Satisficing: A New Approach to Constructive Nonlinear Control

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***Abstract*—The main contribution of this paper is a constructive parameterization of the class of almost smooth universal formulas which render a system asymptotically stable with respect to a known control Lyapunov function (CLF), and a constructive parameterization of a class of inverse optimal universal formulas having Kalman-like stability margins. The novelty of the parameterization is that it is given in terms of two function which are constrained to be locally Lipschitz and satisfy convex constraints. The implication of this result is that the CLF/universal formula approach can be combined with *a priori* performance objectives to design high performance control strategies. Two examples illustrate the approach.**

***Index Terms*—Control Lyapunov functions (CLFs), inverse optimality, nonlinear control, stability margins.**

I. Introduction

LYAPUNOV theory plays a major role in stability analysis. Given a nonlinear ordinary differential equation without inputs, if a Lyapunov function candidate can be shown to be negative definite along the trajectories of the system, then the system is guaranteed to be asymptotically stable [1], [2]. One of the traditional criticisms of Lyapunov theory is that it is not constructive: one must propose a feedback function and then search for an appropriate Lyapunov function. Traditional Lyapunov theory has been used for synthesis purposes by proposing a Lyapunov function candidate, and then finding a feedback strategy that renders it negative definite [1], [3].

The synthesis problem was made more formal by the introduction of control Lyapunov functions (CLFs) [4]–[6]. A CLF is a positive definite, radially unbounded function that can be made negative definite at each state, by some feasible input. In contrast with traditional Lyapunov functions, a CLF can therefore be defined for a system with inputs, without specifying a particular feedback function.

The synthesis problem is completed by using the CLF to choose a (typically smooth) feedback function that renders the derivative of the CLF negative definite along trajectories of the system [7]–[9]. Sontag has shown that if a CLF is known for a nonlinear system that is affine in the control, then the CLF and the system equations can be used to find formulas that render the system asymptotically stable [7]. These formulas are called universal formulas because they depend only upon the CLF and the system equations and not on the particular structure of those equations.

There are several known universal formulas, in particular, Sontag's formula [7], and Freeman and Kokotovic's min-norm formula [10], [11]. It is obvious that there is a large class of universal formulas, however the size and structure of the set of universal formulas has not yet been identified.

One of the contributions of this paper is to completely parameterize the set of universal formulas. In particular, our parameterization is constructive in that the parameterization is given in terms of two, state dependent selection, or tuning, functions that are only required to satisfy mild constraints. Any functions satisfying these conditions can be used to construct a universal formula.

It has been shown that Sontag's formula and the min-norm formula are "good" universal formulas in the sense that they enjoy certain stability margins and are inverse optimal [9], [11], [12]. It is natural to wonder if the set of universal formulas that enjoy these properties can also be parameterized. Another contribution of this paper is to show that this is the case. In addition, the parameterization is shown to be convex.

One of the drawbacks with Sontag's formula is that it does not provide any convenient parameters to tune the performance of the control. The only available tuning device is to modify the control Lyapunov function itself. Freeman and Kokotovic's min-norm formula addresses this problem by adding an additional scalar function that specifies the minimum rate of decrease of the CLF. Performance of the closed loop system can be "tuned" by modifying this function [11]. The parameterization introduced in this paper can be used to address closed-loop performance in a natural way. Performance can be achieved by specifying an auxiliary optimization problem that chooses the selection functions at each state. As long as the selection functions satisfy mild continuity and convex boundedness constraints, the resulting "optimized" system will be asymptotically stable. Other approaches along these lines include [13], [14] which use CLFs to guarantee stability of receding horizon approaches.

Our parameterization of universal formulas is derived using the recently introduced notion of satisficing decision theory [15]–[17]. Satisficing decision theory can be seen as a formal application of cost–benefit analysis to decision making problems. The basic idea is to define two utility functions that quantify the benefits and costs of an action. At each state, the benefits of choosing a control action are given by a "selectability" function. Similarly, at each state, the costs associated with choosing the control action are given by a

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"rejectability" function. The "satisficing" set is defined to be those options for which selectability (benefits) exceeds rejectability (costs) [17].

The first application of the satisficing approach to feedback control was derived in [16]. The selectability function was chosen as the distance from the predicted state at the next time instant to the origin, and the rejectability function was chosen to be proportional to the control effort. The resulting control strategy is reminiscent of model predictive control. There are two limitations of the control strategies derived in [16]: First, closed-loop stability was not guaranteed analytically, and second, at each state, a search needed to be performed to find the satisficing set. This paper solves both of those problems. First, by linking the "selectability" function to a CLF, closed-loop asymptotic stability is ensured. Second, by imposing an affine-in-the-control structure on the nonlinear system, the structure of the satisficing set is used to derive a closed-form description of the satisficing set at each state.

This paper is organized as follows. In Section II, we define a state dependent subset of the control space which we call the satisficing set and show that this set is convex and can be parameterized by state dependent selection functions. In Section III, we define satisficing controls to be continuous selections from the satisficing set, and derive a constructive formula for these controls. We show that all satisficing controls render the closed-loop system asymptotically stable. In Section IV, we show that continuous selections from a convex subset of the satisficing set, which we call the robust satisficing set, result in closed-loop control strategies that enjoy Kalman-like stability margins in the spirit of [9], [18], and [19]. In Section V, we show that these closed-loop strategies are also inverse optimal in the sense of [11], [20]–[22]. Section VI contains the main result which shows that the satisficing framework completely parameterizes all universal formulas that are locally Lipschitz, and zero at the origin. Section VII illustrates the ideas with two simple examples. In Section VIII we offer perspective and concluding remarks.

Throughout this paper, we will denote the partial derivative with a subscript: $V_x \triangleq \partial V/\partial x$, where V_x is assumed to be a column vector. R^T denotes the transpose of the matrix R. $\|x\|$ denotes the Euclidean norm of the vector x, and $\|A\|$ denotes the induced Euclidean norm of the matrix A.

II. Satisficing Set

Consider the affine nonlinear system

$$\dot{x} = f(x) + g(x)u \qquad (1)$$

where $x \in \mathbb{R}^n$, $f : \mathbb{R}^n \to \mathbb{R}^n$, $g : \mathbb{R}^n \to \mathbb{R}^{n\times m}$ and $u \in \mathbb{R}^m$. We will assume throughout this paper that f and g are locally Lipschitz functions and that $f(0) = 0$.

Definition 1: A twice continuously differentiable function $(C^2)V : \mathbb{R}^n \to \mathbb{R}$ is said to be a CLF for system (1), if V is positive definite, radially unbounded, and if

$$\inf_u V_x^T(f + gu) < 0$$

for all $x \neq 0$.

The existence of a CLF implies that there exists a, possibly discontinuous, control law such that the CLF is a Lyapunov function for the closed-loop system. Hence, the CLF can be viewed as a candidate Lyapunov function, where the control law which will render the system stable has not yet been specified.

It has been shown in [4], [5] that system (1) is asymptotically controllable to the origin, if and only if there exists a CLF for the system. In general, finding a CLF is an open problem, however constructive techniques are known for a large class of practically important systems [8] including feedback linearizable systems and systems which are amenable to integrator backstepping.

A CLF V is said to satisfy the *small control property* [12] for (1) if there exists a control law $\alpha_c(x)$ continuous in $\mathbb{R}^n$ such that

$$V_x^T(f + g\alpha_c) < 0 \qquad \forall x \neq 0.$$

The satisficing paradigm calls for the definition of two utility functions: the selectability function $p_s(u, x)$, and the rejectability function $p_r(u, x)$[17]. Following [16], selectability should be large for control values that are desirable in some sense. Similarly, rejectability should be large for control values that are expensive to implement. In addition, define $0 < b(x) < \infty$ to be the selectivity, or boldness, index.

Definition 2: The **satisficing set $S_b(x)$** is defined to be the set of control values such that the selectability times the selectivity index is greater than the rejectability, i.e.,

$$S_b(x) = \left\{ u \in \mathbb{R}^m : p_s(u, x) > \frac{1}{b(x)} p_r(u, x) \right\}.$$

In other words, the satisficing set is the set of all point-wise control values where the instantaneous benefits of applying that action outweigh the instantaneous costs. For practical reasons, we are interested in the case when $S_b(x)$ is a convex set.

Lemma 3: If for each x, $p_s(u, x)$ is a concave function of u and $p_r(u, x)$ is a convex function of u, then $S_b(x)$ is a convex (and, hence, connected) set.

Proof: The lemma follows directly from the definition $S_b(x)$. ■

Note that we only require convexity in u and not in x. Therefore, we do not impose any convexity restrictions on the system, only on the incremental measures of benefit and cost.

