Nonlinear tracking control in the presence of state and control constraints: a generalized reference governor☆

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Abstract

This paper proposes a new approach to reference governor design. As in prior literature, the governor accepts input commands and modifies their evolution so that specified pointwise-in-time constraints on state and control variables are satisfied. The new approach applies to general discrete-time and continuous-time nonlinear systems with uncertainties. It relies on safety properties provided by sublevel sets of equilibria-paramaterized functions. These functions need not be Lyapunov functions, and the corresponding sublevel sets need not be positively invariant. Technical conditions that capture the bare essentials of what is needed are identified and the usual desirable properties of reference governors are established. The new approach significantly broadens the class of methods available for constructing the nonlinear function that is required in the implementation of the reference governors. This advantage is illustrated in a nonlinear control problem where off-line, computer-based simulation is the basis for constructing the nonlinear function.

Keywords: Reference governors; State and control constraints; Nonlinear systems; Disturbance inputs; Parametric uncertainties

1. Introduction

This paper presents a reference governor approach to the problem of tracking control in the presence of system nonlinearities and constraints on state and control variables. To make the situation specific, consider the following nonlinear discrete-time system and constraint condition:

\[ x(t + 1) = f(x(t), v(t), w(t)), \]
\[ (x(t), v(t)) \in \mathcal{C} \quad \forall t \in \mathbb{Z}^+. \]

Here \( f: \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^l \to \mathbb{R}^n \), \( \mathbb{Z}^+ \) is the set of nonnegative integers, \( v(t) \) is the input, \( w(t) \) is a disturbance and \( \mathcal{C} \subset \mathbb{R}^n \times \mathbb{R}^m \) is the constraint set. Although other interpretations are possible, we suppose that (1) models a closed-loop system that, in the absence of (2), has good response characteristics. For example, there may be an output,

\[ y(t) = g(x(t), v(t), w(t)) \in \mathbb{R}^m, \]

and (1) and (3) exhibits good disturbance-tolerant tracking performance, i.e., \( y(t) \approx r(t) \) where \( r(t) \) is the reference command and \( v(t) = r(t) \). Or, (1) may describe a regulator, with a family of set points \( v(t) \equiv r \) designed so that \( x(t) \) converges to a neighborhood of \( x_e(r) \), a desired equilibrium state corresponding to \( w(t) \equiv 0 \).

Many methods now exist for designing feedback systems such as (1) or their corresponding continuous-time systems. A common approach is to synthesize a set of linear controllers, each based on the linearization of the open-loop system at a given equilibrium, and then generate an overall controller by interpolating over the set of linear controllers by gain scheduling (Rugh & Shamma, 2000). Alternatively, the interpolation may be over nonlinear regulators designed by feedback linearization (Isidori, 1995) or by the use of Control Lyapunov functions (Artstein, 1983; Clarke, Ledyaev, Rifford, & Stern, 2000; Freeman & Kokotović, 1996). In these design philosophies and most others (Sepulchre, Janković, & Kokotović, 1997; Slotine

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that they are variant sets. An advantage of linearity is that the positively invariant sets are defined by a parameterized family of Lyapunov functions, \( V(x, r) \). The parameter \( r \) is a constant input, \( v(t) \equiv r \), and \( V(x, r) \) is a Lyapunov function corresponding to a stable equilibrium, \( x_0(r) \). The positively invariant sets are sublevel sets of \( V \) defined by \( \{ x : V(x, r) - \gamma(r) \leq 0 \} \), where \( \gamma(r) \) is chosen so that \( V(x, r) \leq \gamma(r) \) implies \( (x, r) \in \mathcal{G} \). Once \( V(x, r) \) and \( \gamma(r) \) are determined they lead directly to the scalar optimization problem. There are practical difficulties in implementing this Lyapunov function approach: tools for constructing \( V(x, r) \) and \( \gamma(r) \) are limited in scope, the resulting invariant sets are often poor approximations of the maximal invariant sets, disturbance inputs and system modeling errors are not considered.

Results in this paper address these difficulties and other questions as well. A function \( \tilde{V}(x, r) \) is introduced, that plays the role of \( V(x, r) - \gamma(r) \) in the Lyapunov function approach. However, the determination of \( \tilde{V}(x, r) \) does not require Lyapunov theory and the corresponding sublevel sets, \( \Pi(r) \triangleq \{ x : \tilde{V}(x, r) \leq 0 \} \), do not have to be positively invariant. Moreover, technical conditions on \( \tilde{V} \) are weaker than those in the Lyapunov function approach of (Gilbert & Kolmanovsky, 1999a). They have an abstract form that captures the bare essentials of what is needed for the operation of the reference governor. The weaker conditions lead to: novel computational methods for the construction of \( \tilde{V} \), simplifications in the functional form of \( \tilde{V} \), faster response to reference commands, and larger sets of permissible initial states for the closed-loop system.

The paper is organized as follows. Section 2 introduces and motivates notations and assumptions required in subsequent sections. The key assumptions are (A5) and (A6). Respectively, they define, for \( v(t) \equiv r \), two notions: \( \Pi(r) \) is “safe,” initial states in \( \Pi(r) \) lead to constraint-admissible motions; \( \Pi(r) \) is “strongly returnable,” state trajectories beginning in \( \Pi(r) \) eventually return to the \( \Pi(r) \), entering a set \( \Pi(r) \) that belongs to the interior of \( \Pi(r) \). Section 3 contains the main results, including the conditions for finite settling time. Methods for constructing suitable functions \( \tilde{V} \) are considered in Section 4. An illustrative application, based on the control problem described in Miller et al. (2000), is given in Section 5. Concluding remarks appear in Section 6.
2. Basic ideas and assumptions

The main ideas behind the operation of the reference governor are quite simple. The joint objective of constraint satisfaction and minimization of tracking error is achieved by defining

\[ v(t) = v(t-1) + \kappa(t)(r(t) - v(t-1)), \]

and maximizing \( \kappa(t) \in [0, 1] \) subject to appropriate constraint-related conditions. Fig. 2 illustrates the process when \( n = 2 \) and \( m = 1 \). As shown in Fig. 2(a), the sets \( \Pi(r) \), which are “safe” and “strongly returnable”, shift continuously to the right as \( r \) increases from \( a \) to \( b \). Suppose \( r(t) \equiv b \) and \( x(t-1) \in \Pi(v(t-1)) \). Since \( \Pi(v(t-1)) \) is safe, \( (x(t-1), v(t-1)) \in \mathcal{E} \). Subsequently, two situations can occur. The first is shown in Fig. 2(b); there exists a \( \kappa(t) \geq 0 \) that is the maximum of \( \kappa \) subject to \( \kappa \in [0, 1] \) and \( x(t) \in \Pi(v(t-1) + \kappa(r(t) - v(t-1))) \). Usually \( \kappa(t) > 0 \), \( v(t) \) is closer to \( r(t) = b \) and \( x(t) \in \Pi(v(t)) \). If the sets \( \Pi(r) \) are positively invariant, the second situation does not occur. It is shown in Fig. 2(c); there is no \( \kappa(t) \in [0, 1] \) such that \( x(t) \in \Pi(v(t)) \). However, since \( \Pi(v(t-1)) \) is safe, setting \( \kappa(t), \kappa(t+1), \ldots = 0 \) implies \( x(t), v(t), x(t+1), v(t+1), \ldots \in \mathcal{E} \). This process continues until, by strong returnability, there is an \( \hat{r} \) such that there exists a maximizing \( \kappa(\hat{r}) > 0 \) and \( x(\hat{r}) \in \Pi(v(\hat{r})) \).

