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Consensus Seeking in Multiagent Systems Under Dynamically Changing Interaction Topologies

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***Abstract*—This note considers the problem of information consensus among multiple agents in the presence of limited and unreliable information exchange with dynamically changing interaction topologies. Both discrete and continuous update schemes are proposed for information consensus. This note shows that information consensus under dynamically changing interaction topologies can be achieved asymptotically if the union of the directed interaction graphs have a spanning tree frequently enough as the system evolves.**

***Index Terms*—Cooperative control, graph theory, information consensus, multiagent systems, switched systems.**

I. Introduction

The study of information flow and interaction among multiple agents in a group plays an important role in understanding the coordinated movements of these agents. As a result, a critical problem for coordinated control is to design appropriate protocols and algorithms such that the group of agents can reach consensus on the shared information in the presence of limited and unreliable information exchange and dynamically changing interaction topologies. Consensus problems have recently been addressed in [1]–[7], to name a few. In this note, we extend the results of [2] to the case of directed graphs and present conditions for consensus of information under dynamically changing interaction topologies.

In contrast to [2], directed graphs will be used to represent the interaction (information exchange) topology between agents, where information can be exchanged via communication or direct sensing. A preliminary result for information consensus is presented in [8], where a linear update scheme is proposed for directed graphs. However, the analysis in [8] was not able to utilize all available communication links. A solution to this issue was presented in [4] for time-invariant communication topologies. Information consensus for dynamically evolving information was addressed in [9] in the context of spacecraft formation flying where the exchanged information is the configuration of the virtual structure associated with the (dynamically evolving) formation.

In many applications, the interaction topology between agents may change dynamically. For example, communication links between agents may be unreliable due to disturbances and/or subject to communication range limitations. If information is being exchanged by direct sensing, the locally visible neighbors of a vehicle will likely change over time. In [2], a theoretical explanation is provided for the observed behavior of the Vicsek model [10]. Possible changes over time in each agent's nearest neighbors is explicitly taken into account, and is an example of information consensus under dynamically changing interaction topologies. Furthermore, it is shown in [2] that consensus can be achieved if the union of the interaction graphs for the team are connected frequently enough as the system evolves.

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However, the approach in [2] is based on bidirectional information exchange, modeled by undirected graphs. Extensions of this work to second-order dynamics are discussed in [16] and [17]

There are a variety of practical applications where information only flows in one direction. For example, in leader-following scenarios, the leader may be the only vehicle equipped with a communication transmitter. For heterogeneous teams, some vehicles may have transceivers, while other less capable members only have receivers. There is a need to extend the results reported in [2] to interaction topologies with directional information exchange.

In addition, in [2] certain constraints are imposed on the weighting factors in the information update schemes, which may be extended to more general cases. For example, it may be desirable to weigh the information from different agents differently to represent the relative confidence of each agent's information or relative reliability of different communication or sensing links.

The objective of this note is to extend [2] to the case of directed graphs and explore the minimum requirements to reach consensus by using graph theory and matrix theory. As a comparison, [5] applies a set-valued Lyapunov approach to consider discrete-time consensus algorithms with unidirectional time-dependent communication links. In addition, [3] solves the *average-consensus* problem with directed graphs, which requires the graph to be strongly connected and balanced. We show that under certain assumptions *consensus*[1] can be achieved asymptotically under dynamically changing interaction topologies if the union of the collection of interaction graphs across some time intervals has a spanning tree frequently enough. The spanning tree requirement is a milder condition than connectedness and is therefore more suitable for practical applications. We also allow the relative weighting factors to be time-varying, which provides additional flexibility. As a result, the convergence conditions and update schemes in [2] are shown to be a special case of a more general result.

An additional contribution of this note is that we explicitly show that a nonnegative matrix with the same positive row sums has its spectral radius (its row sum in this case) as a simple eigenvalue if and only if the directed graph of this matrix has a spanning tree. In contrast, the Perron–Frobenius Theorem [11] for nonnegative matrices only deals with irreducible matrices, that is, matrices with strongly connected graphs. Besides having a spanning tree, if this matrix also has positive diagonal entries, we show that its row sum is the unique eigenvalue of maximum modulus.

The note is organized as follows. In Section II, we establish the notation and formally state the problem. Section III contains the main results, and Section IV offers our concluding remarks.

II. Problem Statement

Let $\mathcal{A} = \{A_i \mid i \in \mathcal{I}\}$ be a set of n agents, where $\mathcal{I} = \{1, 2, \ldots, n\}$. A directed graph $\mathcal{G}$ will be used to model the interaction topology among these agents. In $\mathcal{G}$, the ith node represents the ith agent A_i and a directed edge from A_i to A_j denoted as (A_i, A_j) represents a unidirectional information exchange link from A_i to A_j, that is, agent j can receive or obtain information from agent i, $(i, j) \in \mathcal{I}$. If there is a directed edge from A_i to A_j, A_i is defined as the parent node and A_j is defined as the child node. The interaction topology may be dynamically changing, therefore let $\bar{\mathcal{G}} = \{\mathcal{G}_1, \mathcal{G}_2, \ldots, \mathcal{G}_M\}$ denote the set of all possible directed interaction graphs defined for $\mathcal{A}$. In applications, the possible interaction topologies will likely be a subset of $\bar{\mathcal{G}}$. Obviously, $\bar{\mathcal{G}}$ has finite elements. The union of a group of directed graphs $\{\mathcal{G}_{i_1}, \mathcal{G}_{i_2}, \ldots, \mathcal{G}_{i_m}\} \subset \bar{\mathcal{G}}$ is a directed graph with nodes given by A_i, $i \in \mathcal{I}$ and edge set given by the union of the edge sets of $\mathcal{G}_{i_j}$, $j = 1, \ldots, m$.

[1]Not necessarily average-consensus.