We will associate the notion of selectability with stability, and the notion of rejectability with instantaneous cost. In particular, let

$$p_s(u, x) = -V_x^T(x)(f(x) + g(x)u) \qquad (2)$$

where V is a CLF. Note that stabilizing control values make $p_s(u, x)$ positive. We choose the rejectability criterion to be

$$p_r(u, x) = l(x) + u^T R(x) u \qquad (3)$$

where $R(x) = R(x)^T > 0$ is a positive–definite matrix function whose elements are locally Lipschitz and $l : \mathbb{R}^n \to \mathbb{R}$ is a locally Lipschitz nonnegative function. Note that $p_s(u, x) = -V_x^T f - V_x^T g u$ is a linear function in u and is, hence, concave in u. Additionally, $p_r(u, x) = l(x) + u^T R(x) u$ is convex in u. For these choices of p_s and p_r the satisficing set becomes

$$S_b(x) = \left\{ u \in \mathbb{R}^m : -V_x^T(f + gu) > \frac{1}{b}(l + u^T R u) \right\} \qquad (4)$$

which by Lemma 3, is guaranteed to be a convex set.

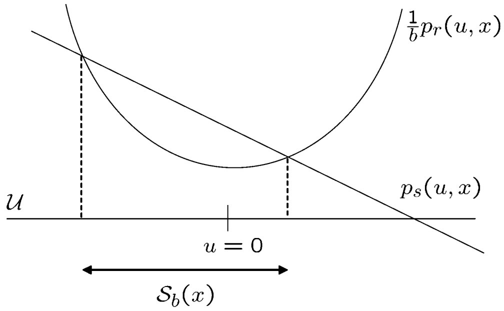

Fig. 1. Selectability and rejectability functions as a function of u, for a particular x, and the resulting satisficing set.

Fig. 1 shows p_s, p_r and $S_b(x)$ for a particular state x for the case of a single input $u \in \mathbb{R}$.

The following theorem completely characterizes the satisficing set for the particular selectability and rejectability functions chosen previously.

Theorem 4: If $p_s(u,x) = -V_x^T(f+gu)$ and $p_r(u,x) = l + u^T Ru$, then the satisficing set at state x is nonempty if and only if $b(x)$ satisfies the inequality

$$l(x) + b(x)V_x^T(x)f(x) - \frac{1}{4}b(x)^2 V_x^T(x)g(x)R^{-1}(x)g^T(x)V_x(x) < 0 \quad (5)$$

at that state.

Furthermore, if $S_b(x) \subset \mathbb{R}^m$ is nonempty, it is given by

$$S_b(x) = \left\{ -\frac{1}{2}bR^{-1}g^TV_x + R^{-1/2} \dot{\nu}\sqrt{\frac{1}{4}b^2V_x^TgR^{-1}g^TV_x - l - bV_x^Tf} : \|\nu\| < 1 \right\}. \quad (6)$$

Thus, $S_b(x)$ is the set of control values, defined at each state, that satisfy the condition

$$-V_x^T(f+gu) > \frac{1}{b}(l + u^TRu).$$

The benefit of (6) is that it provides an explicit formula for control values which satisfy this condition whenever S_b is nonempty. Note that this formula provides a mapping from the open unit ball (ν is a free parameter whose only constraint is that it lie in the unit ball) to the satisficing set. Note also that when $V_x^T g = 0$, the satisficing set is well defined and given by

$$S_b(x) = \left\{ R^{-1/2}\nu\sqrt{-l - bV_x^Tf} : \|\nu\| < 1 \right\}.$$

The proof of Theorem 4 depends upon the following lemma which provides a generalization of the quadratic formula.

Lemma 5: If $A = A^T > 0$, then the set of solutions to the quadratic inequality

$$\xi^TA\xi + d^T\xi + c < 0$$

where $\xi \in \mathbb{R}^s$, is nonempty if and only if

$$\frac{1}{4}d^TA^{-1}d - c > 0$$

and is given by

$$\xi = -\frac{1}{2}A^{-1}d + A^{-1/2}\nu\sqrt{\frac{1}{4}d^TA^{-1}d - c}$$

where $\nu \in \{\xi \in \mathbb{R}^s : \|\xi\| < 1\}$.

Proof: Since $A = A^T > 0$, it is invertible and can be factored as $A = A^{1/2}A^{1/2}$ where $A^{1/2}$ is also symmetric and invertible [23]. By completing the square, we get that

$$\xi^TA\xi + d^T\xi + c = \left(A^{1/2}\xi + \frac{1}{2}A^{-1/2}d\right)^T \cdot \left(A^{1/2}\xi + \frac{1}{2}A^{-1/2}d\right) + c - \frac{1}{4}d^TA^{-1}d.$$

Therefore

$$\begin{aligned}
&\xi^TA\xi + d^T\xi + c < 0 \\
&\quad\iff \left(A^{1/2}\xi + \frac{1}{2}A^{-1/2}d\right)^T \left(A^{1/2}\xi + \frac{1}{2}A^{-1/2}d\right) < \frac{1}{4}d^TA^{-1}d - c \\
&\quad\iff \left\|A^{1/2}\xi + \frac{1}{2}A^{-1/2}d\right\|^2 < \frac{1}{4}d^TA^{-1}d - c.
\end{aligned}$$

Obviously the left hand side of this expression is positive which implies that a solution exists if and only if $1/4d^TA^{-1}d - c > 0$, in which case we have

$$\left\|A^{1/2}\xi + \frac{1}{2}A^{-1/2}d\right\| < \sqrt{\frac{1}{4}d^TA^{-1}d - c}.$$

Note that the aforementioned expression constrains the magnitude but not the direction of $A^{1/2}\xi + 1/2A^{-1/2}d$. Therefore

$$\begin{aligned}
&\left\|A^{1/2}\xi + \frac{1}{2}A^{-1/2}d\right\| < \sqrt{\frac{1}{4}d^TA^{-1}d - c} \\
&\quad\iff A^{1/2}\xi + \frac{1}{2}A^{-1/2}d = \nu\sqrt{\frac{1}{4}d^TA^{-1}d - c} \\
&\|\nu\| < 1 \\
&\quad\iff A^{1/2}\xi = -\frac{1}{2}A^{-1/2}d + \nu\sqrt{\frac{1}{4}d^TA^{-1}d - c} \\
&\|\nu\| < 1 \\
&\quad\iff \xi = -\frac{1}{2}A^{-1}d + A^{-1/2}\nu\sqrt{\frac{1}{4}d^TA^{-1}d - c} \\
&\|\nu\| < 1.
\end{aligned}$$

■

Proof of Theorem 4: The satisficing set is given by

$$\begin{aligned}
S_b(x) &= \left\{u \in \mathbb{R}^m : -V_x^Tf - V_x^Tgu \geq \frac{1}{b}l + \frac{1}{b}u^TRu\right\} \\
&= \left\{u \in \mathbb{R}^m : u^T\left(\frac{R}{b}\right)u + (g^TV_x)^Tu + \left(\frac{1}{b}l + V_x^Tf\right) \leq 0\right\}.
\end{aligned}$$

The theorem therefore follows from Lemma 5 with $A = R/b$, $d = g^TV_x$, and $c = 1/bl + V_x^Tf$. ■

Theorem 4 shows that the selectivity index $b(x)$ plays a critical role in the size of $S_b(x)$. The next lemma shows that for each $x \neq 0$, b can always be chosen such that the satisficing set is nonempty. Toward that end, define

$$\underline{b}(x) \triangleq \begin{cases} \frac{l}{-V_x^T f}, & \text{if } V_x^T g = 0 \\ \frac{2V_x^T f + 2\sqrt{(V_x^T f)^2 + lV_x^T gR^{-1}g^T V_x}}{V_x^T gR^{-1}g^T V_x}, & \text{otherwise.} \end{cases} \quad (7)$$

Lemma 6: If V is a CLF for system (1), $\underline{b}$ is given by (7), and S_b is given by (6), then for each $x \neq 0$

1) $\underline{b}(x) \geq 0$;
2) $b > \underline{b}(x)$ implies that $S_b(x) \neq \emptyset$;
3) if $l : \mathbb{R}^n \to \mathbb{R}^+$ satisfies the property

$$\left(g^T V_x \neq 0 \text{ and } V_x^T f = 0\right) \Longrightarrow l > 0 \quad (8)$$

then $\underline{b}(x)$ is locally Lipschitz on $\mathbb{R}^n \setminus \{0\}$.

