated and at least informally proved by Willems [9]. Our primary purpose here is to make these ideas more accessible to circuit theorists by translating them into a more appropriate language and illustrating them with a simple network example. Secondarily, we have stated them in a form which makes rigorous proofs possible. Internal energy functions will be the central tool in the theory of passive realizations in the next section.

*Definition 20.* Given an $n$-port $\mathfrak{N}$ with a state representation $S$, we say that a function $E_I$: $\Sigma \to \mathbf{R}^+$ is an *internal energy function* for $\mathfrak{N}$ if

$$E_I(x(t_2)) - E_I(x(t_1)) \leq \int_{t_1}^{t_2} p(x(t), u(t))\,dt \quad (5\text{-}1)$$

for all input-trajectory pairs $\{u(\cdot), x(\cdot)\}$ (Definition 8) and all $0 \leq t_1 \leq t_2$, where $p(\cdot,\cdot)$ is the power input function (Definition 2).

In other words, an internal energy function is just a nonnegative[5] function on $\Sigma$ which increases along trajectories more slowly than the rate at which energy is delivered to the ports (or decreases more rapidly than the rate at which energy is extracted from the ports). We shall show shortly (Theorem 10) that under mild technical assumptions an $n$-port $\mathfrak{N}$ is passive if and only if there exists an internal energy function for $\mathfrak{N}$. It is clear from Definition 20 that if $E_I(\cdot)$ attains its lower bound at some point $x^* \in \Sigma$, then the $n$-port $\mathfrak{N}$ is strongly passive with relaxed state $x^*$.

Definition 20 can be viewed as the integral version of the definition of an internal energy function, and it is the most general version. Under somewhat more restrictive assumptions the following lemma provides a differential version of Definition 20.

*Lemma 3.* Given an $n$-port $\mathfrak{N}$ with a state representation $S$, and suppose that $\Sigma \subset \mathbf{R}^m$ is open. Suppose further that $\mathfrak{U}$ satisfies the following mild technical assumption: for each $u_0 \in U$, there exists an input $u(\cdot) \in \mathfrak{U}$ such that $u(0) = u_0$ and $u(\cdot)$ is continuous at $t = 0$. Then a $C^1$ function $\psi$: $\Sigma \to \mathbf{R}^+$ is an internal energy function for $\mathfrak{N} \Leftrightarrow$

$$\langle \nabla\psi(x), f(x,u)\rangle \leq p(x,u), \qquad \text{for all } (x,u) \in \Sigma \times U. \quad (5\text{-}2)$$

The proof is given in [2].

Note that there could exist an internal energy function which satisfies (5-1) but not (5-2) by failing to be differentiable. One might conjecture that a passive, completely controllable system in which $f(\cdot,\cdot)$ and $p(\cdot,\cdot)$ are $C^\infty$ would have at least one internal energy function which is differentiable everywhere. But a counterexample given in [21] shows that this conjecture is false.

*Lemma 4.* Let $\mathfrak{N}$ be a passive $n$-port with a state representation $S$. Then the available energy $E_A(\cdot)$ is an internal energy function for $\mathfrak{N}$.

The proof is given in [2].

The following theorem shows why internal energy functions are of such importance in the study of passivity.

*Theorem 10.* Let $\mathfrak{N}$ be an $n$-port with a state representation $S$, and suppose that $\mathfrak{U}$ is translation invariant and closed under concatenation. Under these conditions, $\mathfrak{N}$ is passive $\Leftrightarrow$ there exists an internal energy function $E_I(\cdot)$ defined on $\Sigma$.

The proof is in [2].

In view of Theorem 10, we could just as well take the existence of an internal energy function as our definition of passivity. This is in fact the approach that Willems [9] has adopted.

It is an important and somewhat surprising fact that the internal energy function for a passive $n$-port will often be nonunique. The following example will make this clear.

*Example 8.* Consider the linear 1-port shown in Fig. 5. If we choose the voltage as input, its state and output equations are

[5] We could equally well require only that $E_I$ be bounded from below.

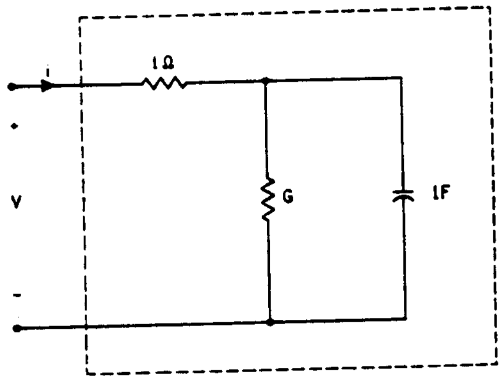


Fig. 5. The value of the shunt resistor is $R=(1/G)$ Ω. The 1-port is of course strongly passive provided $G>0$.