To implement the preceding ideas rigorously we introduce assumptions on the class of disturbances and the function \( \tilde{V}(x, r) \) that determines the \( \Pi(r) \).

The disturbance functions \( w(\cdot): Z^+ \rightarrow \mathbb{R}^l \) belong to a class \( \mathcal{W} \) that is stationary in the following nonconventional sense: \( w(\cdot) \in \mathcal{W} \) implies \( w(\cdot + \sigma) \in \mathcal{W} \) for all \( \sigma \in Z^+ \). This definition includes a variety of disturbance models. For example, let \( W \subset \mathbb{R}^l \). Then, \( \mathcal{W}_p \triangleq \{ w(\cdot): w(t) = \tilde{w} \in W, \forall t \in Z^+ \} \), \( \mathcal{W}_N \triangleq \{ w(\cdot): w(t) \in W \forall t \in Z^+ \} \) and \( \mathcal{W}_{NR} \triangleq \{ w(\cdot): w(t) \in W, \|w(t+1) - w(t)\|_\infty \leq \rho, \forall t \in Z^+ \} \) are stationary classes. In particular, \( \mathcal{W}_p \) models parametric uncertainty, \( \mathcal{W}_N \) is general disturbance family where the only constraint is on the values of \( w(t) \) and \( \mathcal{W}_{NR} \) adds a “rate limit” to the components of \( w(t) \) (Kolmanovsky, Gilbert, McClamrock, & Maizenberg, 1998b).

The disturbance-free situation corresponds to \( W = \{0\} \). If each component of \( w(t) \) is stationary, then \( w(t) \) is also stationary. Hence, different types of stationarity may be mixed within a class \( \mathcal{W} \). Since there is no probabilistic structure in the preceding models of \( \mathcal{W} \), satisfaction of (2) is a worst-case treatment of disturbances.

Denote the solution of (1), defined for all \( t \in Z^+ \), \( x = x(0) \in \mathbb{R}^n, v(t) \equiv r \in S \subset \mathbb{R}^m \) and \( w(\cdot) \in \mathcal{W} \) by \( \phi(t,x,r,w(\cdot)) \). Let

\[ \Pi_x(r) \triangleq \{ x: \tilde{V}(x, r) \leq -\varepsilon \}. \]  (6)

We make the following assumptions on \( S \), the continuity of \( \tilde{V} \) and the existence and boundness of \( \Pi(r) \) and \( \Pi_x(r) \):

(A1) \( S \) is compact and convex.

(A2) \( \tilde{V}: \mathbb{R}^n \times S \rightarrow \mathbb{R} \) is continuous.

(A3) There exists an \( \varepsilon_0 > 0 \) such that \( \Pi_{x_0}(r) \neq \emptyset \) for all \( r \in S \).

(A4) There exists a compact set \( X \subset \mathbb{R}^m \) such that \( \Pi(r) \subset X \) for all \( r \in S \).

Our remaining assumptions on \( \tilde{V} \) implement our notions of safe and strongly returnable:

(A5) For all \( r \in S \) and \( x \in \mathbb{R}^n \) satisfying \( \tilde{V}(x, r) \leq 0 \),

\[ \phi(t,x,r,w(\cdot)), r) \in \mathcal{E} \quad \forall w(\cdot) \in \mathcal{W} \quad \text{and} \quad t \in Z^+. \]  (7)

(A6) There exist \( \varepsilon, 0 < \varepsilon \leq \varepsilon_0 \), and \( \hat{r} = \tilde{r}(x,r,w(\cdot)) \in Z^+ \) such that for all \( r \in S \) and \( x \in \mathbb{R}^n \) satisfying \( \tilde{V}(x,r) \leq 0 \),

\[ \tilde{V}(\phi(\hat{r},x,r,w(\cdot)), r) \leq -\varepsilon \quad \forall w(\cdot) \in \mathcal{W}. \]  (8)

Fig. 3 illustrates how the assumptions apply to system (1) when the \( r \)-constraint set \( \mathcal{E} \) is defined by \( \{ x: h_q(x,r) \leq 0 \ \forall q \in Q \} \). If there is no disturbance, \( x_q(r) \) is the desired stable equilibrium point and \( M(r) \) is a compact region of attraction for \( x_q(r) \) that contains \( x_q(r) \) in its interior. Presence of disturbances requires generalized notions of robust stability such as those described in Freeman and Kokotović (1996), Lin, Sontag, and Yang (1996), McConley, Appleby, Dahleh, and Feron (2000), McConley, Appleby, Dahleh, and Feron (1998). We do not dwell on these issues here but simply assume that \( \mathcal{M}(r) \) is a region of attraction for the asymptotic motions generated by the disturbances. More specifically, \( \mathcal{M}(r) \) is compact and positively invariant.
If $K$ is a specific procedure for evaluating an attraction. There is also a maximum constraint-admissible region of attraction. Usually, $\mathcal{N}(r)$ is relatively small and well within $\mathcal{G}_r$. Thus, $\mathcal{N}(r)$ is a constraint-admissible region of attraction. There is also a maximum constraint-admissible region of attraction: $\mathcal{M}(r) \triangleq \{ x : (7) is satisfied, there exists $i \in \mathbb{Z}^+$ such that $\phi(t, x, r, w(\cdot)) \in \mathcal{N}(r)$ for all $t \geq i$ and $w(\cdot) \in \mathcal{W} \}$. Suitable choices for $\Pi(r)$ and $\Pi_t(r)$ satisfy the inclusions $\mathcal{N}(r) \subset \Pi_t(r) \subset \Pi_t(r) \subset \mathcal{M}(r)$. If (A1) and (A2) hold and $\mathcal{G}_r$ is compact for all $r \in S$, assumptions (A3)–(A6) then follow automatically. Usually $\mathcal{N}(r)$ is much smaller than $\mathcal{M}(r)$, so there is much freedom for the choice of $\mathcal{P}$.