Proof: If $g^T V_x = 0$, then since V is a CLF, $V_x^T f < 0$, therefore, (5) is satisfied if and only if

$$\begin{aligned} -l - bV_x^T f &> 0 \\ \Longleftrightarrow\ b > \frac{l}{-V_x^T f} &\geq 0. \end{aligned}$$

If $g^T V_x \neq 0$, then (5) is satisfied if and only if

$$\begin{aligned} b^2 - \left(\frac{4V_x^T f}{V_x^T gR^{-1}g^T V_x}\right) b - \left(\frac{4l}{V_x^T gR^{-1}g^T V_x}\right) &> 0 \\ \Longleftrightarrow\ \left(b - \left(\frac{2V_x^T f}{V_x^T gR^{-1}g^T V_x}\right)\right)^2 & \\ > \left(\frac{4l}{V_x^T gR^{-1}g^T V_x}\right) & \\ + \left(\frac{2V_x^T f}{V_x^T gR^{-1}g^T V_x}\right)^2. & \end{aligned}$$

Restricting attention to positive solutions, this inequality is true if and only if

$$\begin{aligned} b > &\left(\frac{2V_x^T f}{V_x^T gR^{-1}g^T V_x}\right) \\ &+ \sqrt{\left(\frac{4l}{V_x^T gR^{-1}g^T V_x}\right) + \left(\frac{2V_x^T f}{V_x^T gR^{-1}g^T V_x}\right)^2} \\ \Longleftrightarrow\ b > &\frac{2V_x^T f + 2\sqrt{(V_x^T f)^2 + lV_x^T gR^{-1}g^T V_x}}{V_x^T gR^{-1}g^T V_x} \end{aligned}$$

which is clearly greater than or equal to zero.

To show that $\underline{b}(x)$ is locally Lipschitz on $\mathbb{R}^n \setminus \{0\}$ we follow the arguments used in [12, pp. 8–10] to show the continuity of Sontag's formula. Following their arguments, we show that the function

$$\phi(a, c, l) = \begin{cases} -\frac{l}{a}, & \text{if } c = 0 \text{ and } a < 0 \\ \frac{2a + 2\sqrt{a^2 + lc}}{c}, & \text{elsewhere} \end{cases}$$

is smooth ($\mathcal{C}_\infty$) on the set $P = \{(a, c, l) \in \mathbb{R}^3 | l \geq 0 \text{ and } c \geq 0$ and $(c = 0 \Longrightarrow a < 0)$ and $((c \neq 0 \text{ and } a = 0) \Longrightarrow l > 0)\}$.

Define the function

$$F(a, c, l, p) = l + pa - \frac{1}{4}p^2 c$$

which is smooth on P in all of its arguments. By direct substitution, it is straightforward to show that $F(a, c, l, \phi(a, c, l)) = 0$ for all $(a, c, l) \in P$. If $c = 0$, then

$$\frac{\partial F(a, c, l, \phi(a, c, l))}{\partial p} = a$$

which is strictly less than zero since V is a CLF. If $c \neq 0$, then

$$\frac{\partial F(a, c, l, \phi(a, c, l))}{\partial p} = -\sqrt{a^2 + lc}$$

which, by (8), is nonzero on P. Therefore, by the implicit function theorem, $\phi(a, c, l)$ is smooth on P. Since $V_x^T f$, $V_x^T gR^{-1}g^T V_x$, and l are locally Lipschitz on $\mathbb{R}^n \setminus \{0\}$, $\underline{b} = \phi(V_x^T f, V_x^T gR^{-1}g^T V_x, l)$ is also locally Lipschitz on $\mathbb{R}^n \setminus \{0\}$. ∎

Letting

$$\sigma_1(x, b) \triangleq \frac{1}{2}bR^{-1}g^T V_x \quad (9)$$

$$\sigma_2(x, b) \triangleq R^{-1/2}\sqrt{\frac{1}{4}b^2 V_x^T gR^{-1}g^T V_x - l - bV_x^T f} \quad (10)$$

we can take the union of $S_b(x)$ over all $b \geq \underline{b}(x)$ for all $x \neq 0$ to obtain

$$\begin{aligned} S(x) &= \cup_{b \geq \underline{b}(x)} S_b(x) \\ &= \{-\sigma_1(x, b) + \sigma_2(x, b)\nu : b > \underline{b}(x), \|\nu\| < 1\}. \end{aligned} \quad (11)$$

Lemma 6 guarantees that $S(x)$ is nonempty for $x \neq 0$. In addition, we have shown that the satisficing set can be parameterized by the selection functions $b : \mathbb{R}^n \to \mathbb{R}$ and $\nu : \mathbb{R}^n \to \mathbb{R}^m$, where $b(x) \geq \underline{b}(x)$ and $\|\nu(x)\| < 1$.

III. Satisficing Controls

In this section, we define satisficing controls to be locally Lipschitz selections from the satisficing set. It is shown that satisficing controls asymptotically stabilize the closed-loop system.

Definition 7: The mapping $k : \mathbb{R}^n \to \mathbb{R}^m$ is called a *satisficing control* for system (1) if

1) $k(0) = 0$;
2) $k(x) \in S(x)$ for each $x \in \mathbb{R}^n \setminus \{0\}$;
3) k is locally Lipschitz on $\mathbb{R}^n \setminus \{0\}$.

Theorem 8: If $k(x)$ is a satisficing control for system (1), then the closed loop system $\dot{x} = f + gk$ is globally asymptotically stable.

The proof uses the following lemma which is stated as an exercise in [24, p. 247].

Lemma 9: Suppose that $F(x)$ is locally Lipschitz on $\mathbb{R}^n \setminus \{0\}$ and $F(0) = 0$. If there exist a continuously differentiable, positive–definite, radially unbounded function $V : \mathbb{R}^n \to \mathbb{R}$ such that $V_x^T F < 0$ for all $x \in \mathbb{R}^n \setminus \{0\}$, then the origin is globally asymptotically stable.

Proof of Theorem 8: Since f, g, and k are locally Lipschitz on $\mathbb{R}^n \setminus \{0\}$, $f + gk$ is also locally Lipschitz on $\mathbb{R}^n \setminus \{0\}$ [1]. Since $f(0) = 0$ and $k(0) = 0$, $(f + gk)(0) = 0$. Since $k(x) \in S(x)$ for all $x \neq 0$

$$V_x^T(f + gk) < -\frac{1}{b}\left(l + k^T Rk\right) \leq 0.$$

The theorem therefore follows from Lemma 9. ∎

The next theorem parameterizes the set of satisficing controls via two locally Lipschitz selection functions.

Theorem 10: If

1) V is a CLF for system (1);
2) $\nu : \mathbb{R}^n \to \mathbb{R}^m$ is locally Lipschitz on $\mathbb{R}^n \setminus \{0\}$ and satisfies $\|\nu(x)\| < 1$;
3) $b : \mathbb{R}^n \to \mathbb{R}^+$ is locally Lipschitz on $\mathbb{R}^n \setminus \{0\}$ and satisfies $\underline{b}(x) < b(x)$, where $\underline{b}$ is defined by(7);

then

$$k(x) = \begin{cases} 0, & \text{if } x = 0 \\ -\sigma_1(x, b(x)) + \sigma_2(x, b(x))\nu(x), & \text{otherwise} \end{cases} \tag{12}$$

where σ_1 and σ_2 are given by (9) and (10), is a satisficing control for system (1). Furthermore, if V satisfies the small control property, and in a neighborhood close to the origin, $b(x) = \eta(x)\underline{b}(x)$ where $1 < \eta(x) < N < \infty$, and $R(x)$ satisfies

$$\underline{r}I \leq R(x) \leq \bar{r}I \qquad \forall\, x$$

where $\underline{r}$ and $\bar{r}$ are positive constants, then k is continuous at the origin.

Proof: From (12) and the definition of $S(x)$, it is clear that $k(0) = 0$ and $k(x) \in S(x)$ for all $x \neq 0$. Since the multiplication, addition, and composition of locally Lipschitz functions is locally Lipschitz, σ_1 and σ_2 are locally Lipschitz on $\mathbb{R}^n \setminus \{0\}$. Therefore, $k(x)$ is a satisficing control.

Suppose that V satisfies the small control property. We will show that near the origin $k(x)$ is bounded above by a continuous function that is zero at the origin. Since

$$\begin{aligned} \|k(x)\| &\leq \|\sigma_1(x, b(x))\| + \|\sigma_2(x, b(x))\nu(x)\| \\ &\leq \|\sigma_1(x, b(x))\| + \|\sigma_2(x, b(x))\| \end{aligned}$$

we will derive bounds separately on σ_1 and σ_2. To simplify the notation let $a = V_x^T f$ and $d = g^T V_x$.