A directed path in graph $\mathcal{G}$ is a sequence of edges $(A_{i_1}, A_{i_2}), (A_{i_2}, A_{i_3}), (A_{i_3}, A_{i_4}), \ldots$ in that graph. Graph $\mathcal{G}$ is called strongly connected if there is a directed path from A_i to A_j and A_j to A_i between any pair of distinct nodes A_i and A_j, $\forall (i, j) \in \mathcal{I}$. A directed tree is a directed graph, where every node, except the root, has exactly one parent. A spanning tree of a directed graph is a directed tree formed by graph edges that connect all the nodes of the graph (cf. [12]). We say that a graph has (or contains) a spanning tree if a subset of the edges forms a spanning tree. Let $M_n(\mathbb{R})$ represent the set of all $n \times n$ real matrices. Given a matrix $A = [a_{ij}] \in M_n(\mathbb{R})$, the directed graph of A, denoted by $\Gamma(A)$, is the directed graph on n nodes V_i, $i \in \mathcal{I}$, such that there is a directed edge in $\Gamma(A)$ from V_j to V_i if and only if $a_{ij} \neq 0$ (cf. [11]).

Let $\xi_i \in \mathbb{R}$, $i \in \mathcal{I}$, represent the ith information state associated with the ith agent. The set of agents $\mathcal{A}$ is said to achieve consensus asymptotically if for any $\xi_i(0)$, $i \in \mathcal{I}$, $\|\xi_i(t) - \xi_j(t)\| \to 0$ as $t \to \infty$ for each $(i, j) \in \mathcal{I}$.

Given T as the sampling period, we propose the following discrete-time consensus scheme:

$$\xi_i[k+1] = \frac{1}{\sum_{j=1}^{n} \alpha_{ij}[k] G_{ij}[k]} \sum_{j=1}^{n} \alpha_{ij}[k] G_{ij}[k] \xi_j[k] \tag{1}$$

where $k \in \{0, 1, 2, \ldots\}$ is the discrete-time index, $(i, j) \in \mathcal{I}$, $\alpha_{ij}[k] > 0$ is a weighting factor chosen from any finite set, $G_{ii}[k] \triangleq 1$, and $G_{ij}[k]$ equals one if information flows from A_j to A_i at time $t = kT$ and zero otherwise, $\forall j \neq i$. Equation (1) can be written in matrix form as

$$\xi[k+1] = D[k]\xi[k] \tag{2}$$

where $\xi = [\xi_1, \ldots, \xi_n]^T$, $D = [d_{ij}]$, $(i, j) \in \mathcal{I}$, with $d_{ij} = \alpha_{ij}[k] G_{ij}[k] / \sum_{j=1}^{n} \alpha_{ij}[k] G_{ij}[k]$.

In addition, we propose the following continuous-time consensus scheme:

$$\dot{\xi}_i(t) = -\sum_{j=1}^{n} \sigma_{ij}(t) G_{ij}(t) (\xi_i(t) - \xi_j(t)) \tag{3}$$

where $(i, j) \in \mathcal{I}$, $\sigma_{ij}(t) > 0$ is a weighting factor chosen from any finite set, $G_{ii}(t) \triangleq 1$, and $G_{ij}(t)$ equals one if information flows from A_j to A_i at time t and zero otherwise, $\forall j \neq i$. Equation (3) can be written in matrix form as

$$\dot{\xi}(t) = C(t)\xi(t) \tag{4}$$

where $C = [c_{ij}]$, $(i, j) \in \mathcal{I}$, with $c_{ii} = -\sum_{j \neq i} (\sigma_{ij}(t) G_{ij}(t))$ and $c_{ij} = \sigma_{ij}(t) G_{ij}(t)$, $j \neq i$.

Note that the interaction topology $\mathcal{G}$ may be dynamically changing due to unreliable transmission or limited communication/sensing range. This implies that $G_{ij}[k]$ in (1) and $G_{ij}(t)$ in (3) may be time-varying. We use $\mathcal{G}[k]$ and $\mathcal{G}(t)$ to denote the dynamically changing interaction topologies corresponding to (1) and (3), respectively. We also allow the weighting factors $\alpha_{ij}[k]$ in (1) and $\sigma_{ij}(t)$ in (3) to be dynamically changing to represent possibly time-varying relative confidence of each agent's information state or relative reliabilities of different information exchange links between agents. As a result, both matrix $D[k]$ in (1) and matrix $C(t)$ in (3) are dynamically changing over time.

Compared to the models in [2], we do not constrain the weighting factors $\alpha_{ij}[k]$ in (1) other than to require that they are positive. This provides needed flexibility for some applications. The Vicsek model and simplified Vicsek model used in [2] can be thought of as special cases of our discrete-time consensus scheme. If we let $\alpha_{ij}[k] \triangleq 1$ in (1), we obtain the Vicsek model. Also the simplified Vicsek model can be obtained if we let $\alpha_{ij}[k] \triangleq 1/g, \forall j \neq i$, and $\alpha_{ii}[k] \triangleq 1 - \sum_{j\neq i}(1/g)G_{ij}[k]$, where $g > n$ is a constant. Compared to [8], where the interaction graph is assumed to be time-invariant and weighting factors σ_{ij} are specified *a priori* to be constant and equal to each other, we study continuous-time consensus scheme with dynamically changing interaction topologies and weighting factors. The continuous update rule in [2] can also be regarded as a special case of our continuous update scheme by letting $\sigma_{ij} \triangleq 1/n$.

The main result of this note is that the update schemes (1) and (3) achieve asymptotic consensus for $\mathcal{A}$ if the union of the collection of directed interaction graphs across some time intervals has a spanning tree frequently enough as the system evolves. Toward that end, we have the following preliminary results.

Lemma 2.1: The discrete update scheme (1) achieves asymptotic consensus for $\mathcal{A}$ if and only if

$$D[k-1]D[k-2]\cdots D[2]D[1]D[0] \to \mathbf{1}c^T \quad (5)$$

as $k \to \infty$, where $\mathbf{1}$ denote the $n \times 1$ column vector with all the entries equal to 1, and c is an $n \times 1$ vector of constant coefficients.