Since $S$ is convex, $\tilde{V}(x, v + \lambda(r - v))$ is defined for all $x \in \mathbb{R}^n$, $v \in S$, $r \in S$ and $\lambda \in [0, 1]$. However, by (A2) $\lambda^*(x, v, r)$ is defined if and only if

$$\tilde{V}(x, v + \lambda(r - v)) \leq 0 \text{ for some } \lambda \in [0, 1].$$

Ideally, when (11) is satisfied $K = \lambda^*$. When (11) is not satisfied

$$K(x, v, r) \triangleq 0.$$  

To prove our main results it is necessary to accept small increments, $v(t) - v(t - 1)$, only when $\tilde{V}(x, v(t), v(t - 1)) \leq -\varepsilon$. We achieve this objective by modifying the conclusion $K = \lambda^*$. Let $\delta > 0$. Then, when (11) is satisfied

$$K(x, v, r) \triangleq \lambda^*(x, v, r), \quad \text{if } \lambda^*(x, v, r)||r - v||_{\infty} \geq \delta,$$

and

$$\tilde{V}(x, v) \leq -\varepsilon,$$

and

$$K(x, v, r) \triangleq 0, \quad \text{if } \lambda^*(x, v, r)||r - v||_{\infty} < \delta.$$

In normal operation of the reference governor, the situation corresponding to (15) occurs infrequently, particularly when $\delta$ is small. When it does occur (15) forces $v(t)$ to be held constant until $\tilde{V}(x, v, r)\leq -\varepsilon$ or $||v(t) - v(t - 1)||_{\infty} \geq \delta$ for some $i > t$.

Theorem 1. Suppose $K$ is defined by (10)–(15) and $\tilde{V}$ satisfies (A1)–(A6). Assume there exists a $\tilde{v} \in S$ such that $\tilde{V}(x(0), \tilde{v}) \leq 0$. Set $v(0) = \tilde{v}$ and suppose $r(t) \in S$ for all $t \in \mathbb{Z}^+$. Then, $x(t) \in \mathbb{R}^n$ and $v(t) \in S$ are well defined for all $t \in \mathbb{Z}^+$. Furthermore, constraint (2) is satisfied.

Proof. The first result is an obvious consequence of (1),(5) and (9)–(15) and the convexity of $S$. Suppose $i \in \mathbb{Z}^+$ is such that $\tilde{V}(x(i), v(i)) \leq 0$. Then by (A5), the time invariance of (1) and the stationarity of $w^r$, it follows that $(x(i), v(i)) \in \mathcal{G}_r$. Consider two alternatives: (a) $v(t) = v(i)$ for $t = i + 1, \ldots, i + \sigma$; (b) $v(i + 1) \neq v(i)$. By (7), alternative (a) implies $(x(t), v(t)) \in \mathcal{G}_r$ for $t = i + 1, \ldots, i + \sigma$; by (5) and the definition of $K$, alternative (b) implies $\tilde{V}(x(i + 1), v(i + 1)) < 0$, which in turn implies $x(i + 1), v(i + 1)) \in \mathcal{G}_r$. Alternative (a) holds for all $\sigma \geq 1$ or there exists a $i > \tilde{i}$ such that $v(i + 1) \neq v(i)$. Assembling these results with $\tilde{V}(x(0), v(0)) \leq 0$ completes the proof. $\square$

Remark 1. In principle it is possible to handle any initial state $x(0) \in \Pi_{IS}$ where

$$\Pi_{IS} \triangleq \bigcup_{r \in S} \Pi_t(r).$$

Often $\Pi_{IS}$ is considerably larger than $\Pi_t(r)$ for any $r \in S$. This is an advantage that the reference governor has over the nondynamic tracking controller considered in Blanchini and Miani (2000). Even if $r(t) \equiv r_{set}$, a fixed set-point,
the reference governor is useful because it enlarges the safe domain of attraction. One procedure for picking $\hat{v}$ so that $x(0) \in \Pi(\hat{v})$ is to algorithmically minimize $\|r(0) - \hat{v}\|$ subject to $\hat{v} \in S$ and $\hat{P}(x(0), \hat{v}) \leq 0$.

**Theorem 2.** Suppose all the assumptions of Theorem 1 are satisfied and $r(t) = r_0 \in S$ for all $t \geq t_0$. Then, there exists $\hat{t} \geq t_0$ such that $v(t) = r_0$ for all $t \geq \hat{t}$.

**Proof.** Suppose $v(\hat{t}) = r_0$ for some $\hat{t} \geq t_0$. Then, by (5), it is obvious that $v(t) = r_0$ for all $t \geq \hat{t}$. Thus, the theorem is false only if $\kappa(t) < 1$ for all $t \geq t_0$. We show that this condition leads to a contradiction. Let $I = \{t \geq t_0; \kappa(t) > 0\}$. It is easy to see that $I = I_1 \cup I_2$ where $I_1 = \{t \geq t_0; \|v(t) - v(t-1)\| \geq \delta\}$ and $I_2 = \{t \geq t_0; \hat{P}(x(t), v(t-1)) \leq -\varepsilon, \|v(t) - v(t-1)\| \leq \delta\}$. From (5) it follows that for $t \geq t_0$, $v(t)$ can be represented as $v(t) = v(t_0) + \sigma_r(r_0 - v(t_0))$, where $0 \leq \sigma_r \leq 1$. Since for $t \in I_1$, $\|v(t) - v(t-1)\| = (\sigma_r - 1)\|r_0 - v(t_0)\| \leq \delta$, it follows that $I_1$ is a finite set. Also, by (8), the condition $\kappa(t) = 0$ cannot hold for more than a finite number of consecutive time steps. Thus, $I_2$ must be an infinite set. Note from $0 < \kappa(t) < 1$ and (9) that $\hat{P}(x(t), v(t)) = 0$ for all $t \in I_2$. Since $(x(t), v(t))$ belongs to the compact set $X \times S$ there exists an infinite subset $I_2 \subset I_2$ and $(\tilde{x}, \tilde{v}) \in X \times S$ such that $t \in \tilde{I}_2$, $t \rightarrow \infty$, implies $x(t) \rightarrow \tilde{x}$, $v(t) \rightarrow \tilde{v}$. Because $v(t)$ moves monotonically toward $r_0$ it also follows that $v(t) \rightarrow \tilde{v}$. Hence by the continuity of $\hat{P}$, $\hat{P}(x(t), v(t-1)) \rightarrow \hat{P}(\tilde{x}, \tilde{v}) = 0$. This contradicts the requirement that $\hat{P}(x(t), v(t-1)) \leq -\varepsilon$ for all $t \in I_2$. □

Theorem 2 shows that the reference governor has finite settling time if the reference command eventually becomes constant. Further, by the definition of $\mathcal{N}(r_0)$ and $x(t) \in \Pi(r_0) \subset \mathcal{M}(r_0)$, $x(t)$ has the desired asymptotic behavior associated with $v(t) \equiv r_0$. There exist corresponding results for linear systems (Gilbert & Kolmanovsky, 1999b) that are less restrictive, in the sense that the reference governor has finite settling time even when $r(t)$ does not eventually become constant. Our present result in this direction is the following theorem.

