Stability of Flocking Motion

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Abstract

This paper investigates the aggregated stability properties of of a system of multiple mobile agents described by simple dynamical systems. The agents are steered through local coordinating control laws that arise as a combination of attractive/repulsive and alignment forces. These forces ensure collision avoidance and cohesion of the group and result to all agents attaining a common heading angle, exhibiting flocking motion. Two cases are considered: in the first, position information from all group members is available to each agent; in the second, each agent has access to position information of only the agents laying inside its neighborhood. It is then shown that regardless of any arbitrary changes in the neighbor set, the flocking motion remains stable as long as the graph that describes the neighboring relations among the agents in the group is always connected.

1 Introduction

Over the past decade a considerable amount of attention has been focused on the problem of coordinated motion of multiple autonomous agents. From biological sciences to systems and controls, and from statistical physics to computer graphics, researchers have been trying to develop an understanding of how a group of moving objects such as flocks of birds, schools of fish, crowds of people [35, 20], or man-made mobile autonomous agents, can move in a formation *only* using local interactions and without a global supervisor.

Such problems have been studied in ecology and theoretical biology, in the context of animal aggregation and social cohesion in animal groups [1, 23, 37, 13, 9]. A computer model mimicking animal aggregations was generated in [27]. Following the work in [27] several other computer models have appeared in the literature (cf. [14] and the references therein), and has led to creation of a new area in computer graphics known as *artificial life* [27, 32].

At the same time, several researchers in the area of statistical physics and complexity theory have addressed flocking and schooling behavior in the context of non-equilibrium phenomena in many-degree-of-freedom dynamical systems and self organization in systems of self-propelled particles [36, 34, 33, 21, 18, 30, 6, 16].

Similar problems have become a major thrust in systems and control theory, in the context of cooperative control, distributed control of multiple vehicles and formation control; see for example [17, 24, 25, 7, 19, 10, 31, 15, 22].

The main goal of the above papers is to develop a distributed control strategy for each individual vehicle or agent, such that the global objective, such as a tight formation with fixed pair-wise inter vehicle distances is achieved.

In 1986 Craig Reynolds [27] made a computer model of coordinated animal motion such as bird flocks and fish schools. It was based on three dimensional computational geometry of the sort normally used in computer animation or computer aided design. He called the generic simulated flocking creatures “boids”. The basic flocking model consists of three simple steering behaviors which describe how an individual boid maneuvers based on the positions and velocities its nearby flockmates:

- **Separation:** steer to avoid crowding local flockmates.
- **Alignment:** steer towards the average heading of local flockmates.

- **Cohesion:** steer to move toward the average position of local flockmates.

Each boid has direct access to the whole scene's geometric description, but flocking requires that it reacts only to flockmates within a certain small neighborhood around itself. The neighborhood is characterized by a distance (measured from the center of the boid) and an angle, measured from the boid's direction of flight. Flockmates outside this local neighborhood are ignored. The neighborhood could be considered a model of limited perception (as by fish in murky water) but it is probably more correct to think of it as defining the region in which flockmates influence a boids steering. The super position of these three rules results in all boids moving in a formation, while avoiding collision.

Several variation on this model were also proposed. An example of such generalizations is a leader follower strategy, in which one agent acted as a group leader and the other boids would just follow the same rules as before. Simulations indicate that in this case, all agents tend to follow the leader.

In 1995, a similar model was proposed by Vicsek *et al.* [36]. Vicsek's model, though developed independently, turns out to be a special case of the latter model, in which all agents move with the same speed (no dynamics), and only follow an alignment rule. In this scenario, each agents heading is updated as the average of the headings of agent itself with its nearest neighbors plus some additive noise. Numerical simulations in [36] indicate the spontaneous development of coherent collective motion, resulting in the headings of all agents to converge to a common value. This was quite a surprising result in the physics community and was followed by a series of papers [5, 34, 33, 28, 21]. A proof of convergence for Vicsek's model (in the noise-free case) was given in [15].

The goal of this paper is to develop a mathematical model for the flocking phenomenon in [27], and provide a system theoretic justification by combining results from classical control theory, mechanics, algebraic graph theory, nonsmooth analysis and Lyapunov stability for nonsmooth systems . We will show that the cohesion and separation rules can be *decoupled* from alignment. It will be shown that under assumptions on connectivity of the graph representing the nearest neighbor relation, all agents headings converge to the same value, and all velocities will eventually become the same. Moreover, all pairwise distances converge, resulting in a flocking behavior.

In [15], similar nearest neighbor alignment control laws have been proposed and the stability of the resulting flocking motion was analyzed. The models used in [15] to describe the agent dynamics were kinematic, whereas collision avoidance was not an issue. Further, in [15], it was also shown that stability can be recovered even when connectivity in the network of agent is temporarily lost. While the proof techniques are totally different from those in [15], the end result is similar, suggesting that addition of cohesion and separation forces in addition to alignment as well as addition of dynamics, does not affect the stability of the flocking motion. In the end, all that is needed for flocking to happen is connectivity in the network of agents, although the analysis in this paper requires connectivity to be maintained at all times.

This paper is organized as follows: in Section 2 we describe the system, the problem that we are addressing and we highlight the approach that we are going to follow. Section 3 introduces some basic facts in algebraic graph theory that are used in the stability analysis. The first case, in which each agent uses state information from all other agents in the group, is treated in Section 4. Then, Section 5 presents some important results from Lyapunov stability theory for nonsmooth systems. The case where only local information is used in the control law of each agent is addressed in Section 6. Section 7 follows with numerical simulations, verifying our stability results and Section 8 summarizes and points out our main contributions.

2 Problem Formulation and Overview

Consider N agents, moving on the plane (generalization to three dimensions is straightforward) with the following dynamics:

$$\dot{r}_i = v_i \tag{1a}$$

$$\dot{v}_i = a_i \quad i = 1, \dots, N\,, \tag{1b}$$

where $r_i = (x_i, y_i)^T$ is the position of agent i, $v_i = (\dot{x}_i, \dot{y}_i)^T$ is its velocity and $a_i = (u_x, u_y)^T$ its control (acceleration) inputs. Define the orientation angle (heading) of agent i, θ_i, as:

$$\theta_i = \arctan2(\dot{y}_i, \dot{x}_i)\,. \tag{2}$$

The relative positions between the agents are denoted $r_{ij} = r_i - r_j$.

Each agent is steered via its acceleration input with the objective being to achieve group coordination through local, decentralized control action. The acceleration input consists of two components (Figure 1):

$$a_i = a_{r_i} + a_{\theta_i} \; . \qquad (3)$$

The first component, a_{r_i}, is the negated vector field of an artificial potential function, V_i that encodes the relative position of agent i with respect to its neighbors and is designed to have a minimum at a desired configuration or relative distances between them and to increase when agent i comes very close to its neighbors or moves too far away from them. Term a_{r_i}, therefore, is thought of ensuring collision avoidance and cohesion in the group. The second component, a_{θ_i} is assumed that it regulates the heading of agent i to the weighted average of that of its neighbors by rotating its velocity vector. Acting in a direction normal to the agent velocity, these control forces do not contribute to the kinetic energy of the system and can be regarded as *gyroscopic* [2].

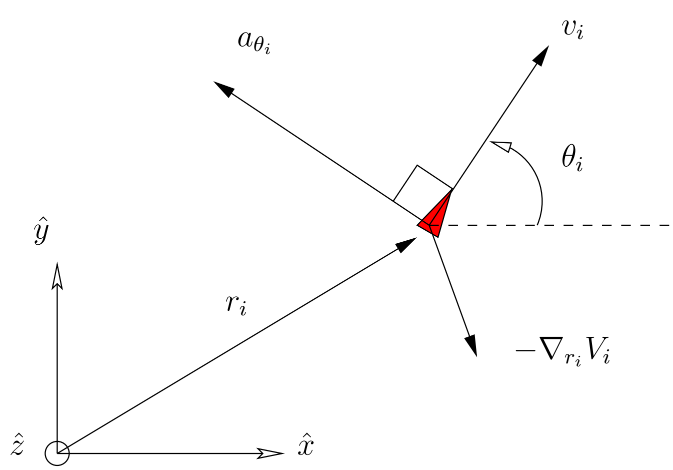

Figure 1: Control forces acting on agent i.

The objective is to determine the control input components so that the group exhibits a stable, collision free flocking motion. This is being understood technically as a convergence property on the heading differences and the relative distances of the agents in the group.

In the following Sections it will be shown that for a very generic class of artificial potential functions, stable flocking motion can be established using local control schemes. Stability can still be established even when the set of neighboring agents providing feedback information to a particular agent controller, changes arbitrarily during motion. The only requirement for stable motion turns out to be the connectivity of the graph representing the neighboring relations in the group.