Proof: Note that the set of agents $\mathcal{A}$ reaches consensus asymptotically if and only if the set

$$\boldsymbol{\mathcal{S}} = \{\xi \in \mathbb{R}^n : \xi_1 = \xi_2 = \cdots = \xi_n\}$$

is attractive and positively invariant.

Since

$$\xi[k] = D[k-1]D[k-2]\cdots D[1]D[0]\xi[0]$$

(5) implies that

$$\lim_{k\to\infty} \xi[k] = \mathbf{1}c^T\xi[0] = \begin{pmatrix} c^T\xi[0] \\ \vdots \\ c^T\xi[0] \end{pmatrix}$$

which implies that $\boldsymbol{\mathcal{S}}$ is attractive and positively invariant.

Conversely, if $\boldsymbol{\mathcal{S}}$ is attractive and positively invariant, then

$$\lim_{k\to\infty} \xi[k] = \lim_{k\to\infty} D[k-1]D[k-2]\cdots D[1]D[0]\xi[0] = \mathbf{1}\alpha$$

where α is a constant coefficient. Which, in turn, implies that

$$\lim_{k\to\infty} D[k-1]D[k-2]\ldots D[1]D[0] = \mathbf{1}c^T.$$

■

Lemma 2.2: The continuous update scheme (3) achieves asymptotic consensus for $\mathcal{A}$ if and only if

$$\Phi(t,0) = I + \int_0^t C(\sigma_1)\,d\sigma_1 + \int_0^t C(\sigma_1)\int_0^{\sigma_1} C(\sigma_2)\,d\sigma_2\,d\sigma_1 + \cdots \to \mathbf{1}c^T \quad (6)$$

as $t \to \infty$.

Proof: Noting that $\xi(t) = \Phi(t,0)\xi(0)$, the proof is similar to that of Lemma 2.1. ■

III. Consensus of Information Under Dynamically Changing Interaction Topologies

Let I_n denote the $n \times n$ identity matrix. A vector is nonnegative if all its elements are nonnegative. A matrix $A = [a_{ij}] \in M_n(\mathbb{R})$ is nonnegative, denoted as $A \geq 0$, if all its entries are nonnegative. Furthermore, if all its row sums are $+1$, A is said to be a (row) stochastic matrix [11]. A stochastic matrix P is called indecomposable and aperiodic (SIA) if $\lim_{n\to\infty} P^n = \mathbf{1}y^T$, where y is some column vector [13]. For nonnegative matrices, $A \geq B$ implies that $A - B$ is a nonnegative matrix. It is easy to verify that if $A \geq \rho B$, for some $\rho > 0$, and the directed graph of B has a spanning tree, then the directed graph of A has a spanning tree.

We need the following two lemmas. The first lemma is from [2] and the second lemma is originally from [13] and restated in [2].

Lemma 3.1: [2] Let $m \geq 2$ be a positive integer and let $P_1, P_2, \ldots, P_m$ be nonnegative $n \times n$ matrices with positive diagonal elements, then

$$P_1P_2\ldots P_m \geq \gamma(P_1 + P_2 + \cdots + P_m),$$

where $\gamma > 0$ can be specified from matrices P_i, $i = 1, \ldots, m$.

Lemma 3.2: [13] Let $S_1, S_2, \ldots, S_k$ be a finite set of SIA matrices with the property that for each sequence $S_{i_1}, S_{i_2}, \ldots, S_{i_j}$ of positive length, the matrix product $S_{i_j}S_{i_{j-1}}\ldots S_{i_1}$ is SIA. Then, for each infinite sequence $S_{i_1}, S_{i_2}, \ldots$ there exists a column vector y such that

$$\lim_{j\to\infty} S_{i_j}S_{i_{j-1}}\ldots S_{i_1} = \mathbf{1}y^T.$$

We also need the following lemmas to derive our main results.

Lemma 3.3: Given a matrix $A = [a_{ij}] \in M_n(\mathbb{R})$, where $a_{ii} \leq 0$, $a_{ij} \geq 0, \forall i \neq j$, and $\sum_{j=1}^n a_{ij} = 0$ for each j, then A has at least one zero eigenvalue and all of the nonzero eigenvalues are in the open left half plane. Furthermore, A has exactly one zero eigenvalue if and only if the directed graph associated with A has a spanning tree.

Proof: For the first statement, note that A is diagonally dominant, has zero row sum, and nonpositive diagonal elements. Therefore, from the Gersgorin disc theorem (cf. [11]), A has at least one zero eigenvalue and all the other nonzero eigenvalues are in the open left half plane.

The second statement will be shown using an induction argument.

Sufficiency: Step 1: The first step is to show that A has exactly one zero eigenvalue if the directed graph associated with A is itself a spanning tree.

Noting that the graph associated with A is a spanning tree, renumber the agents consecutively by depth in the spanning tree, with the root numbered as agent A_1. In other words, children of A_1 are numbered A_2 to A_{q_1}, children of A_2 to A_{q_1} are labeled A_{q_1+1} to A_{q_2} and so on. Note that the associated matrix A is lower diagonal with only one diagonal entry equal to zero.

Step 2: Let $Q = [q_{ij}] \in M_n(\mathbb{R})$, where $q_{ii} \leq 0$, $q_{ij} \geq 0$, $\forall i \neq j$, and $\sum_{j=1}^n q_{ij} = 0$ for each j. Let $S = [s_{ij}] \in M_n(\mathbb{R})$ satisfy the same properties as matrix Q. Also let $\mathcal{G}_1$ and $\mathcal{G}_2$ be the interaction graphs associated with Q and S, respectively. We assume that $s_{\ell\ell} = q_{\ell\ell} - \sigma_{\ell m}$, $s_{\ell m} = q_{\ell m} + \sigma_{\ell m}$, and $s_{ij} = q_{ij}$ otherwise, where $\sigma_{\ell m} > 0$ denotes the weighting factor for the information link from agent m to agent ℓ, $m \neq \ell$. That is, $\mathcal{G}_2$ corresponds to an interaction graph where one more directed link from node m to node ℓ is added to graph $\mathcal{G}_1$, where $m \neq \ell$. Denote $p_Q(t) = \det(tI - Q)$ and $p_S(t) = \det(tI - S)$ as the characteristic polynomial of Q and S, respectively. Let $Q_t = tI - Q$ and $S_t = tI - S$. Given any matrix M, denote $M([i,j])$ as the sub-matrix of M formed by deleting the ith row and jth column.