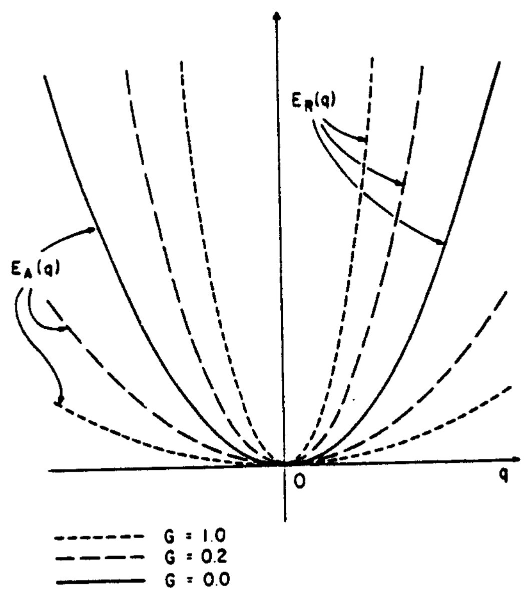


Fig. 6. Available energy and required energy for the 1-port in Fig. 5, plotted for several values of $G$. The solid line in the center represents the function $q^2/2$.

$$\dot{q}=v-(G+1)q$$
$$i=v-q$$

and we suppose that $\mathcal{U}=L^2_{loc}(\mathbf{R}^+\rightarrow\mathbf{R})$ and $G>0$.

It is immediately clear that the system is passive and the electric energy stored in the capacitor, $E_c(q)=q^2/2$, is an internal energy function. But there are other possible internal energy functions of the form $\psi(q)=\alpha\cdot(q^2/2)$. Inequality (5-2) becomes in this case $\alpha q[v-(G+1)q]<v(v-q)$, or $v^2-[(\alpha+1)q]v+\alpha(G+1)q^2>0$, $\forall v, q\in\mathbf{R}$. It is simple to verify that this inequality always holds if and only if $(\alpha+1)^2-4(G+1)\alpha<0$; hence, $\alpha\cdot(q^2/2)$ is an internal energy function if and only if

$$(2G+1)-2\sqrt{G(G+1)}<\alpha<(2G+1)+2\sqrt{G(G+1)}\,. \tag{5-3}$$

There are two features of this result worth noticing. The first is that there is a range of possible values of $\alpha$, and hence of internal energy functions, for each $G>0$. The second is that $\alpha=1$ lies in this interval for any value of $G>0$, as a simple calculation will verify. Therefore, the electrical energy $q^2/2$ is always a valid internal energy function.

Willems' theory for general linear system [9], based on the algebraic Riccati equation, shows that the least value of $\alpha$ satisfying (5-3) defines the available energy. In other words, $E_A(q)=[(2G+1)-2\sqrt{G(G+1)}\,]\cdot q^2/2$ for this system. This function is plotted in Fig. 6 for three different values of $G$. When $G=0$, i.e., when the shunt resistor is an open circuit, we can extract energy from the capacitor with arbitrarily small losses by letting $i$ be very small and the discharge time very long. Therefore, $E_A(q)=q^2/2$ when $G=0$, and all the electrical energy is available at the ports.

Willems' theory further shows that the largest value of $\alpha$ satisfying (5-3) determines the "required energy" function $E_R(q)$, which gives the minimum energy required to reach the state $q$ from the zero initial state. Further theorems, proofs, examples, and discussion concerning internal energy functions, their nonuniqueness, and the required energy function can be found in [2].

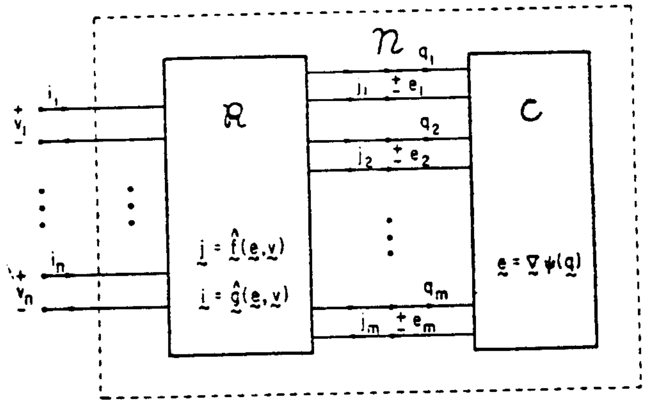


Fig. 7. Every voltage-controlled state representation has a realization of this form in which $\mathcal{C}$ is passive. If $\mathcal{R}$ and $\mathcal{C}$ are both passive it is called a passive realization.