3 Graph Theory Preliminaries

In order to make more precise statements about the enabling conditions for stability we will use some graph theory terminology which we briefly introduce in this Section. The interested reader is referred to [11].

An (undirected) graph $\mathcal{G}$ consists of a vertex set, $\mathcal{V}$, and an edge set $\mathcal{E}$, where an edge is an unordered pair of distinct vertices in $\mathcal{G}$. If $x, y \in \mathcal{V}$, and $(x, y) \in \mathcal{E}$, then x and y are said to be adjacent, or neighbors and we denote this by writing $x \sim y$. A graph is called complete if any two vertices are neighbors. The number of neighbors of each vertex is its valency. A path of length r from vertex x to vertex y is a sequence of $r+1$ distinct vertices starting with x and ending with y such that consecutive vertices are adjacent. If there is a path between any two vertices of a graph $\mathcal{G}$, then $\mathcal{G}$ is said to be connected.

The adjacency matrix $\mathcal{A}(\mathcal{G}) = [a_{ij}]$ of an (undirected) graph $\mathcal{G}$ is a symmetric matrix with rows and columns indexed by the vertices of $\mathcal{G}$, such that $a_{ij} = 1$ if vertex i and vertex j are neighbors and $a_{ij} = 0$, otherwise. The valency matrix $\Delta(\mathcal{G})$ of a graph $\mathcal{G}$ is a diagonal matrix with rows and columns indexed by $\mathcal{V}$, in which the (i, i)-entry is the valency of vertex i. An orientation of a graph $\mathcal{G}$ is the assignment of a direction to each edge, meaning that in each edge one vertex is named as head and the other as tail. We denote by $\mathcal{G}^\sigma$ the graph $\mathcal{G}$ with orientation σ. The incidence matrix $D(\mathcal{G}^\sigma)$ of an oriented graph $\mathcal{G}^\sigma$ is the matrix whose rows and columns are indexed by the vertices and edges of $\mathcal{G}$ respectively, such that the i, j entry of $D(\mathcal{G})$ is equal to 1 if vertex i is the head of edge j, -1 if i is the tail of j, and 0 otherwise.

The symmetric matrix defined as:

$$L(\mathcal{G}) = \Delta(\mathcal{G}) - A(\mathcal{G}) = D(\mathcal{G}^\sigma) D(\mathcal{G}^\sigma)^T$$

is called the Laplacian of $\mathcal{G}$ and is independent of the choice of orientation σ. It is known that the Laplacian matrix captures many topological properties of the graph. Among those, it is the fact that L is always positive semidefinite and the algebraic multiplicity of its zero eigenvalue is equal to the number of connected components in the graph. The eigenvector associated with the zero eigenvalue is the vector of ones, $\mathbf{1}$. If we associate each edge with a positive number and form the diagonal matrix W with rows and columns indexed by the edges of $\mathcal{G}$, then the matrix

$$L_w(\mathcal{G}) = D(\mathcal{G}^\sigma) W D(\mathcal{G}^\sigma)$$

is a weighted Laplacian of $\mathcal{G}$. The weighted Laplacian also enjoys the above properties.

In what follows, we will use graph theoretic terminology to represent the control interconnections between the agents in the group. The connectivity properties of the resulting graph will prove crucial for establishing the stability of the flocking group motion.

4 Global Agent Interaction

By "global interaction" we mean the dependence of the control law of any agent on the state of every other agent in the group. If this interaction is being understood as a neighboring relation, each agent neighbors any other agent in the group By driving the orientation of each agent to the (weighted) average of the headings of all the group members we can achieve flocking behavior. Collision avoidance and group cohesion can be established using an artificial potential field that is solely dependent on the relative distances between the agents. In steady state, it is shown that the agents are going to attain a common orientation and configure themselves in positions corresponding to a point of minimum potential energy.

The graph $\mathcal{G}$, that captures the neighboring relations in the global interaction case is the complete graph with N vertices, $\mathcal{G} = K_N$:

Definition 4.1 (Neighboring graph). *The neighboring graph* $\mathcal{G} = \{\mathcal{V}, \mathcal{E}, \mathcal{W}\}$ *is a labeled graph consisted of*

- *a set of vertices,* $\mathcal{V}$*, indexed by the agents in the group,*
- *a set of edges,* $\mathcal{E} = \{(v_i, v_j) \mid v_i \sim v_j \text{ for } v_i, v_j \in \mathcal{V}\}$*, containing unordered pairs of vertices, and*
- *a set of labels,* $\mathcal{W}$*, indexed by the edges, and a map associating each edge with a label* $w \in \mathcal{W}$*, equal to the inverse of the squared distance between the agents that correspond to the vertices adjacent to that edge.*

For each agent i define an artificial potential function V_i that depends on the distance between i and all the other agents:

$$V_i \triangleq \sum_{\substack{j=1 \\ j \neq i}}^{N} V_{ij}(\|r_{ij}\|).$$

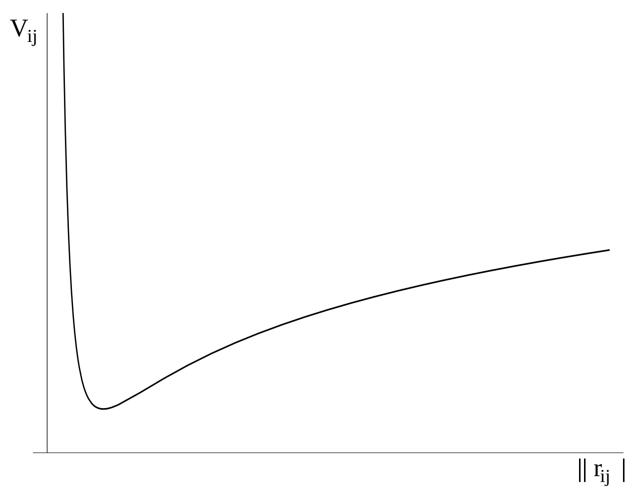

Figure 2: The artificial potential function between two agents under the global interaction regime.

Any potential function $V_{ij}(\|r_{ij}\|)$ can be used, provided that it is symmetric with respect to r_{ij} and r_{ji} [1]. One possible choice could be:

$$V_{ij}(\|r_{ij}\|) = \frac{1}{\|r_{ij}\|^2} + \log \|r_{ij}\|^2 ,$$

giving rise to a potential function that is depicted in Figure 2. The control law a_i is then:

$$a_i = -\nabla_{r_i} V_i + a_{\theta_i}, \tag{4}$$

[1] This will guarantee that the potential field forces generated by V_{ij} satisfy the strong law of action and reaction: internal forces in a system of particles, in addition to being equal and opposite, also lie along the line joining the particles [12].

where a_{θ_i} is given as:

$$a_{\theta_i} = -\sum_{\substack{j=1\\j\neq i}}^{N} \frac{\theta_i - \theta_j}{\|r_{ij}\|^2}(\hat{z} \times v_i). \tag{5}$$

From the cross product $\hat{z} \times v_i$ it is evident that a_{θ_i} acts along a direction normal to the velocity of agent i. From preservation of angular momentum for the system of agents we have:

$$\begin{aligned}
\sum_{i=1}^{N} r_i \times \dot{v}_i &= \sum_{i=1}^{N} r_i \times (-\nabla V_i + a_{\theta_i}) \\
&= \sum_{i=1}^{N} r_i \times a_{\theta_i} - \sum_{i=1}^{N} \sum_{\substack{j=1\\j\neq i}}^{N} r_{ij} \times \nabla V_{ij} \\
&= -\sum_{i=1}^{N} r_i \times \left(\sum_{\substack{j=1\\j\neq i}}^{N} \frac{\theta_i - \theta_j}{\|r_{ij}\|^2}(\hat{z} \times v_i) \right)
\end{aligned} \tag{6}$$

since r_{ij} and ∇V_{ij} are collinear. Equation (6) means that from the angular momentum point of view, the system of interacting agents is equivalent to the one where only the alignment forces are exerted. This fact is to be expected since it is known that "internal forces" (like the potential field forces) in a system of particles do not contribute to the angular momentum of the system. Since these forces are exerted in a direction normal to the agents velocities, we can rewrite (6) using the acceleration component associated with them

$$
\begin{aligned}
&\sum_{i=1}^{N} r_i \times (\omega_i \times v_i) = -\sum_{i=1}^{N} r_i \times \left(\sum_{\substack{j=1 \\ j\neq i}}^{N} \frac{\theta_i - \theta_j}{\|r_{ij}\|^2} (\hat{z} \times v_i) \right) \Leftrightarrow \\
&\sum_{i=1}^{N} \big(\omega_i (r_i \cdot v_i) - v_i (r_i \cdot \omega_i)\big) = \\
&\qquad -\sum_{i=1}^{N} \sum_{\substack{j=1 \\ j\neq i}}^{N} \frac{\theta_i - \theta_j}{\|r_{ij}\|^2} \big(\hat{z}(r_i \cdot v_i) - v_i (r_i \cdot \hat{z})\big) \Leftrightarrow \\
&\sum_{i=1}^{N} \omega_i (r_i \cdot v_i) = -\sum_{i=1}^{N} \sum_{\substack{j=1 \\ j\neq i}}^{N} \frac{\theta_i - \theta_j}{\|r_{ij}\|^2} \hat{z} (r_i \cdot v_i) \Rightarrow \\
&\sum_{i=1}^{N} (r_i \cdot v_i)(\dot{\theta}_i + \sum_{\substack{j=1 \\ j\neq i}}^{N} \frac{\theta_i - \theta_j}{\|r_{ij}\|^2}) = 0
\end{aligned}
$$