Next, we will show that if matrix Q has exactly one zero eigenvalue, then so does matrix S. Without loss of generality, we assume that the new directed information link added to graph $\mathcal{G}_1$ is from node m to node 1, where $m \neq 1$, for simplicity since we can always renumber node ℓ as node 1.

Obviously matrix S has at least one zero eigenvalue and all the other nonzero eigenvalues are in the open left-half plane following the first statement of this Lemma. Later, we will show that S has only one zero eigenvalue.

Assume that $Q_t = [q_{tij}]$, and $S_t = [s_{tij}]$, $(i,j) \in \mathcal{I}$. Accordingly, it can be seen that $s_{t11} = t - s_{11} = t - q_{11} + \sigma_{1m} = q_{t11} + \sigma_{1m}$,

$s_{t1m} = -s_{1m} = -q_{1m} - \sigma_{1m} = q_{t1m} - \sigma_{1m}$, and $s_{tij} = q_{tij}$, otherwise. Also note that $\det S_t([1,j]) = \det Q_t([1,j])$, $j \in \mathcal{I}$. Then, we know that

$$\begin{aligned}\det S_t &= \sum_{j=1}^{n}(-1)^{1+j}s_{t1j}\det S_t([1,j]) \\ &= \sum_{j=1}^{n}(-1)^{1+j}q_{t1j}\det S_t([1,j]) \\ &\quad + \sigma_{1m}\det S_t([1,1]) - (-1)^{1+m}\sigma_{1m}\det S_t([1,m]) \\ &= \det Q_t + \sigma_{1m}(\det S_t([1,1]) + (-1)^m \det S_t([1,m])).\end{aligned}$$

Consider a matrix $E = [e_{ij}]$, $(i,j) = 1,\ldots,n-1$, given by adding $[s_{21}, s_{31}, \ldots, s_{n1}]^T$ to the $(m-1)$th column of matrix $S([1,1])$. Matrix E can be denoted as

$$E = \begin{bmatrix} s_{22} & s_{23} & \cdots & s_{2m}+s_{21} & \cdots & s_{2n} \\ s_{32} & s_{33} & \cdots & s_{3m}+s_{31} & \cdots & s_{3n} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ s_{n2} & s_{n3} & \cdots & s_{nm}+s_{n1} & \cdots & s_{nn} \end{bmatrix}.$$

Thus, $e_{i(m-1)} = s_{(i+1)m} + s_{(i+1)1}$, $i = 1,\ldots,n-1$. Using the properties of determinants, it can be verified that

$$\det(tI - E) = \det S_t([1,1]) + (-1)^m \det S_t([1,m]).$$

Obviously, matrix E has zero row sum and nonpositive diagonal elements. Also matrix E is diagonally dominant. From the Gersgorin disc theorem, we know that E has at least one zero eigenvalue and all the other nonzero eigenvalues are on the open left-half plane. As a result, the Routh stability criterion implies that the characteristic polynomial of E denoted as $\det(tI - E)$ has a nonnegative coefficient in the first power of t. We also know that matrix Q has a positive coefficient for the first power of t in its characteristic polynomial $\det Q_t$ since Q has exactly one zero eigenvalue and all the others are in the open left-half plane.

Noting that $\det S_t = \det Q_t + \sigma_{1m}\det(tI - E)$, it is obvious that $p_S(t)$ has a positive coefficient for the first power of t. Therefore, S can only have one zero eigenvalue.

Step 3: If graph $\mathcal{G}$ associated with A is itself a spanning tree, we know that A has exactly one zero eigenvalue from Step 1. If not, graph $\mathcal{G}$ can be constructed by consecutively adding information links to the spanning tree. Step 2 implies that adding one additional information link to the spanning tree results in an associated matrix that also has exactly one zero eigenvalue. We can recursively add additional information links, where Step 2 implies that the matrix associated with the new graph has exactly one zero eigenvalue, until we obtain the graph $\mathcal{G}$. By induction, we know that A has exactly one zero eigenvalue if graph $\mathcal{G}$ has a spanning tree.

Necessity: If graph $\mathcal{G}$ does not have a spanning tree, then there exist at least two separate subgroups or at least two agents in the group who do not receive any information. For the first case, there is no information exchange between these subgroups and matrix A can be written as block diagonal form by renumbering these agents based on their subgroup. It is straightforward to see that each block has at least one zero eigenvalue. Therefore, A has at least two zero eigenvalues. For the second case, A has at least two zero rows, which implies that A has at least two zero eigenvalues. ■

Lemma 3.4: If a nonnegative matrix $A = [a_{ij}] \in M_n(\mathbb{R})$ has the same positive constant row sums given by $\mu > 0$, then μ is an eigenvalue of A with an associated eigenvector $\mathbf{1}$ and $\rho(A) = \mu$, where $\rho(\cdot)$ denotes the spectral radius. In addition, the eigenvalue μ of A has algebraic multiplicity equal to one, if and only if the graph associated with A has a spanning tree. Furthermore, if the graph associated with A has a spanning tree and $a_{ii} > 0$, then μ is the unique eigenvalue of maximum modulus.

Proof: The first statement follows directly from the properties of nonnegative matrices (cf. [11]).

For the second statement, we need to show both the necessary and sufficient conditions.

Sufficiency: If the graph associated with A has a spanning tree, then the graph associated with $B = A - \mu I_n$ also has a spanning tree. We know that $\lambda_i(A) = \lambda_i(B) + \mu$, where $i = 1,\ldots,n$, and $\lambda_i(\cdot)$ represents the ith eigenvalue. Noting that B satisfies the conditions in Lemma 3.3, we know that zero is an eigenvalue of B with algebraic multiplicity equal to one, which implies that μ is an eigenvalue of A with algebraic multiplicity equal to one.