## VI. The Passive Realization of *n*-Ports

The result reported in this section is related to the work of Anderson and Moylan [22]. Our result is more general because we do not require that the state equations be linear in the control, but for the same reason it is less constructive in nature.

Consider the *n*-port $\mathcal{N}$ in Fig. 7 formed by connecting the capacitive *m*-port $\mathcal{C}$ to the resistive $(n+m)$-port $\mathcal{R}$. We suppose that the constitutive relation of $\mathcal{C}$ is defined by a $C^1$ scalar function $\psi$: $\mathbf{R}^m\rightarrow\mathbf{R}$, i.e., $e=\nabla\psi(q)$. And the constitutive relation of $\mathcal{R}$ is given by $j=\hat{f}(e,v)$ and $i=\hat{g}(e,v)$, where $\hat{f}$: $\mathbf{R}^m\times\mathbf{R}^n\rightarrow\mathbf{R}^m$ and $\hat{g}$: $\mathbf{R}^m\times\mathbf{R}^n\rightarrow\mathbf{R}^n$ are continuous. Substituting the equation $\dot{q}=j$ and the constitutive relation of $\mathcal{C}$ into the above, we arrive at the state and output equations for $\mathcal{N}$:

$$\dot{q}=\hat{f}(\nabla\psi(q),v)$$
$$i=\hat{g}(\nabla\psi(q),v). \tag{6-1}$$

*Technical Assumptions*

We assume throughout the remainder of this subsection that $U=\mathbf{R}^n$, $\Sigma=\mathbf{R}^m$, and that $\mathcal{U}$ satisfies the mild technical assumption given in Lemma 3. Also the phrase "$\mathfrak{R}$ is passive" will mean that $\mathfrak{R}$ is passive when its inputs are restricted to $\nabla\psi[\mathbf{R}^m]\times\mathbf{R}^n\subset\mathbf{R}^m\times\mathbf{R}^n$.

*Lemma 5.* For some value of $m\in\mathbf{R}$ the function $\psi_m(\cdot)\triangleq\psi(\cdot)-m$ is an internal energy function (Definition 20) for the $n$-port $\mathfrak{N}$ in Fig. 7 with the state representation $S$ defined above $\Leftrightarrow$ $\mathfrak{R}$ and $\mathcal{C}$ are both passive.

To see this, we recall from Definition 20 and Lemma 3 that $\psi_m(\cdot)$ is an internal energy function for this system if and only if $\psi_m(\cdot)$ is nonnegative and

$$\langle\nabla\psi_m(q),\hat{f}(\nabla\psi_m(q),v)\rangle\leq\langle v,\hat{g}(\nabla\psi_m(q),v)\rangle \quad (6\text{-}2)$$

for all $(q,v)\in\mathbf{R}^m\times\mathbf{R}^n$. Now $\psi_m(\cdot)$ is nonnegative for some value of $m$ if and only if $\psi(\cdot)$ is bounded from below, a condition which is equivalent to passivity of $\mathcal{C}$ by Theorem 6. Since $\nabla\psi_m=\nabla\psi=e$, (6-2) can be written as

$$\langle v,\hat{g}(e,v)\rangle+\langle e,-\hat{f}(e,v)\rangle=\langle v,i\rangle-\langle e,j\rangle\geq 0$$

which is equivalent to passivity of $\mathfrak{R}$ (Theorem 5) once the reference direction for $j$ is taken into account.

*Definition 21.* The $n$-port $\mathfrak{N}$ in Fig. 7 is a *realization* of the state representation

$$\dot{x}=f(x,v)\quad i=g(x,v) \quad (6\text{-}3)$$

with the technical assumptions listed above if $\hat{f}$, $\hat{g}$, and $\psi$ are chosen so that

$$f(x,v)=\hat{f}(\nabla\psi(x),v)$$
$$g(x,v)=\hat{g}(\nabla\psi(x),v),\qquad\forall(x,v)\in\mathbf{R}^m\times\mathbf{R}^n. \quad (6\text{-}4)$$

It is a *passive realization* if $\mathfrak{R}$ and $\mathcal{C}$ are both passive.