Viewing the last expression as a bilinear form in $r_i \cdot v_i$ and $(\dot{\theta}_i, \theta_i)$ and considering its kernel,

$$\dot{\theta}_i = -\sum_{\substack{j=1 \\ j\neq i}}^{N} \frac{\theta_i - \theta_j}{\|r_{ij}\|^2} \tag{7}$$

Consider now the following positive semi-definite function:

$$V_t = \frac{1}{2} \sum_{i=1}^{N} (V_i + v_i^T v_i + \theta_i^2) = \frac{1}{2} \sum_{i=1}^{N} (\sum_{\substack{j=1 \\ j\neq i}}^{N} V_{ij} + v_i^T v_i + \theta_i^2).$$

Due to V_i being symmetric with respect to r_{ij} and the fact that $r_{ij} = -r_{ji}$,

$$\frac{\partial V_{ij}}{\partial r_{ij}} = \frac{\partial V_{ij}}{\partial r_i} = -\frac{\partial V_{ij}}{\partial r_j}, \tag{8}$$

and therefore it follows:

$$\frac{d}{dt} \sum_{i=1}^{N} \frac{1}{2} V_i = \sum_{i=1}^{N} \nabla_{r_i} V_i \cdot v_i.$$

Let Ω be the set defined as:

$$\Omega \triangleq \{(r_{ij}, v_i, \theta_i) \mid V_t \leq c,\ i, j = 1, \dots, N\},$$

which is nonempty for a sufficiently large choice of c, and closed by continuity of V_t. It is also bounded, because boundedness of V_t implies boundedness of all V_{ij}, which in turns implies the boundedness of every r_{ij}, since V_{ij} increases monotonically with r_{ij}. Bounds for v_i follow trivially and θ_i is always bounded in $[-\pi, \pi]$.

Proposition 4.2. *The system of N boids with dynamics* (1) *steered by control laws* (4)*, with initial conditions in Ω, converges to one of the minima of V_t and a common orientation.*

Proof. Taking the time derivative of V_t, we have:

$$\begin{aligned}
\dot{V}_t &= \sum_{i=1}^{N} \nabla_{r_i} V_i \cdot v_i + \sum_{i=1}^{N} v_i^T a_i + \sum_{i=1}^{N} \theta_i \dot{\theta}_i \\
&\overset{(4)}{=} \sum_{i=1}^{N} \nabla_{r_i} V_i \cdot v_i + \sum_{i=1}^{N} v_i^T (-\nabla_{r_i} V_i^T + a_{\theta_i}) + \sum_{i=1}^{N} \theta_i \dot{\theta}_i \\
&\overset{(5)}{=} \sum_{i=1}^{N} v_i^T \Big(- \sum_{j \in \mathcal{N}_i} \frac{\theta_i - \theta_j}{\|r_{ij}\|^2} \big(\hat{z} \times v_i\big)\Big) + \sum_{i=1}^{N} \theta_i \dot{\theta}_i \\
&= \sum_{i=1}^{N} \theta_i \dot{\theta}_i. \qquad (9)
\end{aligned}$$

Expanding (7) we obtain:

$$\begin{aligned}
\dot{\theta}_i &= - \sum_{\substack{j=1 \\ j \neq i}}^{N} \frac{\theta_i}{\|r_{ij}\|^2} + \sum_{\substack{j=1 \\ j \neq i}}^{N} \frac{\theta_j}{\|r_{ij}\|^2} \\
&= - \left(\tfrac{1}{\|r_{i1}\|^2},\ \cdots,\ \tfrac{1}{\|r_{i\,i-1}\|^2},\ \textstyle\sum_{\substack{j=1 \\ j \neq i}}^{N} \tfrac{1}{\|r_{ij}\|^2},\ \tfrac{1}{\|r_{i\,i+1}\|^2},\ \cdots,\ \tfrac{1}{\|r_{iN}\|^2} \right) \begin{bmatrix} \theta_1 \\ \vdots \\ \theta_{i-1} \\ \theta_i \\ \theta_{i+1} \\ \vdots \\ \theta_N \end{bmatrix},
\end{aligned}$$

so that (9) can be written in matrix form as:

$$\begin{aligned}\dot{V}_t &= -\begin{bmatrix}\theta_1\\ \vdots\\ \theta_N\end{bmatrix}^T \begin{bmatrix}\sum_{\substack{j=1\\ j\neq 1}}^N \|r_{1j}\|^{-2} & -\|r_{12}\|^{-2} & \cdots & -\|r_{1N}\|^{-2}\\ \vdots & \vdots & & \vdots\\ -\|r_{N1}\|^{-2} & -\|r_{N2}\|^{-2} & \cdots & \sum_{\substack{j=1\\ j\neq N}}^N \|r_{Nj}\|^{-2}\end{bmatrix} \begin{bmatrix}\theta_1\\ \vdots\\ \theta_N\end{bmatrix}\\ &= -\theta^T L_w \theta,\end{aligned}$$

where L_w is the weighted Laplacian of the complete graph $\mathcal{G}$, where the weight on each edge is the inverse squared distance between the boids at the vertices adjacent to the edge. Matrix L_w is positive semidefinite and since $\mathcal{G}$ is connected, the only eigenvector associated with the zero eigenvalue is the N dimensional vector of ones, $\mathbf{1}_N$:

$$\mathbf{1}_N \triangleq (\underbrace{1,\ldots,1}_{N})^T.$$

Since $\dot{V}_t \leq 0$, Ω is positively invariant. Applying LaSalle's invariant principle on (1) in Ω we conclude that all trajectories converge to the largest invariant set in $\{(r_{ij}, v_i, \theta_i) \mid \dot{V}_t = 0,\ i,j = 1,\ldots,N\}$. Equality $\dot{V}_t = 0$ holds only at configurations where all agents have the same constant heading, $\theta_1 = \cdots = \theta_N = \bar{\theta}$. Let S_θ be the set where all orientations are the same:

$$S_\theta \triangleq \{(r_1,\theta_1,\ldots,r_N,\theta_N) \mid \theta_1 = \cdots = \theta_N = \bar{\theta}\}.$$

In this set, we have:

$$\tan\bar{\theta} = k = \frac{\dot{y}_1}{\dot{x}_1} = \cdots \frac{\dot{y}_N}{\dot{x}_N}.$$

Differentiating $\frac{\dot{y}_i}{\dot{x}_i} = k$ we get:

$$\frac{d}{dt}\left(\frac{\dot{y}_i}{\dot{x}_i}\right) = 0 \Rightarrow \frac{\ddot{y}_i}{\ddot{x}_i} = \frac{\dot{y}_i}{\dot{x}_i} = k = \frac{a_{y_i}}{a_{x_i}} = \frac{(\nabla_{r_i} V_i)_y}{(\nabla_{r_i} V_i)_x}.$$

This means that the potential force applied on i is aligned with its velocity. Now we distinguish two cases:

1. Case: $-\nabla_{r_i} V_i \cdot v_i \leq 0$. If we take the function $V_{v_i} = \frac{1}{2} v_i^T v_i$, then we see that $\dot{V}_{v_i} = v_i \dot{v}_i = -\nabla_{r_i} V_i v_i \leq 0$. The dynamics of v_i is now $\dot{v}_i = a_i = -\nabla_{r_i} V_i$, which implies that v_i will converge to the largest

invariant set in $S_{v_i} = \{(r_i, v_i) \mid \dot{V}_{v_i} = 0\}$. Since $\nabla_{r_i} V_i$ and v_i are aligned, S_{v_i} contains configurations where either $\nabla_{r_i} V_i = 0$ or $v_i = 0$. The latter configurations are not invariant, unless $\nabla_{r_i} V_i = 0$, because $\dot{v}_i = -\nabla_{r_i} V_i$. Therefore, the system will converge to configurations where $-\nabla_{r_i} V_i = 0$ which correspond to minima of the artificial potential of V_i.

2. Case: $-\nabla_{v_i} V_i \cdot v_i > 0$. Then the time derivative of the artificial potential for i will be $\dot{V}_i = \nabla_{r_i} V_i v_i < 0$, which means that V_i is monotonically decreasing, and eventually reach one of its minima.