Necessity: If the graph associated with A does not have a spanning tree, we know that $B = A - \mu I_n$ has more than one zero eigenvalue from Lemma 3.3, which in turn implies that A has more than one eigenvalue equal to μ.

For the third statement, the Gersgorin disc theorem [11] implies that all the eigenvalues of A are located in the union of the n discs given by

$$\bigcup_{i=1}^{n}\left\{z \in \mathbf{C} : |z - a_{ii}| \le \sum_{j \ne i}|a_{ij}|\right\}$$

where $\mathbf{C}$ is the set of complex numbers. Noting that $a_{ii} > 0$, it is easy to see that this union is included in a circle given by $\{z \in \mathbf{C} : |z| = \mu\}$ and the circular boundaries of the union of n discs has only one intersection with the circle at $z = \mu$. Thus, we know that $|\lambda| < \mu$ for every eigenvalue of A satisfying $\lambda \ne \mu$. Combining the second statement, we know that μ is the unique eigenvalue of maximum modulus. ■

Corollary 3.5: A stochastic matrix has algebraic multiplicity equal to one for its eigenvalue $\lambda = 1$ if and only if the graph associated with the matrix has a spanning tree. Furthermore, a stochastic matrix with positive diagonal elements has the property that $|\lambda| < 1$ for every eigenvalue not equal to one.

Lemma 3.6: If $A \in M_n$ and $A \ge 0$, then the spectral radius of A, denoted as $\rho(A)$, is an eigenvalue of A and there is a nonnegative vector $x \ge 0$, $x \ne 0$, such that $Ax = \rho(A)x$.

Proof: See [11, Th. 8.3.1]. ■

Lemma 3.7: Let $A = [a_{ij}] \in M_n(\mathbb{R})$ be a stochastic matrix. If A has an eigenvalue $\lambda = 1$ with algebraic multiplicity equal to one, and all the other eigenvalues satisfy $|\lambda| < 1$, then A is SIA, that is, $\lim_{m\to\infty} A^m \to \mathbf{1}\nu^T$, where ν satisfies $A^T\nu = \nu$ and $\mathbf{1}^T\nu = 1$. Furthermore, each element of ν is nonnegative.

Proof: The first part of the lemma follows [11, Lemma 8.2.7]. For the second part, it is obvious that A^T is also nonnegative and has $\rho(A^T) = 1$ as an eigenvalue with algebraic multiplicity equal to one. Thus, Lemma 3.6 implies that the eigenspace of A^T associated with eigenvalue $\lambda = 1$ is given by cx, where $c \in \mathbf{C}$, $c \ne 0$, and x is a nonnegative eigenvector. Since ν is also an eigenvector of A^T associated with eigenvalue $\lambda = 1$ and satisfies $\mathbf{1}^T\nu = 1$, it follows that each element of ν must be nonnegative. ■

A. Consensus Using Discrete Time Update Scheme

As a first step toward the general case, we first show necessary and sufficient condition for consensus of information using discrete time update scheme (1) with a time-invariant interaction topology and constant weighting factors, that is, a constant matrix D.

Theorem 3.8: With a time-invariant interaction topology and constant weighting factors, the discrete-time update scheme (1) achieves consensus asymptotically as $k \to \infty$ for $\mathcal{A}$ if and only if the associated interaction graph $\mathcal{G}$ has a spanning tree.

Proof: From Lemma 2.1, we need to show that $D^k \to \mathbf{1}c^T$, where c is a constant column vector.

Sufficiency: Obviously D is a stochastic matrix with positive diagonal entries. The fact that graph $\mathcal{G}$ has a spanning tree also implies that the directed graph of D has a spanning tree. Combining Corollary 3.5 and Lemma 3.7, we know that $\lim_{k\to\infty} D^k \to \mathbf{1}\nu^T$, where ν satisfies the properties defined in Lemma 3.7.

Necessity: If $\mathcal{G}$ does not have a spanning tree, neither does the directed graph of D, which implies, by Corollary 3.5, that the algebraic multiplicity of eigenvalue $\lambda = 1$ of D is $m > 1$. Therefore, the Jordan decomposition of D^k has the form $D^k = MJ^kM^{-1}$, where M is full rank and J^k is lower triangular with m diagonal elements equal to one. Therefore, the rank of $\lim_{k\to\infty} D^k$ is at least $m > 1$ which implies, by Lemma 2.1, that $\mathcal{A}$ cannot reach consensus asymptotically. ■

The next lemma sets the stage for showing that under certain conditions, the existence of a spanning tree is sufficient for consensus under dynamically changing interaction topologies using the discrete update scheme (1).

Lemma 3.9: If the union of a set of directed graphs $\{\mathcal{G}_{i_1}, \mathcal{G}_{i_2}, \ldots, \mathcal{G}_{i_m}\} \subset \bar{\mathcal{G}}$ has a spanning tree, then the matrix product $D_{i_m} \ldots D_{i_2} D_{i_1}$ is SIA, where D_{i_j} is a stochastic matrix corresponding to each directed graph $\mathcal{G}_{i_j}$ in (2).

Proof: From Lemma 3.1, we know that $D_{i_m} \cdots D_{i_2} D_{i_1} \geq \gamma \sum_{j=1}^m D_{i_j}$ for some $\gamma > 0$.

Since the union of $\{\mathcal{G}_{i_1}, \mathcal{G}_{i_2}, \ldots, \mathcal{G}_{i_m}\}$ has a spanning tree, we know that the directed graph of matrix $\sum_{j=1}^m D_{i_j}$ has a spanning tree, which in turn implies that the directed graph of the matrix product $D_{i_m} \ldots D_{i_2} D_{i_1}$ has a spanning tree. Also the matrix product $D_{i_m} \ldots D_{i_2} D_{i_1}$ is a stochastic matrix with positive diagonal entries since stochastic matrices with positive diagonal entries are closed under matrix multiplication.