**Theorem 3.** Suppose all the assumptions of Theorem 1 are satisfied and $\|r(t) - r_0\| \leq \delta_0$ for all $t \geq t_0$ where $r_0 \in S$ and $\delta_0$ satisfies $0 < \delta_0 < \frac{1}{\hat{t}}$. Then, if $\delta$ is sufficiently small, there exists $\hat{t} \geq t_0$ such that $\|v(t) - r_0\| \leq \delta_0$ for all $t \geq \hat{t}$.

**Proof.** Suppose, there exists a $\hat{t} \geq t_0$ such that $\kappa(\hat{t}) = 1$. Then $\|x(\hat{t}) - r_0\| \leq \delta_0$. This result, together with (5), $\kappa(t) \in [0, 1]$, and the convexity of $\{v; \|v - r_0\| \leq \delta_0\} \cap S$ show that $\|v(t) - r_0\| \leq \delta_0$ for all $t \geq \hat{t}$. We prove the existence of $\hat{t}$ by assuming $\kappa(t) < 1$ for all $t \geq \hat{t}$ and showing that this leads to a contradiction. By the Lemma in Gilbert and Kolmanovsky (1999b) it follows from $\|r(t) - r_0\| \leq \delta_0$ and $\kappa(t) \in [0, 1]$ that there exists a $\hat{t} \geq t_0$ such that $\kappa(t)\|r(t) - v(t-1)\| = \|v(t) - v(t-1)\| \leq 2\delta_0 < \delta$

for all $t \geq \hat{t}$. From this result and the definition of $K$ there are only two possibilities for each $t \geq \hat{t}$: (a) $\kappa(t) = 0$; (b) $\kappa(t) > 0$, $\hat{P}(x(t), v(t)) = 0$ and $\hat{P}(x(t), v(t-1)) \leq -\varepsilon$. By (8), (a) cannot hold for more than a finite number of consecutive time steps. Thus, there exists a $t' \geq \hat{t}$ such that (b) holds for $t = t'$. This implies $|\hat{P}(x(t'), v(t')) - \hat{P}(x(t'), v(t' - 1))| \geq \varepsilon$. Since $\hat{P}$ is continuous on the compact set $X \times S$, $\hat{P}$ is uniformly continuous on $\theta$. Thus, there exists $\delta > 0$, dependent only on $\varepsilon$, such that $|\hat{P}(x(t'), v(t')) - \hat{P}(x(t'), v(t' - 1))| < \varepsilon$. This provides the desired contradiction. □

Under additional, reasonable assumptions the result of Theorem 3 implies finite settling time. Since the assumptions are technical in nature and difficult to verify in practice, we omit the proof. The main idea is to show that for large $t$ and small $\delta$ the solution of (1) stays close to the solution of (1) for $v(t) \equiv r_0$. This requires that the norm of the variations in $x(t)$ is bounded by a constant times $\delta$. Hence, for sufficiently large $t$, $x(t)$ is confined to $\mathcal{N}(r_0)$, a neighborhood of $\mathcal{N}(r_0)$ that becomes closer to $\mathcal{N}(r_0)$ as $\delta$ decreases. By choosing $\delta$ so that $\mathcal{N}(r_0) \subset \Pi(r)$ for all $r$ such that $|v - r_0| \leq \delta$ there exists a $\hat{t} \equiv \hat{t}$ such that $x(t) \in \Pi(r(t))$ for all $t \geq \hat{t}$. By our definition of $K$ this implies $\kappa(t) = 1$ for all $t \geq \hat{t}$.

**Remark 2.** The proofs of Theorems 2 and 3 shed light on how $\varepsilon$ and $\delta$ should be chosen. Theorem 2 holds for any choice of $\delta > 0$. If $\delta$ is very large, $\kappa(t) > 0$ if and only if $\hat{P}(x(t), v(t-1)) \leq -\varepsilon$. Since this may unnecessarily cause $\kappa(t) = 0$, the response of the reference governor is slowed. In practice we have found it desirable to pick $\delta$ quite small compared to the expected maximum value of $\mathcal{N}(x, v, r)$ $|v - r| \|v\|_{\infty}$, say $\delta \leq 10^{-2}D$ where $D = \max_{x, r, v \in S} |v - r| \|v\|_{\infty}$, the diameter of $S$. The value of $\varepsilon$ measures the relative “size” of $\Pi(r)$ compared to the “size” of $\Pi(r)$. Its upper value is limited by the requirement that $\mathcal{N}(r) \subset \Pi(r)$. Decreasing $\varepsilon$ has two effects; $\|x, r, v\|_{\infty}$ in (8) decreases and (15) is less likely to cause $\kappa(t) = 0$. Both effects improve the response speed of the reference governor. Conversely, increasing $\varepsilon$ increases the value of $\delta$ in Theorem 3. Thus, fast response of the reference governor and its ability to handle persistently varying reference commands are conflicting requirements. Fortunately, none of the above factors seem to affect strongly the choice of $\varepsilon$. For example, the reference governor often exhibits finite settling time when the variations in $r(t)$ far exceed those imposed by Theorem 3. We have successfully used values of $\varepsilon$ where the diameter of $\Pi(r)$ is between 0.9 and 0.99 times the diameter of $\Pi(r)$.

**Remark 3.** Often it is not possible to restrict user reference commands. There are two methods for removing the condition $r(t) \in S$. The first method follows the pattern used in Bemporad (1998), Gilbert et al. (1995), Gilbert and Kolmanovsky (1999b): define $K: \mathbb{R}^n \times S \times \mathbb{R}^m \rightarrow [0, 1]$ by replacing $[0, 1]$ in (10) and (11) by $[0, \lambda_s(v, r)]$ where $\lambda_s(v, r) \equiv \max\{\lambda \in [0, 1], v + \lambda(r - v) \in S\}$.
Then the proofs of theorems go through essentially as before. If \( r_0 \not\in S \), Theorems 1 and 2 still hold but \( v(t) = v(t_0) + \lambda(t)(v(t_0), r_0)(r_0 - v(t_0)) \neq r_0 \). This result is far from ideal because \( v(t) \) depends on \( v(t_0) \) and may differ greatly from \( r_0 \). In the second method the reference governor is not changed; instead, \( r(t) \) is mapped onto \( S \) before it is sent to the reference governor. The ideal mapping is the projection, \( P : R^m \to S \), that determines the (unique) point in \( S \) that has the least Euclidean distance from \( r(t) \). Obviously, Theorems 1–3 hold since \( P(r(t)) \in S \). In Theorem 2, \( v(t) \) is the point in \( S \) closest to \( r_0 \) and is therefore independent of \( v(t_0) \). The advantages of the second method are tempered by the need to compute \( P(r(t)) \) for each \( t \in \mathbb{Z}^+ \).