In any case therefore, the system converges to a minimum of $\sum_i^N V_i$. □

Collision avoidance is guaranteed since in all configurations where $r_{ij} = 0$ for some $i, j \in \{1, \ldots, N\}$, the function V_t tends to infinity implying that these configurations lay in the exterior of any Ω with c bounded.

5 Stability of Nonsmooth Systems

In this section we present the mathematical tools for the stability analysis of the group in the case where the agents are controlled through local, nearest neighbor rules. Since the set of neighbors may change with time as the agents move with respect to each other, switching in the control laws introduces discontinuities in the closed loop agent dynamics. This new dynamics falls into the category of differential equations with discontinuous right hand sides [8] and stability tools [29, 26] are generally based on nonsmooth analysis [4, 3].

We begin with a definition of our notion of solutions of differential equations with discontinuous right hand sides:

Definition 5.1 ([29],[8]). *Consider the following differential equation in which the right hand side can be discontinuous:*

$$\dot{x} = f(x, t) \tag{10}$$

where $f : \mathbb{R} \times \mathbb{R} \to \mathbb{R}^n$ is measurable and essentially locally bounded. A vector function $x(\cdot)$ is called a solution of (10) *on $[t_0, t_1]$, where if $x(\cdot)$ is absolutely continuous on $[t_0, t_1]$ and for almost all $t \in [t_0, t_1]$*

$$\dot{x} = K[f](x, t)$$

where

$$K[f](x,t) \triangleq \bigcap_{\delta>0} \bigcap_{\mu M=0} \overline{co} f\big(B(x,\delta) - M, t\big)$$

and $\bigcap_{\mu M=0}$ *denotes the intersection over all sets* M *of Lebesgue measure zero,* $\overline{co}$ *denotes the closure of the convex hull and* $B(x,\delta)$ *the ball of radius* δ*, centered at* x*.*

According to this definition, a trajectory $x(t)$ is considered a solution of the discontinuous differential equation (10), if its tangent vector, wherever it can be defined, belongs in the convex closure of the limit of the vector fields defined by (10) in a decreasingly small neighborhood of the solution point. Being able to exclude a set of measure zero, is critically useful since one can thus define solutions even at points where the vector field in (10) is not properly defined. The above definition of solutions, along with the assumption that the vector field f is measurable, guarantees the uniqueness of solutions of (10) [8].

For time invariant vector fields f, in $\mathbb{R}^n$ the differential inclusion $K[f]$ can be given more simply as:

Definition 5.2 ([26]).

$$K[f](x) \triangleq \overline{co}\{\lim_{x_i \to x} f(x_i) \mid x_i \notin M_f \cup M\}$$

where $M_f \subset \mathbb{R}^n$*,* $\mu M_f = 0$ *and* $M \subset \mathbb{R}^n$*,* $\mu M = 0$*.*

A list of useful properties of $K[f]$ is provided in the Appendix.

Lyapunov stability has been extended to nonsmooth systems [29]. Establishing stability results in this framework requires working with generalized derivatives, in all cases where classical derivatives cannot be defined. The generalized derivative is defined using the supremum limit of the ratio of the function value difference to the argument difference, to account for cases where the limit in the classical definition of the derivative does not exist. In addition, this limit is calculated over points in a neighborhood of the point of interest and it thus captures the local behavior of the function. The generalized gradient, on the other hand, is generally a set of vectors, which reduces to the single classical gradient in the case where the function is differentiable. Definitions of the generalized directional derivative and the generalized gradient for nonsmooth functions are given in the Appendix.

The time (generalized) derivative of a function that is either nonsmooth or the dynamics of its arguments are governed by discontinuous differential equations, is given by this special case of the nonsmooth case of the chain rule:

Theorem 5.3 ([29]). *Let $x(\cdot)$ be a Filippov solution to $\dot{x} = f(x,t)$ on an interval containing t and $V : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ be a Lipschitz and in addition, regular function. Then $V(x(t),t)$ is absolutely continuous, $\frac{d}{dt}V(x(t),t)$ exists almost everywhere and*

$$\frac{d}{dt}V\big(x(t),t\big) \in^{\text{a.e.}} \dot{\tilde{V}}(x,t)$$

where

$$\dot{\tilde{V}}(x,t) \triangleq \bigcap_{\xi \in \partial V(x(t),t)} \xi^T \begin{pmatrix} K[f](x(t),t) \\ 1 \end{pmatrix}$$

It can easily be shown that the (global) Lipschitz continuity requirement for $V(x,t)$ can be relaxed to local. Regularity, on the other hand, is a very useful property in the context of nonsmooth analysis. In general, calculations involving generalized gradients lead to set inclusions instead of set equalities. When the arguments are regular (and in some cases also nonnegative) functions, we can replace the inclusions with equalities. Broadly speaking, regularity is related to the convexity of the set of directions in the domain of the function along which the function is non increasing. This will allow the tangent cone to coincide with the contingent tangent cone [4]. A formal definition of regularity is given in the Appendix.

In what follows, we are going to use the nonsmooth version of LaSalle's invariant principle:

Theorem 5.4 ([29]). *Let Ω be a compact set such that every Filippov solution to the autonomous system $\dot{x} = f(x)$, $x(0) = x(t_0)$ starting in Ω is unique and remains in Ω for all $t \geq t_0$. Let $V : \Omega \to \mathbb{R}$ be a time independent regular function such that $v \leq 0$ for all $v \in \dot{\tilde{V}}$ (if $\dot{\tilde{V}}$ is the empty set then this is trivially satisfied). Define $S = \{x \in \Omega \mid 0 \in \dot{\tilde{V}}\}$. Then every trajectory in Ω converges to the largest invariant set, M, in the closure of S.*

6 Local Agent Interaction

Global interaction requires that the state of each agent is available to any other agent. In view of communication power and sensor constraints, this may be a restrictive assumption. A more reasonable approach could be to allow only local interaction between the agents, through nearest neighbor rules. In this regime, an agent regulates its control action based on the state of a subset of other agents that are in within a certain d-neighborhood around its current location. In this setting, we will show that connectivity of the neighboring graph plays a crucial role in establishing stability. The connectivity property of the neighboring graph is here assumed holding at all times The issue of ensuring it by proper coordination is very important and intellectually challenging but is beyond the scope of this paper.

In the case of local interactions, the neighboring relations between each agent do not coincide. These relations are represented by the neighboring graph described in Definition 4.1, which in this case is not necessarily complete. The set of neighbors of an agent i is now denoted

$$\mathcal{N}_i \subseteq \{1, \dots, N\}$$

and is generally changing with time. To account for the lack of interaction when two agents are separated by a distance larger than d, we modify the potential function to (Figure 3):

$$U_{ij} = \begin{cases} V_{ij}(\|r_{ij}\|), & \|r_{ij}\| < d, \\ V_{ij}(d) \equiv V_d, & \|r_{ij}\| \geq d \end{cases}$$

where $N_i = |\mathcal{N}_i|$ and set

$$U_i \triangleq (N - |\mathcal{N}_i|)V_d + \sum_{j=1}^{N} U_{ij}(\|r_{ij}\|)$$

as the artificial potential of agent i. According to this definition, the potential force exerted to agent i from agent j is zero if the distance between i and j grows larger than d.

Any change in the neighboring set $\mathcal{N}_i$, generally introduces discontinuities in the control law (3). One may hope to avoid these discontinuities by smoothening out the function U_{ij} at d, but this will only partly solve the problem: the discontinuities introduced by the alignment control inputs will persist since there is no reason why a new neighbor of agent i should have the same heading with i. These discontinuities will not disappear without *artificially* gluing the new input term using a smooth transition function.

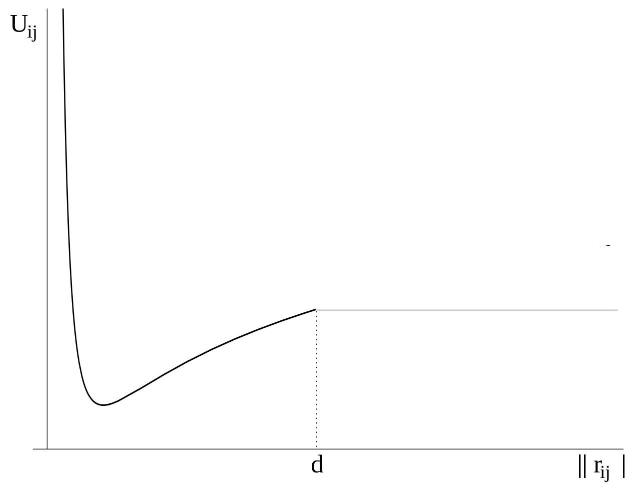

Figure 3: The nonsmooth potential function between two agents under the local interaction regime.