Combining Corollary 3.5 and Lemma 3.7, we know that the matrix product $D_{i_1} D_{i_2} \ldots D_{i_m}$ is SIA. ■

The following theorem extends the discrete-time convergence result of [2].

Theorem 3.10: Let $\mathcal{G}[k] \in \bar{\mathcal{G}}$ be a switching interaction graph at time $t = kT$. Also, let $\alpha_{ij}[k] \in \bar{\alpha}$, where $\bar{\alpha}$ is a finite set of arbitrary positive numbers. The discrete update scheme (1) achieves consensus asymptotically for $\mathcal{A}$ if there exists an infinite sequence of uniformly bounded, nonoverlapping time intervals $[k_jT, (k_j+l_j)T)$, $j = 1, 2, \ldots$, starting at $k_1 = 0$, with the property that each interval $[(k_j + l_j)T, k_{j+1}T)$ is uniformly bounded and the union of the graphs across each interval $[(k_j + l_j)T, k_{j+1}T)$ has a spanning tree. Furthermore, if the union of the graphs after some finite time does not have a spanning tree, then consensus cannot be achieved asymptotically for $\mathcal{A}$.

Proof: Let $\bar{D}$ denote the set of all possible matrices $D[k]$ under dynamically changing interaction topologies and weighting factors $\alpha_{ij}[k]$. We know that $\bar{D}$ is a finite set since both set $\bar{\mathcal{G}}$ and set $\bar{\alpha}$ are finite.

Consider the jth time interval $[k_jT, k_{j+1}T)$, which includes the time interval $[k_jT, (k_j + l_j)T)$ and must be uniformly bounded since both $[k_jT, (k_j + l_j)T)$ and $[(k_j + l_j)T, k_{j+1}T)$ are uniformly bounded. Also, the sequence of time intervals $[k_jT, k_{j+1}T)$, $j = 1, 2, \ldots$, are contiguous.

The union of the graphs across $[k_jT, k_{j+1}T)$, denoted as $\bar{\mathcal{G}}[k_j]$, has a spanning tree since the union of the graphs across $[k_jT, (k_j + l_j)T)$ has a spanning tree. Let $\{D[k_j], D[k_j+1], \ldots, D[k_{j+1}-1]\}$ be the set of stochastic matrices corresponding to each graph in the union $\bar{\mathcal{G}}[k_j]$. Following Lemma 3.9, the matrix product $D[k_{j+1} - 1] \ldots D[k_j + 1]D[k_j]$, $j = 1, 2, \ldots$, is SIA. Then, by applying Lemma 3.2 and mimicking a similar proof for [2, Th. 2], the first part can be proved.

If the union of the graphs after some finite time $\hat{t}$ does not have a spanning tree, then during the infinite time interval $[\hat{t}, \infty)$, there exist at least two agents such that there is no path in the union of the graphs that contains these two agents, which then implies that information of these two agents cannot reach consensus. ■

B. Consensus Using Continuous Time Update Scheme

The continuous-time analog of Theorem 3.8 has been shown in [4]. Therefore, we will focus on demonstrating that under certain conditions, the existence of a spanning tree is also sufficient for consensus under dynamically changing interaction topologies using the continuous time update scheme. To do so, we need the following lemma.

Lemma 3.11: If the union of the directed graphs $\{\mathcal{G}_{t_1}, \mathcal{G}_{t_2}, \ldots, \mathcal{G}_{t_m}\} \subset \bar{\mathcal{G}}$ has a spanning tree and C_{t_i} is the matrix corresponding to each directed graph $\mathcal{G}_{t_i}$ in (4), then the matrix product $e^{C_{t_m}\Delta t_m} \ldots e^{C_{t_2}\Delta t_2} e^{C_{t_1}\Delta t_1}$ is SIA, where $\Delta t_i > 0$ are bounded.

Proof: From (4), each matrix C_{t_i} satisfies the properties defined in Lemma 3.3. Thus, each C_{t_i} can be written as the sum of a nonnegative matrix M_{t_i} and $-\eta_{t_i} I_n$, where η_{t_i} is the maximum absolute value of the diagonal entries of C_{t_i}, $i = 1, \ldots, m$.

From [4, Lemma 1], we know that $e^{C_{t_i}\Delta t_i} = e^{-\eta_{t_i}\Delta t_i} e^{M_{t_i}\Delta t_i} \geq \rho_i M_{t_i}$ for some $\rho_i > 0$. Since the union of the directed graphs $\{\mathcal{G}_{t_1}, \mathcal{G}_{t_2}, \ldots, \mathcal{G}_{t_m}\}$ has a spanning tree, we know that the union of the directed graphs of M_{t_i} has a spanning tree, which in turn implies that the union of the directed graphs of $e^{C_{t_i}\Delta t_i}$ has a spanning tree. From Lemma 3.1, we know that $e^{C_{t_m}\Delta t_m} \ldots e^{C_{t_2}\Delta t_2} e^{C_{t_1}\Delta t_1} \geq \gamma \sum_{i=1}^m e^{C_{t_i}\Delta t_i}$ for some $\gamma > 0$, which implies that the aforementioned matrix product also has a spanning tree.

It can also be verified that each matrix $e^{C_{t_i}\Delta t_i}$ is a stochastic matrix with positive diagonal entries, which implies that the above matrix product is also stochastic with positive diagonal entries.

Combining Corollary 3.5 and Lemma 3.7, we know that the previous matrix product is SIA. ■

In this note, we also apply dwell time (cf. [14] and [2]) to the continuous time update scheme (4), which implies that the interaction graph and weighting factors are constrained to change only at discrete times, that is, the matrix $C(t)$ is piecewise constant.

Equation (4) can be rewritten as

$$\dot{\xi}(t) = C(t_i)\xi(t), \quad t \in [t_i, t_i + \tau_i) \tag{7}$$

where t_0 is the initial time and $t_1, t_2, \ldots$ is an infinite time sequence at which the interaction graph or weighting factors change, resulting in a change in $C(t)$.