We view the multiports $\mathfrak{R}$ and $\mathcal{C}$ as given quantities—we are not concerned with the difficult and unsolved problem of synthesizing these nonlinear multiports. It is clear that any voltage-controlled state representation $S$ has a realization in which $\mathcal{C}$ is passive and linear: if each port of $\mathcal{C}$ is a 1-F capacitor, then $\nabla\psi(q)=q$ and we obtain a realization by choosing $\hat{f}(\cdot,\cdot)=f(\cdot,\cdot)$ and $\hat{g}(\cdot,\cdot)=g(\cdot,\cdot)$; in general, however, the resistive $(m+n)$-port $\mathfrak{R}$ will not be passive for such a realization.

The following theorem is an immediate consequence of the preceding lemma and definition.

*Theorem 11.* Suppose the state representation $S$, given in (6-3) along with the technical assumptions, is passive and further that there exists a $C^1$ internal energy function $\psi$: $\mathbf{R}^m\to\mathbf{R}^+$ such that (6-4) holds. Then the $n$-port in Fig. 7 is a passive realization of $S$.

Since $\mathcal{C}$ is clearly passive under these conditions, the point of Theorem 11 is that $\mathfrak{R}$ is passive as well, precisely because $\psi(\cdot)$ is an internal energy function. The problem with Theorem 11 is of course that we do not generally know how to find $\hat{f}(\cdot,\cdot)$ and $\hat{g}(\cdot,\cdot)$ satisfying (6-4); we do not even know in general when they exist. The following corollary gives us one special case in which these problems do not arise.

*Corollary to Theorem 11.* Suppose the state representation $S$, given in (6-3) along with the technical assumptions, is passive and that there exists a $C^1$ internal energy function $\psi$: $\mathbb{R}^m\to\mathbb{R}^+$ such that $\nabla\psi$: $\mathbb{R}^m\to\mathbb{R}^m$ is 1-1. Then $S$ has a passive realization as in Fig. 7.

In this case we can simply construct $\hat{f}(\cdot,\cdot)$ and $\hat{g}(\cdot,\cdot)$ as follows:

$$\hat{f}(\nabla\psi(x),v)=f\left[(\nabla\psi)^{-1}\circ(\nabla\psi(x)),v\right]$$
$$\hat{g}(\nabla\psi(x),v)=g\left[(\nabla\psi)^{-1}\circ(\nabla\psi(x)),v\right].$$

For simplicity, we have discussed only voltage-controlled state representations in this section. Analogous results hold for any state representation in which $u$ and $y$ form a hybrid pair.

Theorem 11 and its corollary show that the recovery of a $C^1$ internal energy function from a given state representation $S$ is an important first step toward obtaining a passive realization of $S$. This problem is solved for certain classes of state representations in [8].

## VIII. Concluding Remarks

### 7.1 Three Different Interpretations of Passivity

There are three different but logically equivalent ways of interpreting passivity, as defined in this paper.

The first might be called "the thermodynamic point of view," and is related to the various "availability" concepts of thermodynamics [23, ch. 17]. Here we consider an $n$-port as a possible energy source and concern ourselves with how much energy we can hope to extract from it. The maximum amount will generally depend on the initial state and is denoted by $E_A(x_0)$. We have written this paper from the thermodynamic point of view, as Definitions 10 and 11 clearly reflect. This approach seems to us to present the meaning of passivity in the clearest possible light and to be the appropriate one for network synthesis, but it does not make the potential links between passivity and stability especially obvious.

From the second perspective, which we might call the "input–output viewpoint, applied to $n$-ports with state equations," we look on an $n$-port as a family of operators $H_x$, one operator for each initial state $x\in\Sigma$. These operate on an input waveform $u(\cdot)$ to produce an output waveform $y(\cdot)$, i.e. $H_x$: $u(\cdot)\mapsto y(\cdot)$ iff $u(\cdot)$ and $y(\cdot)$ are an input–output pair (Definition 9) with initial state $x$. For this discussion we assume that $u(\cdot)$ and $y(\cdot)$ are a hybrid pair and that the domain and image of $H_x$ are in $L^2_{\mathrm{loc}}(\mathbf{R}^+\to\mathbf{R}^n)$, since we wish to introduce the family of inner products

$$\langle u(\cdot),y(\cdot)\rangle_T=\int_0^T\langle u(t),y(t)\rangle\,dt.$$

We can then think of $H_x$ as passive if $\langle u(\cdot),H_xu(\cdot)\rangle_T$ is bounded below as $u(\cdot)$ varies over $\mathcal{U}$ and $T$ varies over $\mathbf{R}^+$, and we say that the $n$-port is passive if $H_x$ is passive for each $x\in\Sigma$.