Consider the following nonnegative (nonsmooth) function:

$$\begin{aligned} U_t &= \frac{1}{2}\sum_{i=1}^{N}(U_i + v_i^T v_i + \theta_i^2) \\ &= (N^2 - 2|\mathcal{E}|)V_d + \frac{1}{2}\sum_{i=1}^{N}(\sum_{j\in\mathcal{N}_i} U_{ij} + v_i^T v_i + \theta_i^2) \end{aligned} \tag{11}$$

This function will serve as a potential function for the group of agents and will be used similarly to its smooth counterpart in Section 4. Its nonsmooth nature, however, and particularly the discontinuous evolution of the control inputs will require the use of nonsmooth analysis and stability tools.

6.1 Stability under Connectedness

Regardless of the switching occurring in the agents closed loop dynamics, stable flocking motion can still be established provided that the neighboring graph remains connected at all times. Moreover, it is also the case that the agent trajectories are going to converge to some of the minima of the artificial group potential function, U_t. These facts are going to be established in this Section by employing nonsmooth analysis and resorting to the nonsmooth versions of the well known Lyapunov stability theorems. Our analysis also implies that at steady state switching eventually has to stop. In this discussion we do not examine how we can enforce connectedness on the neighboring graph, but instead we consider it given:

Assumption 6.1. *The neighboring graph $\mathcal{G}$ remains connected.*

Since U_{ij} is continuous at d, then it is locally Lipschitz. Then, it can be shown that U_{ij} is also regular [4]:

Lemma 6.2. *The function U_{ij} is regular everywhere in its domain.*

Proof. To show regularity at of U_{ij} at d, we need to establish the equality between the generalized directional derivative and the classical one-sided directional derivative for U_{ij} at d, i.e. $U^{\circ}_{ij}(d;w) = U'_{ij}(d;w)$, for an arbitrary direction w. To simplify the notation in the proof we will drop the subscripts.

For the classical directional derivative we have:

$$U'(d;w) = \lim_{t\downarrow 0} \frac{U(d+tw) - U(d)}{t}.$$

1. If $w \geq 0$ then,

$$U'(d;w) = \lim_{t\downarrow 0} \frac{U(d+tw) - V_d}{t} = \lim_{t\downarrow 0} \frac{V_d - V_d}{t} = 0.$$

2. If $w < 0$ then

$$U'(d;w) = \lim_{t\downarrow 0} \frac{U(d+tw) - V_d}{t} = \lim_{t\downarrow 0} \frac{V(d+tw) - V_d}{t} \equiv c < 0$$

   where c is used to denote the directional derivative of the original artificial potential V_{ij} at d, in a negative direction ($w < 0$).

For the generalized directional derivative, we distinguish the same two cases:

1. If $w \geq 0$, then

$$
\begin{aligned}
U^\circ(d;w) &= \limsup_{\substack{y\to d\\ t\downarrow 0}} \frac{U(y+tw)-U(y)}{t} = \limsup_{\substack{y'\to d\\ t\downarrow 0}} \frac{U(y')-U(y'-tw)}{t} \\
&\leq \limsup_{\substack{y'\to d\\ t\downarrow 0}} \frac{V_d - V\big(y'+t(-w)\big)}{t} \\
&= \limsup_{\substack{y'\to d\\ t\downarrow 0}} \frac{-\big(U\big(y'+t(-w)\big)-V_d\big)}{t} \\
&= -\liminf_{\substack{y'\to d\\ t\downarrow 0}} \frac{U\big(y'+t(-w)\big)-V_d}{t} \\
&= -\lim_{t\downarrow 0} \frac{V_d - V_d}{t} = 0 = U'(d;w).
\end{aligned}
$$

2. If $w < 0$, then,

$$
\begin{aligned}
U^\circ(d;w) &= \limsup_{\substack{y\to d\\ t\downarrow 0}} \frac{U(y+tw)-U(y)}{t} \\
&= \limsup_{\substack{y\to d\\ t\downarrow 0}} \frac{V(y+tw)-U(y)}{t} \\
&= \lim_{t\downarrow 0} \frac{V(d+tw)-V_d}{t} = U'(d;w) \equiv c.
\end{aligned}
$$

Therefore, each U_{ij} is regular at d. The generalized gradient of U_{ij} at d, namely $\partial U_{ij}(d)$, will be determined by the expression:

$$
U_{ij}^\circ(d;w) \triangleq \max\{\langle \zeta, w\rangle \mid \zeta \in \partial U_{ij}(d)\}.
$$

Depending on the sign of w we distinguish the two cases:

1. $w \geq 0$: then, $0 \geq \zeta w \Rightarrow \zeta \leq 0$.

2. $w < 0$: then, $\zeta w \leq c < 0 \Rightarrow \zeta \geq \frac{c}{w} > 0$.

□

Regularity of each potential function U_{ij} is required to ensure the regularity of U_i, as a linear combination of a finite number of regular functions [4]. The latter is a necessary condition for all nonsmooth stability theorems.

From the proof of the preceding Lemma it follows that

Corollary 6.3. *The generalized gradient of U_{ij} at d is empty:*

$$\partial U_{ij}(d) = \emptyset. \tag{12}$$

This result will come handy during the stability analysis.

Since U_t is continuous, the level set defined as:

$$\Omega \triangleq \{(r_{ij}, v_i, \theta_i) \mid U_t \leq c,\ i = 1, \ldots, N,\ j \in \mathcal{N}_i\}$$

will be nonempty for sufficiently large c and in addition, closed. Boundedness in this case follows from connectivity: since the maximal length of a path in $\mathcal{G}$ is $N-1$, it follows that the maximal distance between two agents in a connected group component will be $(N-1)d$, which provides a bound for r_{ij}.

In Ω, we have similarly for U_t:

Corollary 6.4. *Function U_t is regular everywhere in Ω.*

The restriction of U_t in Ω ensures, besides collision avoidance, the differentiability of $\|r_i - r_j\|$. The control inputs for agent i are now defined with reference only to the neighbors of i:

$$a_i = -\sum_{j \in \mathcal{N}_i} \nabla_{r_i} U_{ij} + a_{\theta_i} = -\sum_{j \in \mathcal{N}_i} \nabla_{r_i} U_{ij} - \sum_{j \in \mathcal{N}_i} \frac{\theta_i - \theta_j}{\|r_{ij}\|^2} (\hat{z} \times v_i) \tag{13}$$

Using an argument similar to (8), since for uniform d i.e. common for all agents, if agent i is within the d-neighborhood of agent j so is j with respect to i. Applying Theorem A.3 for U_t as a function of r_i, v_i, θ_i for $i = 1, \ldots, N$, yields:

$$\partial U_t = \left(\textstyle\sum_{j \in \mathcal{N}_1} \partial_{r_1} U_{1j}, \quad v_1^T, \quad \theta_1, \quad \cdots, \quad \sum_{j \in \mathcal{N}_N} \partial_{r_N} U_{Nj}, \quad v_N^T, \quad \theta_N \right).$$

This expression for the generalized gradient of U_t facilitates the computation of its generalized time derivative, for which we have:

Lemma 6.5. *All elements in the generalized time derivative of* U_t, $\dot{\tilde{U}}_t$, *when its arguments evolve according to* (1) *under the control law* (13), *are non positive.*

Proof. For the function U_t we have:

$$\dot{\tilde{U}}_t = \bigcap_{\xi \in \partial U_t} \xi^T \begin{pmatrix} v_1 \\ K\left[-\sum_{j\in\mathcal{N}_1} \nabla_{r_1} U_{1j} + a_{\theta_1}\right] \\ K\left[-\sum_{j\in\mathcal{N}_1} \frac{\theta_1-\theta_j}{\|r_{1j}\|^2}\right] \\ \vdots \\ v_N \\ K\left[-\sum_{j\in\mathcal{N}_N} \nabla_{r_N} U_{Nj} + a_{\theta_N}\right] \\ K\left[-\sum_{j\in\mathcal{N}_N} \frac{\theta_N-\theta_j}{\|r_{Nj}\|^2}\right] \end{pmatrix} \tag{14}$$

Assume that $\xi = (\xi_{1r}, v_1^T, \theta_1, \ldots, \xi_{Nr}, v_N^T, \theta_N)$. Then using the fact that $K[f+g] \subset K[f] + K[g]$, [26], we can write $\dot{\tilde{U}}$ as follows:

$$\begin{aligned} \dot{\tilde{U}}_t \subset \bigcap_{\xi_{ir} \in \partial_{r_i} U_i} \Bigg\{ & \xi_{1r} v_1 + v_1^T K\left[-\sum_{j\in\mathcal{N}_1} \nabla_{r_1} U_{1j}\right] + v_1^T K\left[a_{\theta_1}\right] \\ & + \theta_1 K\left[-\sum_{j\in\mathcal{N}_1} \frac{\theta_1-\theta_j}{\|r_{1j}\|^2}\right] + \cdots + \xi_{Nr} v_N + v_N^T K\left[-\sum_{j\in\mathcal{N}_N} \nabla_{r_N} U_{Nj}\right] \\ & + v_N^T K\left[a_{\theta_N}\right] + \theta_N K\left[-\sum_{j\in\mathcal{N}_N} \frac{\theta_N-\theta_j}{\|r_{Nj}\|^2}\right] \Bigg\} \end{aligned}$$