Let $\tau_i = t_{i+1} - t_i$ be the dwell time, $i = 0, 1, \ldots$. Note that the solution to (7) is given by $\xi(t) = e^{C(t_k)(t-t_k)} e^{C(t_{k-1})\tau_{k-1}} \ldots e^{C(t_1)\tau_1} e^{C(t_0)\tau_0}\xi(0)$, where k is the largest nonnegative integer satisfying $t_k \leq t$. Let $\bar{\tau}$ be a finite set of arbitrary positive numbers. Let Υ be an infinite set generated from set $\bar{\tau}$, which is closed under addition, and multiplications by positive integers. We assume that $\tau_i \in \Upsilon$, $i = 0, 1, \ldots$. By choosing the set $\bar{\tau}$ properly, dwell time can be chosen from an infinite set Υ, which somewhat simulates the case when the interaction graph $\mathcal{G}$ changes dynamically over time.

The following theorem extends the continuous time convergence result in [2].

Theorem 3.12: Let $t_1, t_2, \ldots$ be an infinite time sequence at which the interaction graph or weighting factors switch and $\tau_i = t_{i+1} - t_i \in \Upsilon$, $i = 0, 1, \ldots$. Let $\mathcal{G}(t_i) \in \bar{\mathcal{G}}$ be a switching interaction graph at time $t = t_i$ and $\sigma_{ij}(t_i) \in \bar{\sigma}$, where $\bar{\sigma}$ is a finite set of arbitrary positive numbers. The continuous-time update scheme (3) achieves consensus asymptotically for $\mathcal{A}$ if there exists an infinite sequence of uniformly bounded, nonoverlapping time intervals $[t_{i_j}, t_{i_j+l_j})$, $j = 1, 2, \ldots$, starting at $t_{i_1} = t_0$, with the property that each interval $[t_{i_j+l_j}, t_{i_{j+1}})$

is uniformly bounded and the union of the graphs across each interval interval $[t_{i_j+l_j}, t_{i_{j+1}})$ has a spanning tree. Furthermore, if the union of the graphs after some finite time does not have a spanning tree, then consensus cannot be achieved asymptotically for $\mathcal{A}$.

Proof: The set of all possible matrices $e^{C(t_i)\tau_i}$, where $\tau_i \in \Upsilon$, under dynamically changing interaction topologies and weighting factors can be chosen or constructed by matrix multiplications from the set $\bar{E} = \{e^{C(t_i)\tau_i}, \tau_i \in \bar{\tau}\}$. Clearly, $\bar{E}$ is finite since $\bar{\mathcal{G}}$, $\bar{\sigma}$, and $\bar{\tau}$ are all finite.

Consider the jth time interval $[t_{i_j}, t_{i_{j+1}})$, which includes the time interval $[t_{i_j}, t_{i_j+l_j})$ and must be uniformly bounded since both $[t_{i_j}, t_{i_j+l_j})$ and $[t_{i_j+l_j}, t_{i_{j+1}})$ are uniformly bounded. Also, the sequence of time intervals $[t_{i_j}, t_{i_{j+1}})$, $j = 1, 2, \ldots$, are contiguous.

The union of the graphs across $[t_{i_j}, t_{i_{j+1}})$, denoted as $\bar{\mathcal{G}}(t_{i_j})$, has a spanning tree since the union of graphs across $[t_{i_j}, t_{i_j+l_j})$ has a spanning tree. Let $\{C(t_{i_j}), C(t_{i_j+1}), \ldots, C(t_{i_{j+1}-1})\}$ be a set of matrices corresponding to each graph in the union $\bar{\mathcal{G}}(t_{i_j})$. Following Lemma 3.11, the matrix product $e^{C(t_{i_{j+1}-1})\tau_{i_{j+1}-1}} \cdots e^{C(t_{i_j+1})\tau_{i_j+1}} e^{C(t_{i_j})\tau_{i_j}}$, $j = 1, 2, \ldots$, is SIA. Then, the first statement follows from Lemma 3.2 and an argument similar to the proof of [2, Th. 2].

The proof of the second statement is similar to the argument used in Theorem 3.10. ■

C. Discussion

The contribution of this note is that the results in [2], which are limited to undirected graphs, are extended to directed graphs. Therefore, unidirectional information exchange is allowed instead of requiring bidirectional information exchange. This will be important in applications where bidirectional communication or sensing is not available.

Reference [2] shows that consensus of information (the heading of each agent in their context) can be achieved if the union of a collection of graphs is connected frequently enough. This note demonstrates that the same result can be achieved as long as the union of the graphs has a spanning tree, which is a milder requirement than being connected and implies that one half of the information exchange links required in [2] can be removed without adversely affecting the convergence result. In this sense, the results for convergence in [2] can be thought of as a special case of a more general result. Of course, the final achieved equilibrium points will depend on the property of the directed graphs. For example, compared to strongly connected graphs, graphs that are not strongly connected will reach different final equilibrium points (see [4] for an analysis of the final equilibrium points).

The leader following scenario described in [2] can also be thought of as a special case of our result. If there is one agent in the group which does not have any incoming link, but the union of the interaction graphs has a spanning tree frequently enough, then this agent must be the root of the spanning tree, i.e, the leader. Since consensus is guaranteed, the information state of the other agents asymptotically converges to the information state of the leader. Therefore, the scenario discussed in [2] of being linked to a leader frequently enough is a special case of having a spanning tree, frequently enough, with the leader as the root.

For the continuous model used in [2], the switching times of the interaction graph is constrained to be separated by τ_D time units, where τ_D is a constant dwell time. Our continuous update scheme allows the switching times to be within an infinite set of positive numbers generated by any finite set of positive numbers, which is better suited to simulating the random switching of interaction graphs. Therefore, the continuous scheme in [2] can be thought of a special case of our result by letting $\bar{\tau} = \{\tau_d\}$ and $\Upsilon = \{k\tau_d \mid k = 1, 2, \ldots\}$.