We should mention that the usual practice in works on feedback systems or input–output theory is to model a physical system by a single operator $H$ and to ignore its dependence on the initial state. This practice is reasonable enough in the linear case where the intended initial state is clearly the origin. But in the nonlinear case it is incomplete, especially for those systems that have a multiplicity of locally stable equilibrium points when the input is zero. Furthermore, this omission has the effect of making the gap between the input–output and state space viewpoints seem even wider than it actually is.

The thermodynamic approach is logically equivalent to the input–output viewpoint described above. Input–output theory is extremely important as the setting for many stability theorems from control theory which are based on functional analysis [14]. It is important to note, however, that such theorems always require conditions which are stronger than or distinct from passivity alone, e.g., incremental passivity.

The third way of looking at passivity might be called "the internal energy point of view." From this perspective we would say that an $n$-port is passive if there exists a nonnegative function on the state space which decreases along trajectories at least as rapidly as the rate at which energy leaves the ports. We have shown in Theorem 10 that under mild technical assumptions this point of view is equivalent to the first two. Many stability theorems in circuit theory take this view of passivity because the internal energy function can often serve as a Lyapunov function; it is important to note, however, that even a smooth internal energy function need not be a genuine Lyapunov function because the concept of passivity alone imposes no requirements on its shape. The internal energy function in Fig. 3(b) is an example which shows that additional requirements are necessary, since it does not qualify as a Lyapunov function.

Finally, we would like to mention a recent paper [24] which also addresses the relationship between various input–output and state space concepts of passivity. In terms of the notation and assumptions adopted earlier in this subsection, [24] defines passivity as follows. For any subset $\Omega\subset\Sigma$, let $H_\Omega$ be the family of operators $\{H_x|x\in\Omega\}$. Then $H_\Omega$ is "passive" according to [24] if $\langle u(\cdot), H_x u(\cdot)\rangle_T > 0$ for all $T\in\mathbf{R}^+$, $u(\cdot)\in\mathcal{U}$, and $x\in\Omega$. And the (mathematical model of the) $n$-port is passive according to [24] if there exists a nonempty $\Omega\subset\Sigma$ such that $H_\Omega$ is passive. This notion of passivity is quite precise about the role played by the initial state, but we feel it is not fully adequate for $n$-ports for reasons which the following example makes clear. Let $H_0$ be the zero state input–output map for Example 2, i.e. $H_0$: $\{i_1(\cdot), i_2(\cdot)\}\mapsto\{v_1(\cdot), v_2(\cdot)\}$ when $v_C(0)=0$. Then $H_0$ is passive and, therefore, the $n$-port in Fig. 2 is passive according to [24] even when $C<0$, i.e., under conditions when $v_C=0$ is an unstable state, the system is an infinite energy source for all $v_C\neq 0$, and $|v_2(t)|\to\infty$ as $t\to\infty$ for all nonzero initial states and all inputs. For this reason we feel that an $n$-port should be classified as passive only on the basis of its behavior over *all* of $\Sigma$.

Passivity (Definition 11) is equivalent to "weak passivity" of $H_\Sigma$ in the sense of [24]. But [24] incorrectly states that strong passivity (Definition 15) is equivalent to "passivity" of $H_\Sigma$ in the sense of [24]. The correct statement would be that an $n$-port is strongly passive (Definition 15) if $H_\Sigma$ is "weakly passive" and $H_\Omega$ "passive" for some nonempty $\Omega\subset\Sigma$, both in the sense of [24].

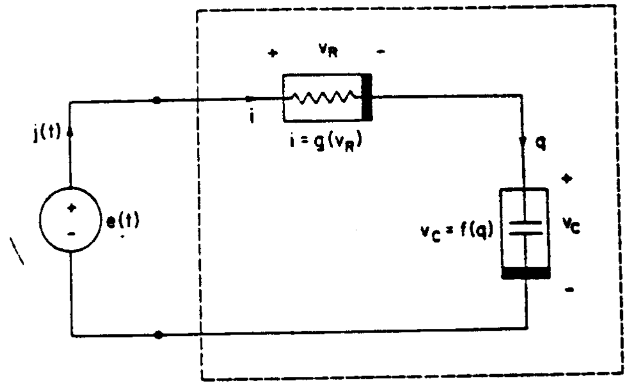

Fig. 8. Example 9 gives various choices of $f$ and $g$ for which this 1-port is passive but not stable.