Noting that $v_i^T \chi = 0, \forall \chi \in K[a_{\theta_i}]$, since, χ is always along a direction normal to v_i, we can simplify the above:

$$\begin{aligned} \dot{\tilde{U}}_t \subset \bigcap_{\xi_{ir} \in \partial_{r_i} U_i} \Bigg\{ & \xi_{1r} v_1 + v_1^T K\left[-\sum_{j\in\mathcal{N}_1} \nabla_{r_1} U_{1j}\right] + \theta_1 K\left[-\sum_{j\in\mathcal{N}_1} \frac{\theta_1-\theta_j}{\|r_{1j}\|^2}\right] + \cdots \\ & + \xi_{Nr} v_N + v_N^T K\left[-\sum_{j\in\mathcal{N}_N} \nabla_{r_N} U_{Nj}\right] + \theta_N K\left[-\sum_{j\in\mathcal{N}_N} \frac{\theta_N-\theta_j}{\|r_{Nj}\|^2}\right] \Bigg\} \end{aligned} \tag{15}$$

Since $\xi_{ir} \in \sum_{j\in\mathcal{N}_i} \partial_{r_i} U_{ij}$, each artificial potential function U_{ij} is a composition of the norm of the relative position vector, $\|r_{ij}\|$ with the function $U_{ij} : \mathbb{R}_+ \setminus \{0\} \to \mathbb{R}_+$. The norm is a smooth (hence strictly differentiable) function of both position vectors r_i, r_j when $r_i \neq r_j$. Recall from Section 4 that $r_i = r_j$ corresponds to configurations in the exterior of Ω, for any bounded c. On the other hand, $U_{ij}(s)$ is locally Lipschitz and regular for all $s > 0$. Thus, the conditions for equality in case (3) of Theorem A.6 are satisfied, and therefore:

$$\partial_{r_i} U_{ij}(\|r_{ij}\|) = \partial_{r_{ij}} U_{ij}(\|r_{ij}\|) \cdot \frac{\partial \|r_{ij}\|}{\partial r_i}$$

At the point d where U_{ij} is not differentiable, $\partial_{r_{ij}} U_{ij}(d) = \emptyset$ by (12), and thus,

$$\partial_{r_i} U_{ij}(d) = \emptyset$$

Note also that:

$$\begin{aligned} K[-\sum_{j\in\mathcal{N}_i} \nabla_{r_i} U_{ij}](x) &= K[\sum_{j\in\mathcal{N}_i} \nabla_{r_i}(-U_{ij})](x) \\ &\subset \sum_{j\in\mathcal{N}_i} K[\nabla_{r_i}(-U_{ij})](x) = \sum_{j\in\mathcal{N}_i} \partial_{r_i}\big(-U_{ij}\big)(x) \\ &= -\sum_{j\in\mathcal{N}_i} \partial_{r_i} U_{ij}(x) \end{aligned}$$

Thus, at all points where U_t is differentiable, we have

$$\partial_{r_i} U_i v_i + v_i^T K[-\sum_{j\in\mathcal{N}_i} \nabla_{r_i} U_{ij}] = \left\{ \sum_{j\in\mathcal{N}_i} \nabla_{r_i} U_{ij} v_i \right\} - \left\{ v_i^T \sum_{j\in\mathcal{N}_i} \nabla_{r_i}^T U_{ij} \right\} = \{0\}$$

whereas at points where for some k, U_{ik} (and thus also U_t) is not differentiable, since $\partial_{r_i} U_{ik} = \emptyset$,

$$\partial_{r_i} U_i v_i + v_i^T K[-\sum_{j\in\mathcal{N}_i} \nabla_{r_i} U_{ij}] \subset \left\{ \sum_{\substack{j\in\mathcal{N}_i \\ j\neq k}} \nabla_{r_i} U_{ij} v_i \right\} - \left\{ v_i^T \sum_{\substack{j\in\mathcal{N}_i \\ j\neq k}} \nabla_{r_i}^T U_{ij} \right\} + \emptyset = \{0\}$$

If follows that in any case, (15) reduces to:

$$\dot{\tilde{U}}_t \subset \sum_{i=1}^{N} \theta_i K \left[\sum_{j\in\mathcal{N}_i} \frac{\theta_j - \theta_i}{\|r_{ij}\|^2} \right] = K \left[\sum_{i=1}^{N} \theta_i \sum_{j\in\mathcal{N}_i} \frac{\theta_j - \theta_i}{\|r_{ij}\|^2} \right] \tag{16}$$

because each θ_i is continuous. Now it is clear that we can write (16) using the definition of $K[\cdot]$ as:

$$\dot{\tilde{U}}_t \subset \overline{\text{co}}\left\{-\theta^T\left[\lim L_w(s)\right]\theta \mid (s_1,\ldots,s_N)\to(r_1,\ldots,r_N), \|s_i-s_j\|\neq d\right\} \tag{17}$$

where $L_w(r)$ is the weighted Laplacian of the neighboring graph $\mathcal{G}$ at the configuration $r=(r_1,\ldots,r_N)$.

If the graph $\mathcal{G}$ is always connected, then $L_w(r)$ has a single zero eigenvalue with corresponding eigenvector $\mathbf{1}_N$, for all r. Thus, whenever θ and $\mathbf{1}_N$ are not parallel (which is equivalent to saying that not all agents have the same orientation,) $\theta^T L_w(r)\theta > 0$. From (17) we have that for all $\nu \in \dot{\tilde{U}}_t$, $\nu < 0$, because ν will belong in a closed interval of negative numbers. The only case where $0 \in \dot{\tilde{U}}_t$ is when $\theta^T L_w(r)\theta = 0 \Leftrightarrow \theta \parallel \mathbf{1}_N$. □

Proposition 6.6. *The system of N agents with dynamics* (1) *governed by the discontinuous control laws* (13)*, with initial conditions in Ω, converges to one of the minima of* (11) *with a common orientation.*

Proof. First note that under Assumption 6.1, and in view of Lemma 6.5 the set Ω will be positively invariant. Applying Theorem 5.4 on Ω using regular function U_t, it follows that trajectories converge to the largest invariant set in the closure of the set:

$$S_\theta = \{(r_1,\theta_1,\ldots,r_N,\theta_N) \mid \theta_1=\cdots=\theta_N=\theta_0\}$$

From the definition of θ_i, we have

$$\phi_i \triangleq \dot{y}_i - \dot{x}_i\tan\theta_i = \dot{y}_i - k\dot{x}_i = 0,\; i=1,\ldots,N$$

To find the largest invariant set in S_θ, we investigate the dynamics of ϕ_i:

$$\dot{\phi}_i \in K[a_{y_i} - k a_{x_i}]$$

In the invariance set we will have $K[a_{y_i} - k a_{x_i}] = \{0\}$ which means that $a_{y_i} - k a_{x_i} = 0$. Note that this also implies that $a_{y_i} - k a_{x_i}$ is continuous, i.e. there will be no switching in steady state.

We will now show that $a_{y_i} = k a_{x_i} \neq 0$, implies that $a_{y_i} = a_{x_i} = 0$. Consider the function $V_v = \frac{1}{2}v_i^T v_i$. Then

$$\frac{d}{dt}\left(\frac{1}{2}v_i^T v_i\right) = v_i^T K[a_i] = v_i^T a_i$$

because a_i has to be continuous at steady state (the same can be shown even without the continuity assumption.) Then, having $a_{y_i} = ka_{x_i}$ and $\dot{y}_i = k\dot{x}_i$, the velocity vector and the acceleration vector are collinear, so either $v_i^T a_i = \|v_i\| \|a_i\| \geq 0$ or $v_i^T a_i = -\|v_i\| \|a_i\| < 0$. Consider the latter case: if for some i it is $v_i^T a_i = -\|v_i\| \|a_i\| < 0$, then its kinetic energy is monotonically decreasing:

$$\frac{d}{dt}\left(\frac{1}{2}v_i^T v_i\right) = v_i^T a_i < 0.$$

This means that in the invariant set it will have $v_i = 0$, also implying $a_i = 0$, which contradicts $v_i^T a_i < 0$. In the invariant set, therefore, it can only be $v_i^T a_i = \|v_i\| \|a_i\| \geq 0$ for all i. In that case, we can see from the function $\frac{1}{2}\sum_{i=1}^N U_i$, that the system will approach one of the minima of $\sum_{i=1}^N U_i$:

$$\frac{d}{dt}\sum_{i=1}^{N}\frac{1}{2}U_i = \sum_{i=1}^{N}\nabla_{r_i}U_i v_i = \sum_{i=1}^{N}(-a_i)^T v_i \leq 0$$

Thus, the system will approach the set where $a_i = 0$, since $v_i = 0$ also implies $a_i = 0$, which corresponds to a minimum of the artificial potential function. □

7 Simulations

In this section we verify numerically the stability results obtained in the previous sections. The group is consisted of ten systems with identical second order dynamics. The neighborhood radius of each agent was fixed to a constant $D = 15$[m]. The initial conditions (positions and velocities) were generated randomly within a ball of radius $R_0 = 10$[m].