**Remark 4.** The reference governor can generate large jumps in \( v(t) \). While they are desirable in the sense that they move \( v(t) \) rapidly toward \( r(t) \), they may excite large transients in system (1) or violate unmodeled physical constraints. It is possible to impose the constraint \( \|v(t) - v(t-1)\|_{\infty} \leq \delta_{\text{max}} \) by modifying (10) and (11) in the following way. Assume \( \delta < \delta_{\text{max}} \) and let

\[
\lambda_{\text{max}}(v, r) \triangleq \|r - v\|^{-1}_\infty \delta_{\text{max}} \quad \text{if} \quad \|r - v\|^{-1}_\infty \delta_{\text{max}} < 1,
\]
\[
= 1 \quad \text{if} \quad \|r - v\|^{-1}_\infty \delta_{\text{max}} \geq 1.
\]

In (10) and (11) replace the \([0, 1]\) by \([0, \lambda_{\text{max}}(v, r)]\). Then the proofs of Theorems 1–3 go through again with only minor changes. It is also possible to include either of the modifications described in Remark 3: in the second method, no further change is required; in the first method, \([0, 1]\) is replaced by \([0, \lambda_{\text{max}}(v, r)] \cap [0, \lambda_S(v, r)]\).

In most practical situations, the dependence of \( \bar{F}(x, r) \) on \( r \) is nonlinear, not necessarily smooth, and quite complicated. Hence, it is generally impossible to obtain a formula for testing (11) and evaluating (10) if (11) holds. We now describe an algorithmic definition of \( K(x, r, v) \) that preserves Theorems 1–3 and the remarks. To simplify notations define

\[
F(\lambda) = \bar{F}(x, v + \lambda(r - v)).
\]

Then \( \lambda^*(x, v, r) \), as required in the preceding developments, is obtained by attempting to maximize \( \lambda \) subject to \( \lambda \in [0, \mu] \) and \( F(\lambda) \leq 0 \), where \( \mu = 1, \lambda_S(v, r) \) or \( \lambda_{\text{max}}(v, r) \).

Stated differently, there are three alternatives: (1) \( F(\lambda) > 0 \) for all \( \lambda \in [0, \mu] \), \( \lambda^*(x, v, r) \) is not defined. (2) \( F(\lambda) < 0 \) for all \( \lambda \in [0, \mu] \), \( \lambda^*(x, v, r) = \mu \). (3) \( F(\lambda) = 0 \) for some \( \lambda \in [0, \mu] \), \( \lambda^*(x, v, r) \) is the largest root of \( F(\lambda) = 0 \) for \( \lambda \in [0, \mu] \). The algorithmic definition of \( K(x, v, r) \) involves two changes: in testing \( F(\lambda) \) for the three alternatives \( \lambda \in [0, \mu] \) is replaced by \( \lambda \in A = \{1, \ldots, N \} \) where \( \lambda_i < \lambda_{i+1} \) for \( i = 1, \ldots, N - 1 \), \( \lambda_0 = 0 \) and \( \lambda_N = \mu \); given \( \varepsilon \in (0, \varepsilon) \), the maximum root of \( F(\lambda) = 0 \) in \( \lambda \in [0, \mu] \), is replaced by any solution of \( -\varepsilon \leq F(\lambda) \leq 0 \), \( \lambda \in [0, \mu] \). It is possible that \( -\varepsilon \leq F(\lambda) \leq 0 \) may have solutions on many distinct sub-intervals of \([0, \mu]\). The following algorithm searches first for larger solutions. Because \( F(\lambda) \) is continuous, the needed bisection process is finite and the algorithm stops after evaluating \( F(\lambda) \) finitely many times.

**Algorithm K.** Step 1: Evaluate \( F(\lambda_i) \) for \( i = N, N - 1, \ldots, 1 \) until \( F(\lambda_i) \leq 0 \). If no such \( i \) is found, set \( K(x, v, r) = 0 \) and end. If \( i = N \), set \( \lambda^*(x, v, r) = \mu \), define \( K(x, v, r) \) by (13)–(15) and end. Step 2: By Step 1, \( F(\lambda_i) \leq 0 \) and \( F(\lambda_{i+1}) > 0 \). Use bisection on the root finding problem \( F(\lambda) = 0 \) to obtain \( \lambda \in [\lambda_i, \lambda_{i+1}] \) such that \(-\varepsilon \leq F(\lambda) \leq 0 \). Set \( \lambda^*(x, v, r) = \lambda \), define \( K(x, v, r) \) by (13)–(15) and end.

**Theorem 4.** Assume \( K(x, v, r) \) is determined by Algorithm K. Then the results of Theorems 1–3 and its variants still hold.

**Proof.** The prior arguments go through with minor changes in reasoning and notations. Although the replacement of \([0, \mu] \) by \( A \) makes it more likely that \( \kappa(t) = 0 \), by (8) it is still true that this condition cannot persist for more than a finite number of consecutive time steps. Principal changes in notations of the proofs are: \( \bar{F}(x(t), v(t)) = 0 \) is replaced by \( -\varepsilon \leq \bar{F}(x(t), v(t)) \leq 0 \); \( \bar{F}(\tilde{x}, \tilde{v}) = 0 \) is replaced by \( \bar{F}(\tilde{x}, \tilde{v}) \geq -\varepsilon \); at the end of the proof of Theorem 3, \( \varepsilon \) is replaced by \( \varepsilon - \varepsilon \). □