First, consider the case when $d = g^T V_x \neq 0$. Since V satisfies the small control property, there exist a continuous $\alpha_c(x)$ with $\alpha_c(0) = 0$ such that

$$a + d^T \alpha_c < 0 \qquad \forall\, x \neq 0$$

which implies that $|a| / \|d\| < \|\alpha_c\|$. Therefore, σ_1 can be bounded as follows:

$$\begin{aligned} \|\sigma_1\| &= \left\| \frac{1}{2}\eta\underline{b}R^{-1}d \right\| \\ &\leq N\frac{1}{\underline{r}} \left\| \frac{1}{2}\underline{b}d \right\| \\ &\leq N\frac{1}{\underline{r}} \left\| \left(\frac{a + \sqrt{a^2 + ld^TR^{-1}d}}{d^TR^{-1}d} \right) d \right\| \\ &\leq N\frac{1}{\underline{r}} \left[\frac{|a|\,\|d\|}{\frac{1}{\bar{r}}\|d\|^2} + \frac{\|d\|\sqrt{a^2 + l\frac{1}{\underline{r}}\|d\|^2}}{\frac{1}{\bar{r}}\|d\|^2} \right] \\ &\leq \frac{N\bar{r}}{\underline{r}} \left[\|\alpha_c\| + \sqrt{\|\alpha_c\|^2 + l\frac{1}{\underline{r}}} \right]. \end{aligned}$$

Similarly, σ_2 can be bounded as

$$\begin{aligned} \|\sigma_2\|^2 &\leq \frac{1}{\underline{r}} \left\| \frac{1}{2}\eta^2\underline{b}^2 d^TR^{-1}d - l - \eta\underline{b}a \right\| \\ &\leq \frac{1}{\underline{r}} \left\| (\eta^2 - 2\eta)\left(\frac{a^2}{d^TR^{-1}d} + \frac{a\sqrt{a^2 + ld^TR^{-1}d}}{d^TR^{-1}d} \right) \right. \\ &\qquad \left. + \left(\eta^2 - 1\right) l \right\| \\ &\leq \left(N^2 - 2N\right)\frac{1}{\underline{r}} \left[\bar{r}\,\|\alpha_c\|^2 + \bar{r}\,\|\alpha_c\| \sqrt{\|\alpha_c\|^2 + l\frac{1}{\underline{r}}} \right] \\ &\qquad + \frac{1}{\underline{r}}(N^2 - 1)l. \end{aligned}$$

Alternatively, assume that $d = g^T V_x = 0$. Then, clearly

$$\|\sigma_1\| = 0$$

and σ_2 can be bounded as

$$\begin{aligned} \|\sigma_2\|^2 &\leq \frac{1}{\underline{r}^2}\|-l - ba\| \\ &\leq \frac{1}{\underline{r}^2}\|-l - \eta\underline{b}a\| \\ &\leq \frac{1}{\underline{r}^2}\left\| -l - \eta\left(-\frac{l}{a}\right)a \right\| \\ &\leq \frac{1}{\underline{r}^2}(N - 1)l. \end{aligned}$$

Therefore, k is continuous at $x = 0$. ■

IV. Robustly Satisficing Controls

We have shown that all satisficing controls provide asymptotic stability and that for a given CLF V, the set of controls generated by V are parameterized by two locally Lipschitz functions $b : \mathbb{R}^n \to \mathbb{R}$ and $\nu : \mathbb{R}^n \to \mathbb{R}^m$. In this section, we will show that if the selection process is limited to a convex subset of $S(x)$, which we call the robust satisficing set, that the resulting control strategies have Kalman-like gain margins.

Definition 11: An asymptotically stabilizing control law, $u = q(x)$, has stability margins (m_1, m_2) where

$$-1 \leq m_1 < m_2 \leq \infty$$

if for every $\alpha \in (m_1, m_2)$, $u = (1+\alpha)q(x)$, also asymptotically stabilizes the system.

In particular, it was shown in [18], [19], [25], and [26] that optimal control laws have stability margins of $(-1/2, \infty)$. In fact one of the primary motivations for considering inverse optimal control laws, is that they have guaranteed stability margins of $(-1/2, \infty)$[9], [11]. In this section, we will show that selection from a well defined subset of $S(x)$ results in feedback strategies with stability margins of $(-1/2, \infty)$.

Definition 12: The *robust satisficing set* for system (1), denoted $S_R(x)$, is defined as

$$\begin{aligned} S_R(x) &= \left\{ u \in S(x) : V_x^T g R^{-1/2}\nu \leq 0 \right\} \\ &= \Big\{ -\sigma_1(x, b) + \sigma_2(x, b)\nu : \\ &\qquad b > \underline{b}(x), \|\nu\| < 1, V_x^T g R^{-1/2}\nu \leq 0 \Big\} \end{aligned}$$

where σ_1 and σ_2 are given in (9) and (10).

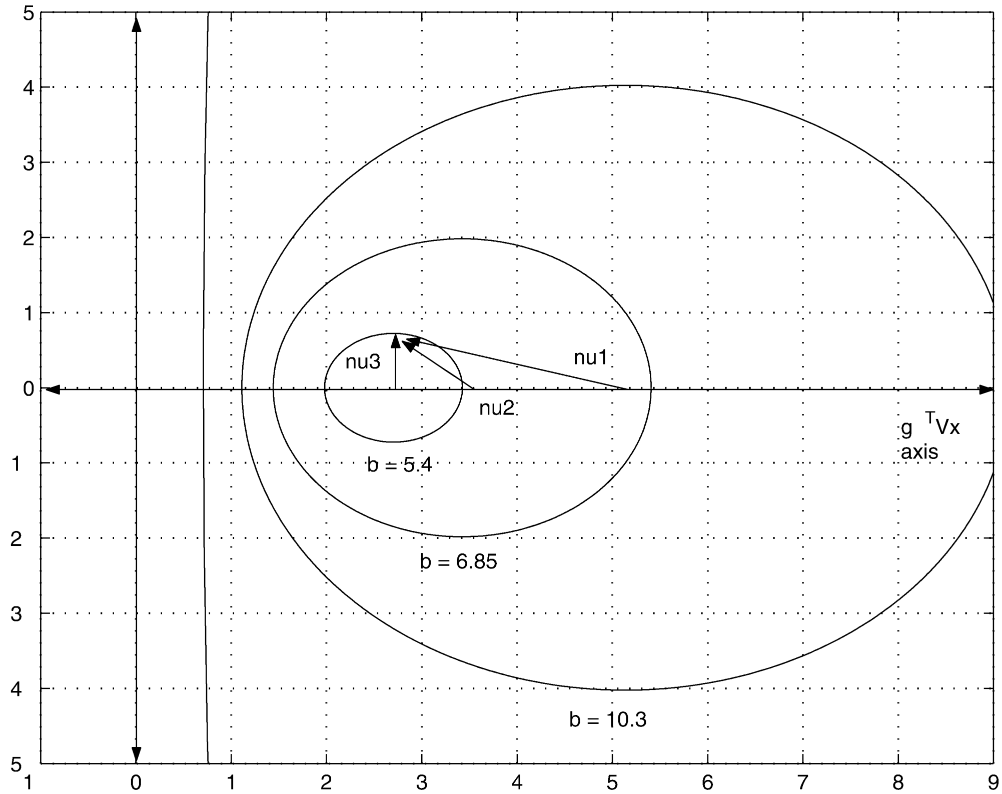

Fig. 2. Satisficing set overparameterization.

Definition 13: The mapping $k_R : \mathbb{R}^n \to \mathbb{R}^m$ is called a *robustly satisficing control* for (1) if

1) $k_R(0) = 0$;
2) $k_R(x) \in S_R(x)$ for each $x \in \mathbb{R}^n \setminus \{0\}$;
3) k_R is locally Lipschitz on $\mathbb{R}^n \setminus \{0\}$.

Theorem 14: If k_R is a robustly satisficing control for (1), then it has stability margins equal to $(-1/2, \infty)$.

Proof: By definition, k_R satisfies the following inequality:

$$V_x^T f + V_x^T g k_R < -\frac{1}{b} l - \frac{1}{b} k_R^T R k_R.$$

Adding $\alpha V_x^T gk$ to both sides gives

$$V_x^T f + (1+\alpha) V_x^T g k_R < -\frac{1}{b} l - \frac{1}{b} k_R^T R k_R + \alpha V_x^T g k_R. \quad (13)$$

A sufficient condition for asymptotic stability is that the right-hand side of (13) be nonpositive for all $x \in \mathbb{R}^n$. We need to show that this condition is satisfied for all $\alpha \in (-1/2, \infty)$. Noting that

$$k_R(x) = -\sigma_1 + \sigma_2 \nu = -\frac{1}{2} b R^{-1} g^T V_x + \hat{\sigma}_2 R^{-1/2} \nu$$

where

$$\hat{\sigma}_2 = \sqrt{\frac{1}{4} b^2 V_x^T g R^{-1} g^T V_x - l - b V_x^T f}$$

gives

$$\begin{aligned} -\frac{1}{b} l - \frac{1}{b} k_R^T R k_R + \alpha V_x^T g k_R &= -\frac{1}{b}\left(l + \hat{\sigma}_2^2 \nu^T \nu\right) - \frac{b}{2}\left(\frac{1}{2} + \alpha\right) V_x^T g R^{-1} g^T V_x \\ &\quad + \hat{\sigma}_2 (1+\alpha) V_x^T g R^{-1/2} \nu. \end{aligned}$$

The first term is always nonpositive. The second term is nonpositive if $\alpha \in (-1/2, \infty)$, in which case the third term is nonpositive if $V_x^T g R^{-1/2} \nu \leq 0$. ■

The parameterization of the satisficing set in terms of (b, ν) is a redundant parameterization since $S(x) \subset \mathbb{R}^m$ and $(b, \nu^T)^T \in \mathbb{R}^{m+1}$. Therefore, if $w \in S(x)$, there may be many (b, ν) pairs such that $w = -\sigma_1(b) + \sigma_2(b)\nu$. For example, if $V_x^T f = .7$, $V_x^T g = [-1\ 0]^T$, $R = I$, and $l(x) = 3$, then Fig. 2 shows three (b, ν) pairs corresponding to a single point in $S(x)$. Note that as b increases, the size of the ellipsoid determined by $\sigma_2(x, b)$ grows. In addition, the center of the ellipsoid, determined by $\sigma_1(x, b)$ moves in the direction of $R^{-1} g^T V_x$. While $S(x)$ contains the entire ellipsoid for every b, $S_R(x)$ only contains half of that ellipsoid. For the values given previously, the robust satisficing set is shown in Fig. 3. S_R is to the right of the shown boundary, where the vector $-R^{-1} g^T V_x = -R^{-1/2} g^T V_x$ lies along the x axis. Note that the intersection of the boundary of S_R with the $-g^T V_x$ vector corresponds to $b = \underline{b}$, in which case $\sigma_2(b) = 0$ and any ν gives the same control value. Fig. 3 suggests a minimal parameterization of S_R, where ν is always chosen perpendicular to $-R^{-1/2} g^T V_x$.