Depending on the direction and magnitude of the initial velocity vectors, the neighboring graph could either remain connected or become disconnected at some point. In the case when the graph remains connected, stability is ensured and the group eventually converges to a compact formation with a common heading. In contrast, when the number of connected components in the graph increases, then each component converges to its own heading.

7.1 Global Agent Interaction

When the neighborhood radius is sufficiently large, the interconnection graph can remain fully connected at all times. In this case, the interconnection graph is complete, having $\frac{N(N-1)}{2}$ edges, where N is the number of vertices in the graph. Since every vertex has all other $N-1$ vertices being adjacent to it at all times, Proposition 4.2 applies and it is expected that the agents are going to converge to a configuration of minimum potential energy and move along a common direction. This is verified in simulation. Figures 4-7 give snapshots of the system's evolution in one case where the neighboring graph is complete. In the Figures, the position of the agents is represented by small red circles. The yellow circle around each agent mark its neighborhood. All other agents inside this region are considered neighbors and a blue line is drawn to indicate an edge in the neighboring graph. With the neighborhood radius being large enough the set of neighbors of each agent includes all its flockmates. The paths followed by the agents as they move under control laws (4) are shown by green curves.

In the simulation example depicted in Figures 4-7 all agents lay within each other's neighborhoods at all times and thus the neighboring graph is always complete. Thus, this simulation example is an instance of the "global interaction" case, analyzed in Section 4. Completeness of the graph ensures that artificial potential functions take values within the region in which they are smooth. Control inputs given by (4) are continuous and the group trajectories converge to the invariant set where the heading angles are the same and the agents are relatively positioned in such a way so that the sum of the artificial potential functions attains a minimum.

7.2 Local Agent Interaction

With smaller neighborhood radii, the set of neighbors for each agent may not contain all its flockmates. This would mean that that the agent's control input are given by (13) and are determined by local information, consisting of the relative position and heading of neighboring flockmates. This set of neighbors changes with time since the relative positions between the agents evolve. At each instant where the neighboring set of an agent changes, its control input exhibits a discontinuity due to the local interaction that was just introduced. Despite this switching, stability of the group motion is still guaranteed under Proposition 6.6, as long as the interconnection graph remains connected. This is seen in the simulation example depicted in Figures 8-11. In this case, the randomly generated initial conditions were such that the neighboring graph could not preserve completeness, yet, connectedness was sufficient to ensure stability.

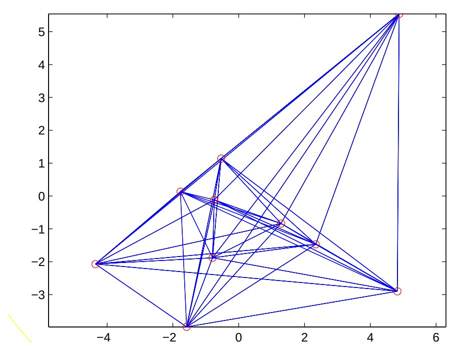

Figure 4: Initial configuration.

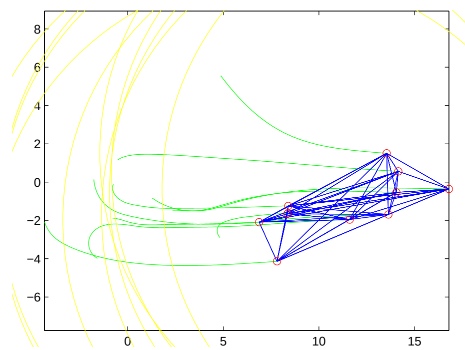

Figure 5: Converging to a common heading.

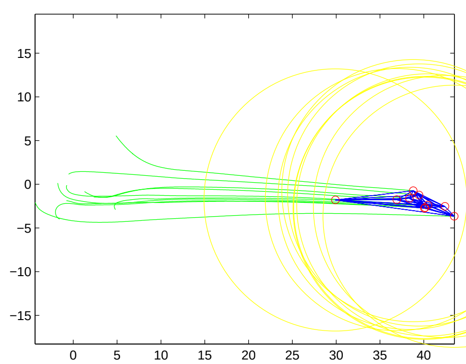

Figure 6: Approaching a potential energy minimum.

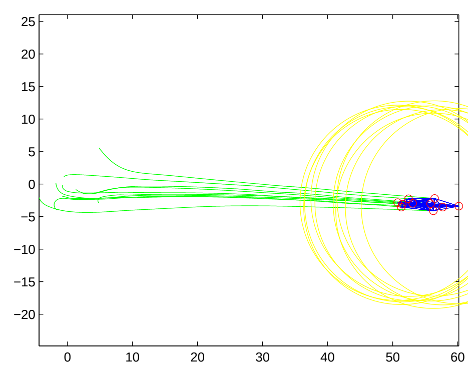

Figure 7: Steady state.

In Figure 8 the agents are initiated with random initial conditions within a bounded set which is such to ensure that the neighboring graph will initially be complete. Control effort is not sufficient to overcome the misalignment of initial velocity vectors and thus the graph soon looses its completeness as seen in Figure 9. However, the graph remains connected and this allows the agents to achieve a formation where the graph is complete and all headings are aligned (Figure 10). This ensures that they will finally flock into a configuration that also minimizes the potential energy of the group (Figure 11).

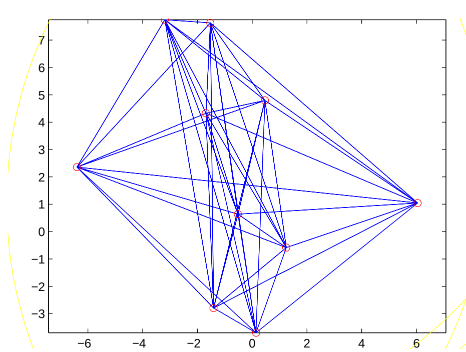

Figure 8: Initial configuration; complete graph.

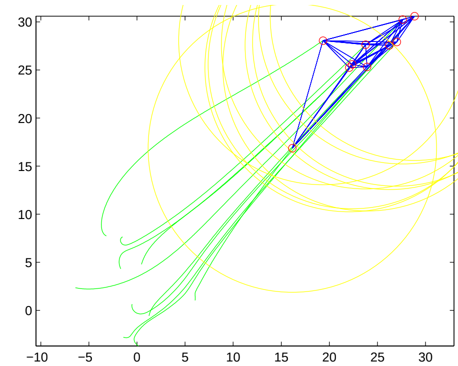

Figure 9: The rear-most boid has lost sight of three of its flockmates.

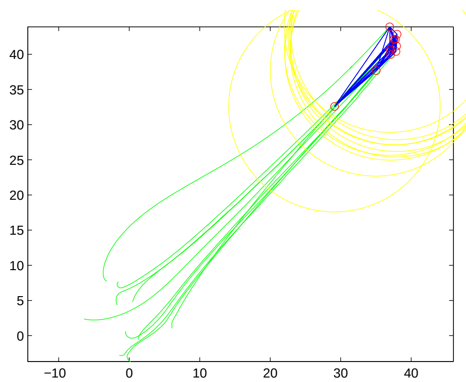

Figure 10: Convergence yields completeness for the graph.

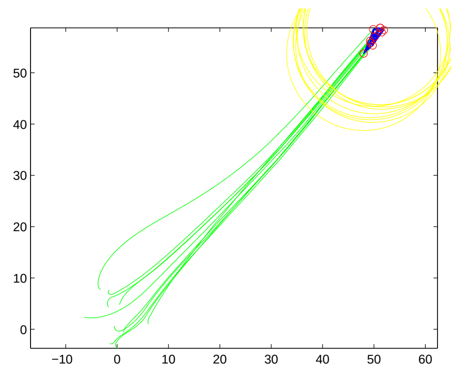

Figure 11: Steady state.

7.3 Losing Connectedness

There exist initial conditions for which the neighboring graph cannot remain connected throughout the group's motion. This could either be attributed to a combination of initial velocity misalignment, insufficient control effort and small neighborhood radius. In this case Proposition 6.6 can no longer ensure flocking and as seen in Figures 12-15 the group splits up and heading is stabilized within the limits of each connected component.