**Remark 5.** With appropriate modifications all the preceding results apply to continuous-time systems. System (1) and constraint (2) are replaced by

\[
\frac{d\tilde{x}}{dt}(\tau) = f(\tilde{x}(\tau), \tilde{v}(\tau), \bar{w}(\tau)), \tag{20}
\]
\[
(\tilde{x}(\tau), \tilde{v}(\tau)) \in \mathcal{G} \quad \forall \tau \in \mathbb{R}^+, \tag{21}
\]
and the reference governor is the sampled-data system defined by: \( \tilde{v}(\tau) = v(T) \) for \( tT \leq \tau < (t + 1)T \), \( x(t) = \tilde{x}(tT) \), \( r(t) = \tilde{r}(tT) \). Here \( T > 0 \) is the sample period, \( t \in \mathbb{Z}^+ \), \( r \in \mathbb{R}^+ \to S \) is the continuous-time reference command and \( v(t) \) is defined by (5) and (9). Assume there are conditions on \( f \) and \( \bar{w} \) such that (20) has a solution, \( \tilde{v}(\tilde{x}, \tilde{r}, \bar{w}(\cdot)) \), defined for all \( \tau \in \mathbb{R}^+ \), \( \tilde{x}(0) = x \in \mathbb{R}^n \), \( \tilde{r}(\cdot) \equiv r \in S \) and \( \bar{w}(\cdot) \in \mathcal{W} \). Let \( \mathcal{W} \) be stationary (same conditions as in Section 2 with \( T \) and \( \mathbb{R}^+ \) replacing \( t \) and \( \mathbb{Z}^+ \)); define \( \phi(t, x, r, \bar{w}(\cdot)) = \bar{v}(T, x, r, \bar{w}(\cdot)) \); replace (7) by

\[
\tilde{v}(\tilde{x}(\tau), \tilde{r}(\tau), \bar{w}(\cdot)) \in \mathcal{G} \quad \forall \tau \in \mathbb{R}^+. \tag{22}
\]

Then under assumptions (A1)–(A6) everything goes through as before with constraint satisfaction corresponding to (21). Our previous construction of the reference governor for continuous-time nonlinear systems without disturbances Gilbert and Kolmanovsky (1999a) uses positively-invariant sets \( \Pi(r) \) that are sublevel sets of an \( r \)-parameterized Lyapunov function \( V(x, r) \). The assumptions in Gilbert and Kolmanovsky (1999a) can be shown to imply the assumptions in Section 2 of the present paper. Thus, the reference governor of Gilbert and Kolmanovsky (1999a) follows as a special case.
4. Determination of $\hat{V}$

We now turn briefly to the problem of identifying functions $\hat{V}(x, r)$ whose sublevel sets $\Pi(r)$ either equal or approximate the maximal sets, $\mathcal{M}(r)$. There is a general procedure for obtaining $\Pi(r) = \mathcal{M}(r)$. Here we consider only discrete-time systems; using Remark 5 the extension to continuous-time systems is straightforward.

**Theorem 5.** Let $\mathcal{C} = \{(x, r): h_q(x, r) \leq 0 \forall q \in Q\}$, where $Q = \{1, \ldots, q_{\text{max}}\}$. Assume: (i) (A1) is satisfied, (ii) $W$ is compact and $W^c$ is one of the classes defined in Section 2, (iii) $\mathcal{C}$ is bounded, (iv) $f: \mathbb{R}^n \times S \times W \rightarrow \mathbb{R}^n$ and $h_q: \mathbb{R}^n \times S \rightarrow \mathbb{R}$, $q \in Q$, are continuous, (v) there exists $\varepsilon_0 > 0$ and $t_0 \in \mathbb{Z}^+$ such that $h_q(\phi(t, x, r, w(\cdot))) \leq -\varepsilon_0$ for all $t \geq t_0$, $q \in Q$, $(x, r) \in \mathcal{C}$, $r \in S$ and $w(\cdot) \in W$. Define $\hat{V}$ on $\hat{\mathcal{C}} = \mathcal{C} \cap (\mathbb{R}^n \times S)$ by

$$
\hat{V}(x, r) = \max \{h_q(\phi(t, x, r, w(\cdot)), r): q \in Q, t = 0, \ldots, t_0 - 1, w(\cdot) \in W\}
$$

and $K$ by (10)–(15). Then Theorems 1–4 remain valid.

**Proof.** It is only necessary to show that (A2)–(A6) are satisfied. Define $H(x, r, \psi) = \max \{h_q(\phi(t, x, r, w(\cdot)), r): q \in Q, t = 0, \ldots, t_0 - 1, \psi \in \mathcal{C}\}$. Then $\hat{V}(x, r) = \max \{H(x, r, \psi): \psi \in \mathcal{C}\}$. From (1)–(iv) it follows that $\mathcal{C}$ and $\mathcal{H}$ are compact and $H: \mathcal{C} \times \mathcal{C} \rightarrow \mathbb{R}$ is continuous. Thus, the maximum of $H$ exists and $\hat{V}$ is continuous on $\mathcal{C}$. Therefore (A2) holds on $\hat{\mathcal{C}}$, a restriction of (A2) that does not affect Theorems 1–4 and their proofs. Let $\mathcal{C} = \mathcal{C}$ and $w(\cdot)$ be any element of $W$. Then by the time invariance of (1) and the stationarity of $W^c$ it follows that $\hat{V}(r) = \phi(t_0, x, r, w(\cdot))$ belongs to $\Pi_{\hat{V}}$. Hence, (A3) is satisfied. Assumption (A4) follows from (vii), and (A5) and (A6) are immediate consequences of (v) and (23). □

**Remark 6.** While $\Pi(r)$ is the maximal constraint admissible set associated with $\mathcal{C}$, it does not take into account desired stability properties of (1) with $v(t) \equiv r$. From the definition of $\mathcal{M}(r)$ in Section 2 the equality $\mathcal{M}(r) = \Pi(r)$ holds only if $\Pi(r)$ is also a region of attraction for the set $\mathcal{M}(r)$. Suppose there exist $\varepsilon_0 > 0$ and $t_0 \in \mathbb{Z}^+$ such that the following conditions are satisfied for all $r \in S$: (a) $\phi(t_0, x, r, w(\cdot)) \in \mathcal{M}(r)$ for all $x \in \mathcal{C}$, and $w(\cdot) \in W^c$, (b) $h_q(x, r) \leq -\varepsilon_0$ for all $x \in \mathcal{M}(r)$ and $q \in Q$. Then it is easy to confirm that assumption (v) in Theorem 5 holds and $\mathcal{M}(r) = \Pi(r)$.

In principle, $\hat{V}(x, r)$ can be computed by solving the finite dimensional optimization problem of maximizing $H(x, r, \psi)$ subject to $\psi \in \mathcal{C}$; in actual on-line applications, the computation of $\hat{V}(x, r)$ may be impractical because of the complexity of $H$ and the likely high dimension of $\psi$. Computation of $\hat{V}(x, r)$ is much easier if system (1) is free of disturbances and parameter uncertainties. It is only necessary to simulate system (1) with $x(0) = x$, $v(t) \equiv r$ and $w(t) \equiv 0$ and set $\hat{V}(r, v) = \max \{h_q(x(t), r): q \in Q, t = 0, \ldots, t_0 - 1\}$. For the continuous-time version of (23), our approach yields results that include, in a somewhat different way, the main results of Bemporad (1998). However, there are significant generalizations. For example: disturbances and parametric uncertainties are considered, rate-limits can be applied to the reference governor output, Algorithm K determines in an efficient and theoretically precise way an approximate evaluation of $K(x, v, r)$.