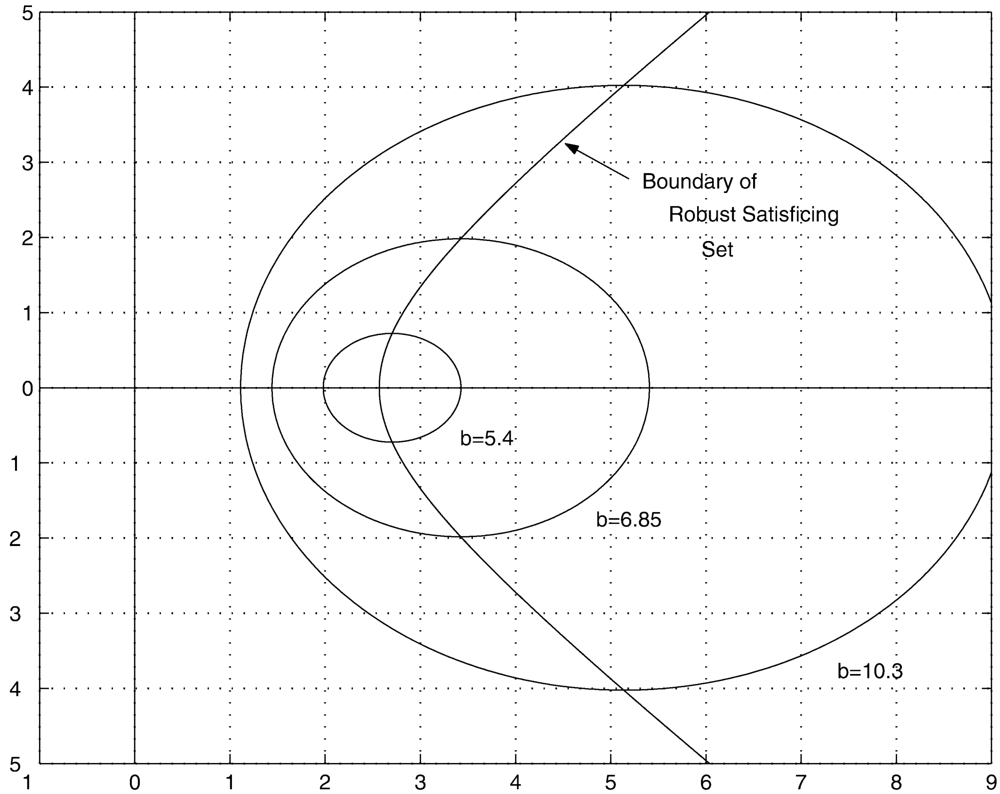

Fig. 3. Robust satisficing set.

Theorem 15: Suppose that $R^{-1/2}g^TV_x \neq 0$, the SVD of $V_x^TgR^{-1/2}$ is given by

$$V_x^TgR^{-1/2} = U_1\begin{pmatrix}\Sigma & 0\end{pmatrix}\begin{pmatrix}V_1^T\\ V_2^T\end{pmatrix}$$

and the null space of $R^{-1/2}g^TV_x$ is denoted by $\mathcal{N}(R^{-1/2}g^TV_x)$. Then $w \in S_R$ if and only if there exists a unique parameterization $(\hat{b}, \hat{\nu}) \in \mathbb{R}^m$ where $\hat{b} > \underline{b}$

$$\hat{\nu} \in \mathcal{N}\left(R^{-1/2}g^TV_x\right) \subset \mathbb{R}^{m-1}$$

and $\|\hat{\nu}\| \leq \left\|R^{-1/2}\right\|$, such that

$$w = -\sigma_1(\hat{b}) + \sigma_2(\hat{b})R^{-1/2}V_2\hat{\nu}.$$

Proof: To show necessity, let $w \in S_R$. By the orthogonality theorem [23], w can be uniquely written as

$$w = w_{\|} + w_{\perp}$$

where $w_{\|} \in \operatorname{span}(R^{-1/2}g^TV_x)$ and $w_{\perp} \in \mathcal{N}(R^{-1/2}g^TV_x)$. Since

$$S_R \cap \operatorname{span}(R^{-1/2}g^TV_x) = \left\{-\frac{b}{2}R^{-1/2}g^TV_x : b > \hat{b}\right\}$$

there exist a unique $\hat{b}$ such that $-\sigma_1(\hat{b}) = w_{\|}$. Since $w \in S_R \subset S$, given $\hat{b}$, there exists a unique $\tilde{\nu} \in B(\mathbb{R}^m)$ such that $w_{\perp} = \sigma_2(\hat{b})R^{-1/2}\tilde{\nu}$. Let $\hat{\nu} = V_2^TR^{-1/2}\tilde{\nu}$ (note that $\hat{\nu}$ is uniquely defined), then

$$\begin{aligned}\|\hat{\nu}\| &= \left\|V_2^TR^{-1/2}\tilde{\nu}\right\|\\ &= \left\|R^{-1/2}\tilde{\nu}\right\|\\ &\leq \left\|R^{-1/2}\right\|\|\tilde{\nu}\|\\ &\leq \left\|R^{-1/2}\right\|.\end{aligned}$$

To show sufficiency, suppose that there is a unique parameterization $(\hat{b}, \hat{\nu})$ such that

$$w = -\sigma_1(\hat{b}) + \sigma_2(\hat{b})R^{-1/2}V_2\hat{\nu}.$$

Let $\nu = R^{1/2}V_2\hat{\nu} \in \mathbb{R}^m$. Since V_2 is unitary, $\|\nu\| \leq \left\|R^{1/2}\right\|\|V_2\hat{\nu}\| = \left\|R^{1/2}\right\|\|\hat{\nu}\| \leq 1$. Since $\hat{b} > \underline{b}$, $w \in S(x)$. In addition, $V_x^TgR^{-1}\hat{\nu} = V_x^TgR^{-1/2}V_2^TR^{-1/2}\nu = 0$. Therefore, $w \in S_R$. ■

The next theorem parameterizes the set of robustly satisficing controls for (1).

Theorem 16: If

1) V is a CLF for system (1);
2) $\nu : \mathbb{R}^n \to \mathbb{R}^{m-1}$ is locally Lipschitz on $\mathbb{R}^n \setminus \{0\}$ and satisfies $\|\nu(x)\| \leq \left\|R^{-1/2}\right\|$;
3) $b : \mathbb{R}^n \to \mathbb{R}^+$ is locally Lipschitz on $\mathbb{R}^n \setminus \{0\}$ and satisfies $b(x) > \underline{b}(x)$, where $\underline{b}$ is given in (7);

4) $V_2(x) : \mathbb{R}^n \to \mathbb{R}^{m\times(m-1)}$ is a locally Lipschitz orthogonal matrix that spans the null space of $V_x^T gR^{-1/2}$; then

$$k_R(x) = \begin{cases} 0, & \text{if } x=0 \\ -\sigma_1(x,b(x)) + \sigma_2(x,b(x))V_2(x)\nu(x), & \text{otherwise} \end{cases} \tag{14}$$

is a robustly satisficing control for (1).

V. Inverse Optimality

In this section, we show that all robustly satisficing control laws are inverse optimal.

Definition 17: A control law $q(x)$ that asymptotically stabilizes the system $\dot{x} = f + gq$ is said to be **inverse optimal** if there exists a positive–definite, radially unbounded $W(x)$, a positive–definite function $m(x)$ and a symmetric positive–definite function $\hat{R}(x)$ such that

$$q(x) = -\frac{1}{2}\hat{R}^{-1}(x)g^T W_x$$

where W satisfies the Hamilton–Jacobi equation

$$W_x^T f + m - \frac{1}{4}W_x^T g\hat{R}^{-1}g^T W_x = 0$$

point-wise at each x.

The following lemma sets the stage for our main result.