In Figure 12 the group starts from a configuration where the neighboring graph is connected but not complete. At first, the headings converge, but due to excessive misalignment the agents move away from each other and for some the set of neighboring flockmates shrinks considerably (Figure 13). With

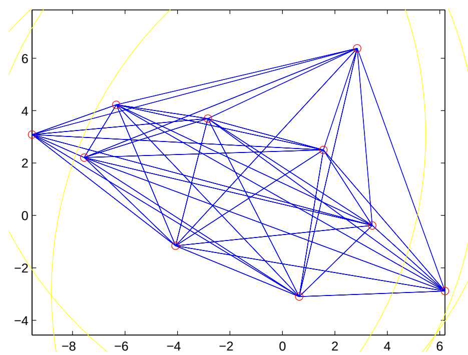

Figure 12: Initial configuration; graph connected but not complete.

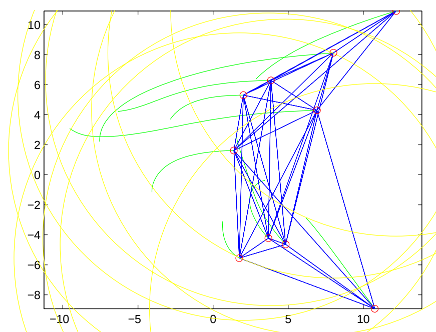

Figure 13: Converging of headings but distant agents loose connectivity.

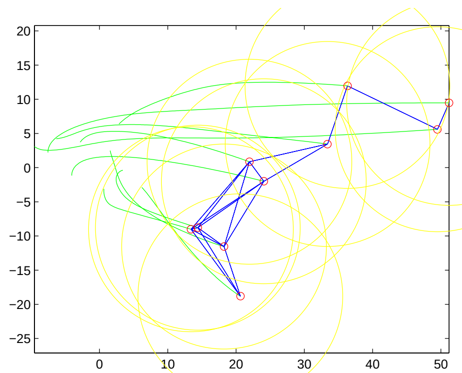

Figure 14: Graph density drops and interaction grows weaker.

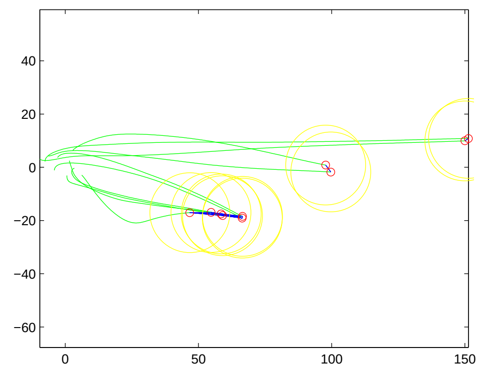

Figure 15: Group splits and each component flocks independently.

the number of their neighbors decreasing, the interaction between the boids agents even weaker (Figure 14) and finally the graph becomes disconnected (Figure 15). After this point, each connected component follows its own way and stability is established at the component level.

8 Conclusions

In this paper we established the stability properties of flocking motion under local coordinating control laws. We showed the importance of maintaining sufficient information flow within the flock as a condition for stability. This condition is formalized in graph theoretic terms as the connectedness of the neighboring graph that encodes agent interaction. In the case where information flow is dependent on proximity between group members, then discontinuities are generally unavoidable. Despite discrete changes in the neighboring graph and control input discontinuities, however, stability is maintained and flocking is guaranteed as long as the connectedness requirement is satisfied.

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A Facts from Nonsmooth Analysis

Definition A.1 ([4]). *Let f be Lipschitz near a given point x and let w be any vector in a Banach space X. The generalized directional derivative of f at x in the direction w, denoted $f^{\circ}(x;w)$ is defined as follows:*

$$f^{\circ}(x;w) \triangleq \limsup_{\substack{y\to x\\ t\downarrow 0}} \frac{f(y+tw)-f(y)}{t}$$

Definition A.2 ([4]). *The generalized gradient of f at x, denoted $\partial f(x)$, is the subset of X^* given by:*

$$\partial f(x) \triangleq \{\zeta \in X^* \mid f^{\circ}(x;w) \geq \langle \zeta, w\rangle,\ \forall w \in X\}$$

In the special case where X is finite dimensional, we have the following convenient characterization of the generalized gradient:

Theorem A.3 ([3]). *Let $x \in \mathbb{R}^n$ and let $f : \mathbb{R}^n \to \mathbb{R}$ be Lipschitz near x. Let Ω be any subset of zero measure in $\mathbb{R}^n$, and let Ω_f be the set of points in $\mathbb{R}^n$ at which f fails to be differentiable. Then*

$$\partial f(x) \triangleq \mathrm{co}\{\lim_{x_i\to x} \nabla f(x_i) \mid x_i \notin \Omega, x_i \notin \Omega_f\}$$

Definition A.4 ([4]). *A function f is said to be regular at x provided,*

1. *For all w, the usual one-sided directional derivative $f'(x;w)$ exists, and*
2. *for all w, $f'(x;w) = f^\circ(x;w)$.*

A differential concept that is relevant to the generalized directional derivative, is the strict derivative:

Theorem A.5 ([4]). *Let F map a neighborhood of $x \in X$ to Y, and let ζ be an element of the space of continuous linear functionals from X to Y, denoted $\mathcal{L}(X,Y)$. The following are equivalent:*

- *F is strictly differentiable at x and $D_s F(x) = \zeta$.*
- *F is Lipschitz near x, and for each w in X one has*

$$\lim_{\substack{x' \to x \\ t \downarrow 0}} \frac{F(x'+tw) - F(x')}{t} = \langle \zeta, w \rangle$$

Then, if F is Lipschitz near x and strictly differentiable, it is also regular at x.

Next, we present some of the basic properties of the generalized gradient which are useful in calculus. The following list is compiled from [4]:

1. If f is Lipschitz of rank L near x, then $f^\circ(x;w) = \max\{\langle \zeta, w \rangle \mid \partial f(x)\}$
2. For any scalar s, $\partial(sf)(x) = s\partial f(x)$.
3. $\partial\left(\sum f_i\right)(x) \subset \sum \partial f_i(x)$ where equality holds if each f_i is regular at x.
4. $\partial\left(\sum s_i f_i\right)(x) \subset \sum \partial s_i f_i(x)$, where equality holds if each f_i is regular at x and all s_i are nonnegative.
5. $\partial(f_1 f_2)(x) \subset f_2(x)\partial f_1(x) + f_1(x)\partial f_2(x)$ and equality holds if each f_1, f_2 is regular at x and $f_1(x) \geq 0$, $f_2(x) \geq 0$.
6. If f_1, f_2 are Lipschitz near x and $f_2(x) \neq 0$, then

$$\partial\left(\frac{f_1}{f_2}\right)(x) \subset \frac{f_2(x)\partial f_1(x) - f_1(x)\partial f_2(x)}{f_2^2(x)}$$

   and if in addition, $f_1(x) \geq 0$, $f_2(x) > 0$ and f_1, $-f_2$ are regular at x, equality holds.

The general case of the chain rule is particularly important and we present it as a theorem:

Theorem A.6 ([4]). *Consider a function $f = g \circ h$, where $h : X \to \mathbb{R}^n$ and $g : \mathbb{R}^n \to \mathbb{R}$, and denote the component functions of h by h_i $(i = 1, \dots, n)$. Assume that each h_i is Lipschitz near x and that g is Lipschitz near $h(x)$. Then,*

$$\partial f(x) \subset \overline{\text{co}} \left\{ \sum \alpha_i \zeta_i \mid \zeta_i \in \partial h_i(x), \alpha \in \partial g\big(h(x)\big) \right\}$$

and equality holds under any one of the following additional hypotheses:

1. *g is regular at $h(x)$, each h_i is regular at x, and every element a of $\partial g(h(x))$ has nonnegative components.*

2. *g is strictly differentiable at $h(x)$ and $n = 1$.*

3. *g is regular at $h(x)$ and h is strictly differentiable at x.*

We hereby list some of the basic properties of the differential inclusion of Definition 5.1, $K[f]$, taken from [26]. Assume that f and g are locally bounded. Then:

1. $K[f + g](x) \subset K[f](x) + K[g](x)$.

2. Let $g : \mathbb{R}^m \to \mathbb{R}^n$ be C^1 with $Dg(x) = n$; then $K[f \circ g](x) = K[f](g(x))$.

3. Let $g : \mathbb{R}^m \to \mathbb{R}^{p \times n}$ be C^0; then $K[fg](x) = g(x)K[f](x)$

4. Let $V : \mathbb{R}^n \to \mathbb{R}$ be locally Lipschitz; then $K[\nabla V](x) = \partial V(x)$.

5. Let $f : \mathbb{R}^m \to \mathbb{R}^n$ be continuous; then $K[f](x) = \{f(x)\}$.