The computations required by (23) may be too expensive for on-line implementation of the reference governor. In such situations approximations of $\mathcal{M}(r)$ play a crucial role. Empirical procedures are a promising approach to the construction of the required functions $\hat{V}(x, r)$. They exploit data generated by simulating or optimizing (1) or (20) over large families of pairs $(x, r) = (x(t_0), r) \in \mathcal{C}$. For example, each pair $(x^i, r^i)$ is either constraint admissible $(i \in I^+)$ or not $(i \in I^-)$. Determination of $\hat{V}(x, r)$ then becomes a classification problem. Admittedly, the overall computational effort may be very large, but the computations are off-line and it is possible to take advantage of cheap computer time. We have had some success in using methods of machine learning (Christianini & Shawe-Taylor, 2000; Schölkopf & Smola, 2002) and further investigations are currently in progress.

5. Example

We apply the preceding ideas to the second-order electromagnetically actuated mass-spring damper system (EAMSD) considered in Hong, Cummings, Wasabaugh, and Bernstein (1997), Miller et al. (2000). The open-loop system is described by the equations:

$$
\frac{d\hat{x}_1}{dt} = \hat{x}_2,
$$

$$
\frac{d\hat{x}_2}{dt} = -\frac{k}{m} \hat{x}_1 - \frac{c}{m} \hat{x}_2 + \frac{z}{m} \frac{u}{(d_0 - d_1)}, \quad u = i^B,
$$

where $\hat{x}_1$ is the position, $\hat{x}_2$ is the velocity, $u = i^B$ represents the control input and $i$ denotes the current applied to the coil. The parameter values are: $\alpha = 4.5 \times 10^{-2}$, $\beta = 1.92$, $\gamma = 1.99$, $c = 0.6590, k = 38.94, d_0 = 0.0102$, $m = 1.54$. Despite its apparent simplicity, system (24) is a basic model for many electromagnetic actuation devices such as solenoids, magnetic suspensions, injectors, electromagnetic valves. Depending on different values of $u$, the mass has different equilibria, $\hat{x}_1 = d, \hat{x}_2 = 0$. For $d > 0.0034$, it can be shown that they are not stable (Miller et al., 2000). Thus, a stabilizing feedback law is necessary.
Practical considerations impose three constraints on $\tilde{x}$ and $w$: $\tilde{x}_1 \leq 0.008$, $0 \leq u$, $u \leq 0.3$. The first constraint prevents collision of the mass and of the electromagnet while the remaining constraints are current limits imposed by the power electronics.

Neglecting the constraints it is possible to obtain an effective stabilizing feedback law by nonlinear inversion:

$$u = \frac{1}{2}(d_0 - \tilde{x}_1)^2(k\tilde{v} - c_d\tilde{x}_2) = g(\tilde{x}, \tilde{v}).$$

(25)

This yields a closed-loop system of the form (20):

$$\frac{d\tilde{x}_1}{dt} = \tilde{x}_2$$

$$\frac{d\tilde{x}_2}{dt} = -\frac{k}{m} \tilde{x}_1 - \frac{c + c_d}{m} \tilde{x}_2 + \frac{k}{m} \tilde{v},$$

(26)

where $c_d = 4.0$. Since there is no disturbance, $w(t) \equiv 0$.

System (26) has three constraints on $\tilde{x}$ and $\tilde{v}$. In the notation of Theorem 5: $h_1(x, r) = x_1 - 0.008$, $h_2(x, r) = -kr + cgx_2$, $h_3(x, r) = g(x, r) - 0.3$. These constraints are tighter and more difficult to handle than those considered in Miller et al. (2000); the upper limit on $u$ is smaller and $c_d \neq 0$ significantly complicates $h_2$ and $h_3$.

Constraint-admissible stable equilibria, $\tilde{x}_c(r) = (r, 0)$, exist in the interior of $\mathcal{C}_c$ for $\tilde{v}(t) \equiv r \in S = [0.0005, 0.0075]$. In all of what follows the reference governor is the sampled-data system described in Remark 5.

We consider three distinctly different means for defining $\tilde{V}(x, r)$: the Lyapunov-function approach described in Gilbert and Kolmanovsky (1999a), Miller et al. (2000); on-line simulation of (26) using the continuous-time version of (23); empirical approximation of $\mathcal{M}(r)$ based on data generated by off-line simulation of (26). Fig. 4 indicates the general situation for $x = \tilde{x}(0)$ and $\tilde{v}(t) \equiv r = 0.007$. The boundaries of $\mathcal{C}_c$, defined by $h_3(x, r) = 0$, $q = 1, 3$, are designated by dashed lines. The boundary defined by $q = 2$ is outside the area shown in the figure. The circles designate safe initial conditions ($\tilde{x}(\tau) \in \mathcal{C}_c$ for all $\tau \in R^+$) while the crosses designate unsafe initial conditions. In the absence of a characterization of $\mathcal{M}(r)$ the classification of such points is determined by multiple simulations of (20) with $\tilde{v}(\tau) \equiv r$ and $x(0) = x^*$.

For each $r \in S$ the energy of the mass-spring system, $V(\tilde{x}, r) = (k/2)(\tilde{x}_1 - r)^2 + (m/2)\tilde{x}_2^2$, is a good choice for the Lyapunov function. From it $\tilde{V}(\tilde{x}, r) = \tilde{V}(\tilde{x}, r) - V(\tilde{r}, r)$, where $\gamma(r) > 0$, $r \in S$, is a continuous function selected so that the positively invariant ellipsoidal sets $\Pi(r) = \Pi^0(r)$ belong to $\mathcal{C}_c$. See the ellipse in Fig. 4. Since $\tilde{x}_c(r) \in \mathcal{C}_c$ it is clear that selections for $\gamma(r)$ exist. In fact, $\gamma(r)$ should be selected so that it maximizes or nearly maximizes the size of $\Pi^0(r)$ subject to $\Pi^0(r) \subset \mathcal{C}_c$. To simplify the evaluation of $\tilde{V}(\tilde{x}, r)$ it is expressed approximately by the linear interpolation in $r$ using values at $r_i$, where $10^5r_i = 0.5, 1, 2, 3, 4, 5, 6, 7, 7.5$. Although the interpolation satisfies (A1)–(A4) there is no guarantee that it satisfies (A5) and (A6) when $r \neq r_i$. To make it likely that (A5) and (A6) do hold, the $\gamma(r)$ are chosen so that the inclusions $\Pi^0(r) \subset \mathcal{C}_c$ are satisfied conservatively. While the sets $\Pi^0(r)$ are positively invariant, this may not be true for $\Pi^0(r)$ when $r \neq r_i$.