Lemma 18: If $g^T V_x \neq 0$ then every robustly satisficing control can be written as $k_R(x) = -(1/2)\tilde{R}^{-1}(x)g^T V_x$, where $\tilde{R}(x)$ is a positive–definite matrix function.

Proof: Let $v_f = -(\underline{b}+1/2)g^T V_x$. $k_R \in S_R$ implies that

$$v_f^T k_R = \frac{(\underline{b}+1)b}{4}V_x^T gR^{-1}g^T V_x - \frac{(\underline{b}+1)\hat{\sigma_2}}{2}V_x^T gR^{-1/2}\nu > 0$$

since $V_x^T gR^{-1/2}\nu > 0$. Therefore, since $v_f^T k_R = \|v_f\|\,\|k_R\|\cos\theta$, where θ is the angle between v_f and k_R, we know that $\cos\theta > 0$ or that $|\theta| < \pi/2$. For the trivial case where $\theta = 0$, we can simply let $\tilde{R}^{-1}(x) = \|k_R\|/(\underline{b}+1)\,\|v_f\|$. Suppose however, that k_R is not parallel to v_f. Our objective is to construct a matrix $P = P^T > 0$ such that $Pv_f = k_R$ for all $x \neq 0$.

We begin by defining a new orthonormal basis for $\mathbb{R}^m$. The first basis vector, $b_1 = k_R/\|k_R\|$, is a unit vector in the direction of k_R. The second basis vector

$$b_2 = \frac{-\frac{v_f^T k_R}{k_R^T k_R}k_R + v_f}{\left\|-\frac{v_f^T k_R}{k_R^T k_R}k_R + v_f\right\|}$$

is a unit vector lying in the plane spanned by k_R and v_f with b_2 orthogonal to k_R. The rest of the new basis vectors $(b_3 \ldots b_m)$ can be generated with a Gram–Schmidt algorithm such that $(b_1,\ldots,b_m)$ constitute a complete orthonormal basis. Define the transformation matrix, $T = [b_1,\, b_2,\, \ldots,\, b_m]$, and note that $T^{-1} = T^T$.

In this new coordinate frame, the vector k_R becomes $\hat{k}_R = [\|k_R\|,\, 0,\, \ldots,\, 0]$. Likewise, v_f becomes $\hat{v}_f = \|v_f\|\,[\cos\theta,\, \sin\theta,\, 0,\, \ldots,\, 0]$. We will now construct $R_o = R_o^T > 0$ to rotate $\hat{v}_f$ into $\hat{k}_R$. Since all but the first two elements of v_f and k_R are zero, let $R_o = \begin{bmatrix}\hat{R}_o & 0\\ 0 & I\end{bmatrix}$, where $\hat{R}_o = \begin{bmatrix}\alpha & \delta\\ \delta & \beta\end{bmatrix}$. Therefore, we must have that

$$\begin{bmatrix}\|k_R\|\\ 0\end{bmatrix} = \begin{bmatrix}\alpha & \delta\\ \delta & \beta\end{bmatrix}\|v_f\|\begin{bmatrix}\cos\theta\\ \sin\theta\end{bmatrix}$$

which implies the following equations:

$$\alpha\cos\theta + \delta\sin\theta = \frac{\|k_R\|}{\|v_f\|}$$
$$\delta\cos\theta + \beta\sin\theta = 0.$$

Additionally, the positive definiteness of R_o requires that

$$\alpha\beta - \delta^2 > 0, \text{ and } \alpha > 0.$$

Letting $\delta = -\sin\theta$, $\beta = \cos\theta$, and

$$\alpha = \frac{\frac{\|k_R\|}{\|v_f\|} + \sin^2\theta}{\cos\theta}$$

where $|\theta| < \pi/2$ ensures that α is well defined, we see that all of the equations are satisfied. Define $P = T^T R_o T$, then

$$Pv_f = T^T R_o T v_f = T^T R_o \hat{v}_f = T^T\,\hat{k}_R = k_R.$$

By construction $P = P^T > 0$. Letting $\tilde{R}(x) = (\underline{b}+1)P^{-1}(x)$ shows that any k_R can be written in the desired form. ■

We can now show the following result.

Theorem 19: Every robustly satisficing control law is inverse optimal.

Proof: The proof follows the arguments in [9, p. 108]. Let $k_R(x)$ be a robustly satisficing control law. From Lemma 18 , there exists a positive–definite matrix function $\tilde{R}$ such that $k_R = -(1/2)\tilde{R}^{-1}g^T V_x$. Since $(1/2)k_R$ is asymptotically stabilizing, we know

$$V_x^T f - \frac{1}{4}V_x^T g\tilde{R}^{-1}g^T V_x < 0.$$

Choosing

$$m(x) = -V_x^T f + \frac{1}{4}V_x^T g\tilde{R}^{-1}g^T V_x > 0$$

$W(x) = V(x)$, and $\hat{R}(x) = \tilde{R}(x)$, it is straightforward to verify that the Hamilton–Jacobi equation

$$W_x^T f - \frac{1}{4}W_x^T g\hat{R}^{-1}g^T W_x + m(x) = 0$$

is satisfied at all x. ■

VI. Universal Formulas

Theorem 10 suggests a new class of universal formulas. In particular, any locally Lipschitz selection function $k(x) \in S(x)$ represents a universal formula given the CLF V. If in addition,

$k(x)$ is selected from $S_R(x)$, then inverse optimality and optimal robustness margins are ensured. Attention can therefore be turned to optimizing performance via the selection functions $b(x)$ and $\nu(x)$.

Definition 20: A **universal stabilizing formula** for system (1) is a continuous function $\alpha : \mathbb{R} \times \mathbb{R}^m \to \mathbb{R}^m$ such that for any CLF V, the following statements hold:

- $V_x^T f + V_x^T g\alpha(V_x^T f, V_x^T g) < 0 \qquad \forall x \neq 0$;
- $\forall \epsilon > 0, \exists \delta > 0$, such that

$$\left.\begin{array}{r} V_x^T f < \delta \left\|V_x^T g\right\| \\ \left|V_x^T f\right| < \delta \\ \left\|V_x^T g\right\| < \delta \end{array}\right\} \Longrightarrow \left|\alpha\left(V_x^T f, V_x^T g\right)\right| < \epsilon.$$

The next theorem shows that the functions $b(x)$ and $\nu(x)$ used in Theorem 10 to parameterize the set of satisficing controls, also parameterizes a new family of universal formulas.

Theorem 21: Assume that the hypothesis of Theorem 10 is satisfied, then (12) is a universal stabilizing formula for (1).

Proof: The fact that $V_x^T f + V_x^T gk(x) < 0$ for all $x \neq 0$ follows from the fact that $k(x) \in S(x)$.

In the proof of Theorem 10, we showed that $\|k\| \leq \|\sigma_1\| + \|\sigma_2\|$, where

$$\|\sigma_1\| \leq \frac{N\bar{r}}{\underline{r}}\left[\left(\frac{a}{\|d\|}\right) + \sqrt{\left(\frac{a}{\|d\|}\right)^2 + l\frac{1}{\underline{r}}}\right]$$

$$\begin{aligned}\|\sigma_2\|^2 \leq &\left(N^2 - 2N\right)\frac{1}{\underline{r}} \\ &\times\left[\bar{r}\left(\frac{a}{\|d\|}\right)^2 + \bar{r}\left(\frac{a}{\|d\|}\right)\sqrt{\left(\frac{a}{\|d\|}\right)^2 + l\frac{1}{\underline{r}}}\right] \\ &+ \frac{1}{\underline{r}}\left(N^2 - 1\right)l.\end{aligned}$$

Assuming that $a < \delta\|d\|$, $|a| < \delta$, and $\|d\| < \delta$ gives

$$\|\sigma_1\| \leq \frac{N\bar{r}}{\underline{r}}\left[\delta + \sqrt{\delta^2 + l\frac{1}{\underline{r}}}\right]$$

$$\begin{aligned}\|\sigma_2\|^2 \leq &\left(N^2 - 2N\right)\frac{1}{\underline{r}}\left[\bar{r}\delta^2 + \bar{r}\delta\sqrt{\delta^2 + l\frac{1}{\underline{r}}}\right] \\ &+ \frac{1}{\underline{r}}\left(N^2 - 1\right)l.\end{aligned}$$

Since l is continuous, for sufficiently small δ, $\|\sigma_1\|$ and $\|\sigma_2\|$ can be bounded by $\epsilon/2$. ■

Corollary 22: If the hypothesis of Theorem 16 is satisfied, then (14) is a universal stabilizing formula for system (1) that is both inverse optimal and has gain margins equal to $(-1/2, \infty)$.