*7.2 Passive Systems are Not Always Stable*

There are so many well-established connections between passivity and stability that it is tempting to suppose that every passive system is stable in some sense. But consider the following example.

*Example 9.* The voltage-controlled state equations for the 1-port in Fig. 8 are

$$\frac{dq}{dt}=g(e-f(q))$$
$$j=g(e-f(q)). \tag{7-1}$$

*Case I:* Suppose the resistor is 1 Ω and the capacitor is the exponential capacitor of Fig. 3(a), i.e., $g(v_R)=v_R$, and $f(q)=\exp(q)$. This system is passive but not strongly passive. The zero-input trajectories are given by $q(t)=-\ln[\exp(-q(0))+t]$ and hence unbounded for *any* initial state $q(0)$.

*Case II:* Suppose the resistor is 1 Ω and the capacitor is characterized by $f(q)=q(2-q^2)\exp(-q^2/2)$. One can easily verify that the available energy for the capacitor alone is $E_A(q)=q^2\exp(-q^2/2)$. Therefore, the origin is a relaxed state (Definition 14) and hence the 1-port in Fig. 8 is strongly passive (Definition 15) in this case. With zero input the origin is (locally) asymptotically stable. But one can easily verify that if $|q(0)|>\sqrt{2}$ and the input is always zero, then $|q(t)|\to\infty$ as $t\to\infty$.

Using the notation of Section 7.1, this example is described by an operator $H_0$ which is passive in the sense of [24], and hence the 1-port of Case II is passive in the sense of [24]. This second case thus shows that neither passivity in the sense of [24] nor strong passivity in the sense of Definition 15 is sufficient to guarantee that trajectories remain bounded for zero input. But neither case above is especially disturbing since the directly observable port variables $v$ and $i$ decay to zero, although $q(\cdot)$ is unbounded.

*Case III: (Pathological Case)* Suppose the capacitor is 1 F but the resistor is the peculiar passive element characterized by $v \cdot i \equiv 1$. Then the state equations (7-1) are not defined when $e = q$. This system violates standing assumption (2) of Section II and therefore, falls, strictly speaking, outside the range of our theory. We have included it, however, to show the violently unstable zero-input behavior some passive nonlinear systems can exhibit. The zero-input solution in this case is $q(t) = \sqrt{q^2(0) - 2t}$ if $q(0) > 0$, and the corresponding output current is $j(t) = -1/\sqrt{q^2(0) - 2t}$, which exhibits *finite escape time*.

Example 9 dealt only with the zero-input stability of systems. Of course passivity alone is not sufficient to guarantee bounded-input bounded-output stability either, as is clear from the familiar example of a lossless *LC* resonant circuit. The purpose of these examples is merely to illustrate a point we made in the previous subsection: all the stability theorems based on passivity involve additional assumptions, e.g., linearity, monotonicity, or local or global requirements on the shape of an internal energy function.

### 7.3 Active Models of Passive Systems

A system can be active, as we have defined the term, only if it is capable of supplying an unbounded amount of energy to the outside world. It is clear that no finite physical object can be active in that sense, yet active elements, e.g., ideal voltage sources, appear frequently in engineering models. It seems to us that there are basically two sorts of approximations which produce active models.

The first comes from multiple time scales. For example, if we were modelling an oscillator circuit powered by a storage battery, we might be concerned with its behavior only over time intervals too short for the battery to discharge significantly. In that case we would approximate the storage battery, a passive device, by an ideal voltage source, an active element. A second sort of idealization is that we are frequently interested only in certain ports of a multiport device, so we delete the other ports from our model. A transistorized power supply is a good example. If we count the electric power plug as one of its ports, it is clearly a passive 2-port, but we might reasonably model it as an active ideal 1-port voltage source.

### 7.4 Possible Generalizations

We have made the standing assumptions in this paper more restrictive than necessary in order to avoid hiding the concepts under a mass of elaborate definitions and unfamiliar notation. A number of possible extensions are discussed at length in [2].

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