The on-line simulation approach yields a function $\tilde{V}(x, r) = \tilde{V}(x, r)$ such that $\Pi(r) = \Pi^0(r) \subset \mathcal{M}(r)$. The algorithm for evaluating $\tilde{V}(x, r)$ has the following form: Step 1: Solve (26) for $\tilde{v}(\tau) \equiv r$ and $\tilde{x}(0) = x$. Step 2: Set $\tilde{V}_S(x, r) = \max\{h_3(\tilde{x}(\tau), r): q = 1, 2, 3, \tau \in [0, \tau_0]\}$. Here, $\tau_0$ is large enough to guarantee that $h_3(\tilde{x}(\tau), r) \leq -\varepsilon$ for all $q = 1, 2, 3$ and $\tau \geq \tau_0$. Numerically, the evaluation of $\tilde{V}(x, r)$ is not exact because $\tilde{x}(\tau)$ is not restricted to a grid of points on $[0, \tau_0]$. To keep the resulting error small the grid must be reasonably fine (we used 201 grid points in $[0, 2]$). Within the small error $\Pi^0(r)$ is positively invariant for all $r \in S$.

Guided by the set of safe test points, our empirical approach covers a large portion of $\mathcal{M}(r_i)$ by a union, $\Pi^\infty(r_i)$, of three rectangular polytopes, $\Pi^\infty(r_i), j = 1, 2, 3$, each containing $\tilde{x}_c(r_i)$ in its interior. See Fig. 4. Each rectangle is generated by a function $\tilde{V}(x, r) = \tilde{V}^\infty(x, r)$ of three rectangular polytopes, $\Pi^\infty(r_i), j = 1, 2, 3, 4$, each containing $\tilde{x}_c(r_i)$ in its interior. See Fig. 4. Each rectangle is generated by a function $\tilde{V}^\infty(x, r) = \max\{x: \nu_{j,k} \cdot (x - c_{j,k}), k = 1, 2, 3, 4\},$ where $c_{j,k}$ is the interior point of $\Pi^\infty(r_i)$. It is easy to confirm that for $r = r_i \in S$, the function $\tilde{V}(x, r)$ satisfies (A2)–(A6). Finally, applying the results of Section 4, we generate $\tilde{V}(x, r) = \tilde{V}^\infty(x, r)$ by linearly interpolating the $\tilde{V}(x, r)$: $\min\{\nu_j(x, r_i): j = 1, 2, 3\}$ on the same set of $r_i$ used for $\tilde{V}^\infty(x, r_i)$. By choosing the rectangles to satisfy $\Pi^\infty(r_i) \subset \mathcal{M}(r_i)$ conservatively, $\tilde{V}_S(x, r)$ satisfies (A1)–(A6). For all $r \in S$, $\Pi^\infty(r)$ is not positively invariant.

The reference governors defined by the preceding $\tilde{V}^\infty, \tilde{V}^\infty$, $\tilde{V}^\infty$ were simulated with $T = 0.1$, $\varepsilon = 10^{-1}$ and $\delta = 5 \times 10^{-5}$. The resulting position responses to $\tilde{v}(\tau) \equiv 0.0074$ and $\tilde{x}(0) = [0, 0.012]^T$ are shown in Fig. 5. The value of $v(0)$ was
determined by Eq. (5) with $t(−1) = 0.002$ and $κ(0)$ determined by (9)–(15). The position constraint is severely violated in absence of the reference governor, but is enforced by all reference governor designs. Control constraints are also enforced. See Fig. 6. The position response of the three reference governors differs in speed, with the one based on $\hat{V}^s(r)$ being the fastest. The on-line computational times for it are, however, larger by a factor of about 30 than for the ones based on $\hat{V}^e(r)$ and $\hat{V}^{rec}(r)$. The position responses of the reference governors based on $\hat{V}^{rec}$ and $\hat{V}^e$ are similar, with the reference governor based on $\hat{V}^{rec}$ noticeably faster. By replacing the rectangles with nonrectangular polytopes the approximation of the $\mathcal{H}(r_2)$ would be better, giving a response closer to the one for $\hat{V}^s$ with little increase in computational complexity. As expected from Theorem 2, the reference governors have finite settling time, with $\bar{z}(\tau)$ reaching 0.0074 at $\tau = 0.6, 0.7, 0.8$, respectively for $\hat{V}^{rec}$, $\hat{V}^{rec}_s$, $\hat{V}^e$.

6. Concluding remarks

We have introduced a new approach for the design of reference governors. The required assumptions are significantly weaker than those in prior treatments of non-linear closed-loop systems (Bemporad, 1998; Gilbert & Kolmanovsky, 1999a; Miller et al., 2000): the sets $Π(r)$ do not need to be positively invariant; disturbance inputs and parametric uncertainties are allowed; it is possible to impose rate-limits on $v(t)$ and $w(t)$; the main results still hold when $K(x,v,r)$ is evaluated approximately by a finite-step algorithm. The resulting reference governors maintain the advantages of past reference governors.

With the exception of Gilbert and Kolmanovsky (1999b), which considers only linear systems, former papers require positive invariance of sets corresponding to our $Π(r)$. Eliminating the assumption of positive invariance creates many opportunities for constructing $\hat{V}(x,r)$ so that $Π(r)$ closely approximates $\mathcal{H}(r)$ and on-line computational costs are manageable. In this respect, the empirical approaches described in Section 5 appear very promising, particularly when they are implemented automatically by machine learning techniques. While parametric uncertainties and disturbance inputs increase greatly the computational cost of determining $\hat{V}(x,r)$, on-line costs associated with the evaluation of $\hat{V}(x,r)$ should not be significantly affected. Also, there is no reason to limit the application of the reference governor to situations where the asymptotic behavior of $x(t)$ for each value of $r$ is a single equilibrium point or a disturbance generated neighborhood of a single equilibrium point. For instance, the desired asymptotic behavior may be an $r$-parameterized family of stable limit cycles; it is only necessary to arrange $\hat{V}(x,r)$ so that the sets $Π(r)$ form appropriate tubes containing the limit cycles.

Comments similar to those of the preceding paragraphs also apply to constrained regulator systems (Burridge, Rizzi, & Koditschek, 1999; Kolmanovsky & Gilbert, 1998a; Kolmanovsky et al., 1998b; Kolmanovsky & Gilbert, 1996; McConley et al., 2000) that expand domains of attraction by chaining together constrained positively invariant sets, each set corresponding to one of several set points. In much the same way and with most of the same advantages, the requirement of constrained positive invariance can be replaced by conditions similar to our assumptions (A5) and (A6).

References


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