Two well-known universal formulas are Sontag's formula [7] and Freeman and Kokotovic's min-norm formula [11]. We will demonstrate that both of these formulas are subsumed in our approach. As described in [7], Sontag's formula is given by

$$k(x) = \begin{cases} 0, & \text{if } V_x^T g = 0 \\ -\frac{V_x^T f + \sqrt{(V_x^T f)^2 + (V_x^T gg^T V_x)^2}}{V_x^T gg^T V_x} g^T V_x, & \text{otherwise .} \end{cases} \tag{15}$$

Note that this is equal to (14) when $R = I$, $l = V_x^T gg^T V_x$, $\nu(x) = 0$, and $b(x) = \underline{b}(x)$. Similarly, Freeman and Kokotovic's min-norm formula [11] is given by

$$k(x) = \begin{cases} 0 & \text{if } V_x^T f \leq 0 \\ \frac{-V_x^T f}{V_x^T gg^T V_x} g^T V_x, & \text{otherwise} \end{cases} \tag{16}$$

which is equal to (14) when $R = 2I$, $l(x) = 0$, $\nu(x) = 0$, and $b(x) = \underline{b}(x)$. Therefore, (14) can be thought of as a generalization of both Sontag's formula and Freeman and Kokotovic's min-norm formula.

We have shown that Theorem 10 parameterizes a new class of universal formulas. One may wonder about the completeness of this parameterization, i.e., are there universal formulas that are not generated by Theorem 10 . Our final result is that the parameterization is complete.

Theorem 23: If α is a universal formula that is locally Lipschitz on $\mathbb{R}^n \setminus \{0\}$ and $\alpha(0) = 0$, then α is a satisficing control.

Proof: Conditions 1 and 3 from Definition 7 are trivially satisfied and it remains to show that $\alpha(x) \in S(x)$ at every $x \neq 0$ for some choice of l and R. This can be done by showing that α satisfies the fundamental satisficing condition $V_x^T f + V_x^T g\alpha < -1/b(x)(l + \alpha^T R(x)\alpha)$ at every $x \neq 0$ with $b(x) > \underline{b}(x) \geq 0$. Since α is a universal formula we know that

$$\begin{aligned} &V_x^T f + V_x^T g\alpha < 0 \\ \iff &V_x^T f + V_x^T g\alpha = -N(x), (N(x) > 0). \end{aligned}$$

Letting $l(x) = 0$ and $R(x) = I$, the satisficing condition requires that $-N(x) < -(1/b(x))\alpha^T\alpha$ which is true if and only if $b(x) > \alpha^T\alpha/N(x)$. The selectivity function $b(x)$ must also satisfy $b(x) > \underline{b}(x) \geq 0$. Letting $b(x) > \max\left(\underline{b}(x), \alpha^T\alpha/N(x)\right)$ completes the proof. ■

VII. EXAMPLES

This section presents two examples that illustrate the potential of satisficing controls. The first example illustrates the application of the ideas to linear systems. The second example illustrates the ideas for a second-order nonlinear system with two inputs.

A. Linear Systems

Consider the linear system given by

$$\dot{x} = Ax + Bu \tag{17}$$

where (A, B) is assumed to be controllable. Let Q be a symmetric, positive semi-definite matrix such that $(A, Q^{1/2})$ is observable, let R be a symmetric positive–definite matrix, and let P be the symmetric positive–definite solution to the Riccati equation

$$PA + A^T P + Q - PBR^{-1}B^T P = 0. \tag{18}$$

(A, B) controllable implies that $V(x) = x^T Px$ is a CLF for system (17) since

$$\inf_u \dot{V} = \inf_u\{-x^T Qx + x^T PB\left(B^T Px + 2u\right)\} < 0$$

for all $x \neq 0$.

From (7) we see after some algebra that for linear systems, $\underline{b}(x)$ is shown in (19) at the bottom of the page. If we choose $l(x) = x^T Qx$, then the expression for $\underline{b}$ simplifies further as $\underline{b} = 1$.

If we let the rejectability control penalty matrix function be equal to R [in the linear quadratic regulator (LQR) sense], then the satisficing parametrization for (17) is

$$u_{\text{satisficing}}(x) = -bR^{-1}B^T Px + R^{-1/2}\nu\sqrt{\frac{1}{4}b^2 V_x^T gR^{-1}g^T V_x - l - bV_x^T f} \quad (20)$$

where the satisficing parameters satisfy $b(x) \geq 1$ and $\|\nu(x)\| \leq 1$.

Note that if $b(x) = 1$ and $\nu(x) = 0$, then $k(x) = -R^{-1}B^T Px$ which is the optimal controller associated with the performance index

$$J = \int_0^\infty (x^T Qx + u^T Ru)dt.$$

Also, note that by making the selection parameters b and ν functions of the state, satisficing controls become nonlinear functions of the state .

As a concrete example, consider the double integrator system

$$A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \quad B \triangleq \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

with $Q = I_2$ and $R = 1$. The solution to the Riccati equation is given by

$$P = \begin{pmatrix} 1.732 & 1 \\ 1 & 1.732 \end{pmatrix}$$

resulting in the CLF $V(x) = x^T Px$.

Fig. 4 shows the phase portrait of the closed loop system using four different controllers. The phase portrait in the upper left corresponds to the LQR controller. The phase portrait in the upper right corresponds to Sontag's formula. It may be desirable in some applications to have high-gain, nonlinear response in certain regions of the state space, but linear response in other regions. This can be achieved by judicious choices of $b(x)$ and $\nu(x)$. The lower left hand phase plot shows the response to the satisficing control with

$$b(x) = \begin{cases} 1 + 5\|x\|, & \text{if } 1 \leq \|x\| \leq 3 \\ 1, & \text{otherwise} \end{cases} \quad (21)$$

$$\nu(x) = 0. \quad (22)$$

Note that the response is high-gain in the region $1 \leq \|x\| \leq 3$, but retains the LQR response on the rest of the state–space. It is interesting to note, that since $b(x)$ increases the gain in the direction of $R^{-1}g^T V_x$, the direction of the eigenspaces are retained in the nonlinear region. The direction of the eigenspaces can be shaped by the function $\nu(x)$. In the lower right-hand plot, $b(x)$ is chosen similarly to the lower left hand plot, but $\nu(x)$ is chosen to minimize the rate of decrease along the function $W(x) = x^T x$, i.e.,

$$\nu(x) = -\text{sign}\left(\sigma_2(x)x^T B\right).$$

Note that the apparent eigenspaces align with the eigenspaces of the identity matrix as we might expect from $W(x)$.

B. Nonlinear Example

Consider the system

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} = \begin{pmatrix} -x_1^3 + x_1x_2 \\ \sin(x_1) + x_2^2 \end{pmatrix} + \begin{pmatrix} x_2 & 0 \\ 0 & x_1 \end{pmatrix}\begin{pmatrix} u_1 \\ u_2 \end{pmatrix}. \quad (23)$$

It can be shown that $V(x) = x^T Px$ is a CLF for the system if $P = P^T$ is positive definite and $p_{12} \neq 0$. Letting

$$P = \begin{pmatrix} 1.73 & 1 \\ 1 & 1.73 \end{pmatrix}$$

the resulting phase portrait using Sontag's formula (15) is shown in the upper left subplot of Fig. 5. The phase portrait of the system using the min-norm control

$$k(x) = \begin{cases} 0, & \text{if } V_x^T f + \ell \leq 0 \\ \frac{-V_x^T f - \ell}{V_x^T gg^T V_x} g^T V_x, & \text{otherwise} \end{cases}$$

where $\ell(x) = x^T x$, is shown in the upper right subplot of Fig. 5

A heuristic technique that is both easy to tune and is known to give good results is the state dependent Riccati equation (SDRE) technique [27]. The basic idea is to factor the term $f(x)$ in (1) as $f(x) = A(x)x$, and then to compute the linear quadratic control gain at each x associated with the system $(A(x), g(x), Q(x), R(x))$, where $Q(x)$ and $R(x)$ are state dependent weighting matrices. The drawback with the SDRE technique is that conditions are not currently known that guarantee that the technique results in a stable and robust closed-loop system. The bottom left subplot of Fig. 5 shows the phase portrait that results from applying the SDRE technique to (23) where

$$A(x) = \begin{pmatrix} -x_1^2 & x_1 \\ \frac{\sin(x_1)}{x_1} & x_2 \end{pmatrix}$$

$Q(x) = I$, and $R(x) = I$.

The satisficing technique can be used to retain the essential behavior of the SDRE controller while ensuring closed-loop stability and robustness properties. If $k_{\text{SDRE}}(x)$ is the SDRE controller at state x, then $b(x)$ and $\nu(x)$ are chosen according to the following optimization problem:

$$k(x) = \arg\min_{u \in S_R(x)} \|u - k_{\text{SDRE}}(x)\|. \quad (24)$$

$$\underline{b}(x) = \begin{cases} \frac{l}{x^T Qx}, & \text{if } x^T PB = 0 \\ \frac{x^T PBR^{-1}B^T Px - x^T Qx + \sqrt{(x^T PBR^{-1}B^T Px - x^T Qx)^2 + 4lx^T PBR^{-1}B^T Px}}{2x^T PBR^{-1}B^T Px}, & \text{otherwise.} \end{cases} \quad (19)$$

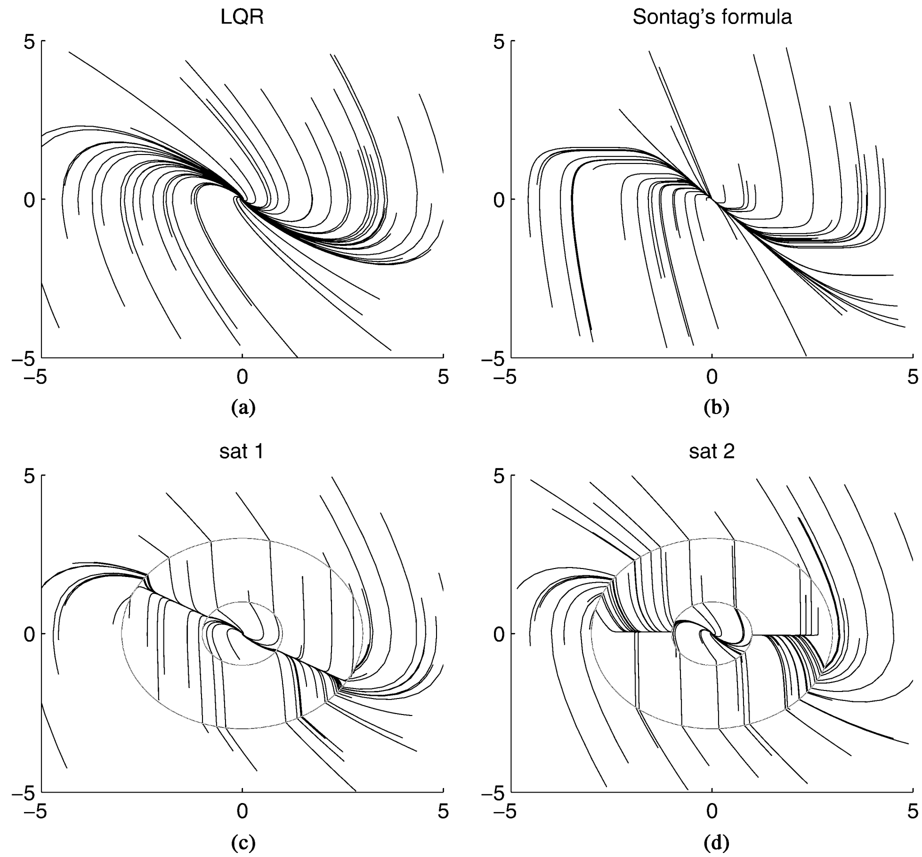

Fig. 4. Phase portrait for a double integrator systems under the control of (a) LQR, (b) Sontag's formula, (c) satisficing #1, and (d) satisficing #2.

Since k is a robust satisficing control, (24) is an inverse optimal universal formula that retains the qualitative performance of the SDRE controller. The phase portrait of the closed loop system using (24) is shown in the bottom right subplot of Fig. 5.

VIII. Discussion

The main results contained in this paper can be summarized as follows. Given an affine nonlinear system $\dot{x} = f + gu$ and an associated CLF V, there exists a convex set $S(x)$ called the satisficing set, which is given by (11), and which is nonempty for each $x \neq 0$. Furthermore, this set is completely parameterized by two selection functions $b(x)$ and $\nu(x)$. Theorem 10 guarantees that if these selection functions are locally Lipschitz and satisfy the constraints $b(x) > \underline{b}(x)$ and $\|\nu(x)\| \leq 1$, then the resulting control strategy, given by (12), globally asymptotically stabilizes the system. In Definition 12 the robust satisficing set, $S_R(x)$ is defined as a convex subset of $S(x)$, and it was shown that $S_R(x)$ is again parameterized by selection functions $b(x)$ and $\nu(x)$. It was shown in Theorems 14, 16, and 19 that if these selection functions are locally Lipschitz and satisfy certain convex constraints, then the resulting control strategy given by (14) has optimal robustness margins and is inverse optimal. Finally, Theorems 21 and 23 show that the satisficing framework completely characterizes all universal formulas that can be derived from a given CLF.

The techniques developed in this paper can be used as both an analysis and as a synthesis tool. As an example of their applications as an analysis tool, suppose that a control strategy has been designed based on Lyapunov techniques. If it is possible to find functions $l(x)$, $R(x)$, $b(x)$, and $\nu(x)$, such that the control strategy takes the form of (14) then Theorem 19 guarantees that the control law is inverse optimal. In addition, as shown in Section VII, the techniques can be used to ensure stability and robustness properties of heuristic control strategies such as the SDRE technique.

As a synthesis tool, the satisficing framework developed in this paper provides a powerful technique for developing new control strategies with guaranteed robustness and stability properties. The satisficing set can be thought of as the set of "safe" or "good" options available at each x. Given a CLF, the synthesis problem reduces to that of finding selection functions $b(x)$ and $\nu(x)$ that lead to desirable performance. Stability, robustness margins, and inverse optimality are provided for *a priori*. For example, an asymptotically stable, inverse optimal model predictive control strategy can be defined as

$$k_{MPC}(x) = \arg \min_{u \in S_R(x)} J_{MPC}(x,u)$$

where $J_{MPC}(x,u)$ is a cost criteria based on model predictive strategies.

Since the satisficing technique is built upon control Lyapunov functions, both local and global properties of the system can be addressed. One way of thinking about the satisficing approach is that it bridges the gap between local and global concerns: it is built upon the comparison of instantaneous cost with

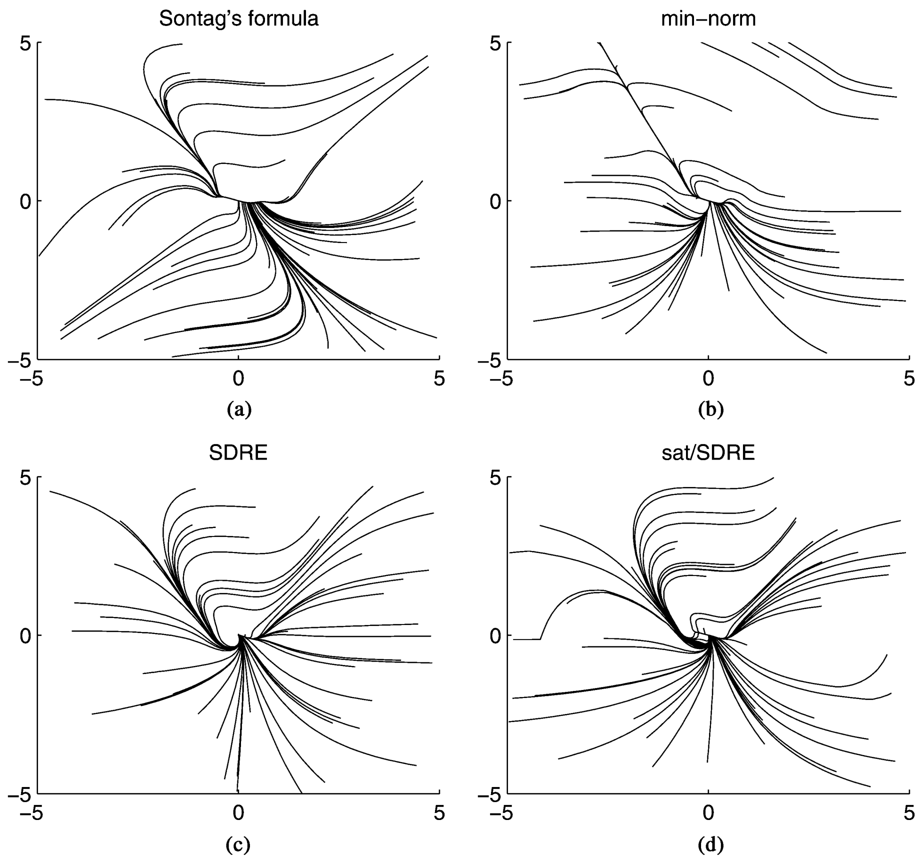

Fig. 5. Phase portrait for system (23) under the control of (a) Sontag's formula, (b) min-norm formula, (c) SDRE, and (d) SDRE projected onto the robust satisficing set.

instantaneous benefit, but by defining the benefit of a control action in terms of a CLF this local decision inherits global consequences.

The strength of the satisficing approach is its flexibility. Instead of providing just another in a list of (possibly inverse-optimal) universal formulas, our approach completely parameterizes the entire class of such control laws. To aid in the choice of selection functions, the designer is free to harness other control techniques such as model prediction, SDRE, fuzzy logic, or neural networks, to find the selection functions of $b(x)$ and $\nu(x)$,

This paper demonstrates that the satisficing approach offers new insights into CLF-based nonlinear control and has the potential to be a powerful tool in the design and analysis of nonlinear control strategies.

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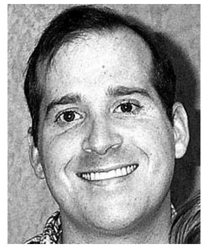

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