Unlike the update schemes in [2], we do not constrain the weighting factors in our discrete and continuous update schemes, other than to require that they be positive. This provides flexibility to account for relative confidence in information from different agents.

An additional contribution of this note is the proof for properties of nonnegative matrices with the same positive row sums. The Perron–Frobenius Theorem states that if a nonnegative matrix A is irreducible, that is, the directed graph of A is strongly connected, then the spectral radius of A is a simple eigenvalue. We show that the irreducibility condition is too stringent for nonnegative matrices with the same positive row sums. Lemma 3.4 explicitly shows that for a nonnegative matrix A with identical positive row sums, the spectral radius of A (the row sum in this case) is a simple eigenvalue if and only if the directed graph of A has a spanning tree. In other words, A may be reducible but retains its spectral radius as a simple eigenvalue. Furthermore, if A has a spanning tree and positive diagonal entries, we know that the spectral radius of A is the unique eigenvalue of maximum modulus.

Note that we assume that weighting factors α_{ij} and σ_{ij} are chosen from any finite set of positive numbers for simplicity of proof. In fact, the results of this note are still valid if this assumption is relaxed to $\alpha_{ij} \in [\alpha_L, \alpha_M]$ and $\sigma_{ij} \in [\sigma_L, \sigma_M]$, where $\alpha_L, \alpha_M, \sigma_L$, and σ_M are arbitrary positive numbers satisfying $\alpha_L < \alpha_M$ and $\sigma_L < \sigma_M$. The argument is based on the concluding remark in [13], which deals with the case when the set of stochastic matrices is infinite.

IV. Conclusion

This note has considered the problem of information consensus under dynamically changing interaction topologies and weighting factors. We have used directed graphs to represent information exchanges among multiple agents, taking into account the general case of unidirectional information exchange. We also proposed discrete and continuous update schemes for information consensus and gave conditions for asymptotic consensus under dynamically changing interaction topologies and weighting factors using these update schemes. The reader is referred to [15] for simulation examples that illustrate the results presented in this note.

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Stabilization of Switched Linear Systems

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***Abstract*—In this note, we study the stabilization problem of systems that switch among a finite set of controllable linear systems with arbitrary switching frequency. For both cases of known and unknown switching functions, feedback laws are designed to achieve exponential stability. For the later case, a method combining on-line adaptive estimation and feedback stabilization is developed in the controller design.**

***Index Terms*—Estimation, excitation, stability, stabilization, switched systems.**

I. Introduction

In recent years, the switched systems have attracted considerable efforts; see, e.g., [2], [6], [10], [11], and [15], among many others. This is because switched systems have strong engineering backgrounds; see, for instance, [16] and [17]. When the switching laws are modeled as finite state Markov chains, the stabilization problem of switched stochastic systems has been investigated by many authors, and necessary and sufficient conditions have been given to solve the problem for both the nonadaptive case where the switchings are available (c.f. [7] and [9] and the adaptive case where the switchings are unavailable (c.f. [19]).

We will consider the stabilization problem for switched linear systems as follows:

$$\dot{x}(t) = A_{\sigma(t)}x(t) + B_{\sigma(t)}u(t), \qquad x(t) \in \mathbb{R}^n,\ u(t) \in \mathbb{R}^m \quad (1)$$

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where the switching law $\sigma(t) : [0, \infty) \rightarrow \Lambda$ is a piecewise constant function that is continuous from the right, and where $\Lambda = \{1, 2, \ldots, N\}$ for some integer N.

When the switching law has no given mode (or is arbitrary), one way to investigate the stability and stabilization problems is to find a common Lyapunov function for all the switching models (c.f. [1], [3], [5], [10], and [14]). The conditions in such an approach tend to be strong because the existence of a common Lyapunov function guarantees the stability of a system under *all* possible switchings.

Another commonly used approach is to assume that a system remains unswitched for a period long enough to allow the overshoots of the closed-loop system in the transient phases to fade (c.f. [8] and [12]).

In this note, we will consider the stabilization problem of systems that switch among a finite set of controllable linear systems at *any* given frequency. To guarantee the stability of such a system at a given switching rate, it is certainly not enough to just stabilize each individual system for the obvious reason that the overshoots may destroy the stability. A feedback should be designed so that the magnitudes of the states of each individual system will decay by half on any interval of a given length. We will achieve this by first developing an estimation on the overshoots of the transition matrices (see Lemma 3.2), which can be considered as an enhancement of the Squashing Lemma in [13].

We will first present a preliminary result for the case when the switching functions are explicitly given. Our design in this case applies whenever the switching frequency is finite and known, in particular when the "average-dwell-time"[8] (instead of just the dwell time) is positive. The way the switching frequency is defined (see Definition 2.1) allows our result to apply to the case when the switching functions have some fast switchings on some intervals, provided that the switching frequency is "bounded on average" in the long run. We will then continue with the case when the switching frequency is finite but unknown. Finally, we will develop a method that combines online adaptive estimation and stabilization to treat the case when the switching functions are not given. We remark that even in the simplest case when the switching law is given the controllability condition cannot be relaxed to stabilizability. It is not hard to find an example of a system that switches between two stable systems that fails to be stable with certain switchings.

II. Main Results

Consider a system as in (1) with a switching function $\sigma(t)$. The switching moments $0 < t_1 < t_2 < \cdots$ of $\sigma(t)$ are defined recursively by $t_{k+1} = \inf\{t > t_k : \sigma(t) \neq \sigma(t_k)\}$, $t_0 = 0$. The switching duration δ_k is defined by $\delta_k = t_k - t_{k-1}$ $(k = 1, 2, \ldots)$.

Definition 2.1: Consider a switching function $\sigma(t) : [0, \infty) \rightarrow \Lambda$.

- The *switching frequency* f of $\sigma(t)$ is defined by

$$f = \overline{\lim_{t\rightarrow\infty}} \frac{\{\text{Number of switches of } \sigma(\cdot) \text{ in } [0, t]\}}{t}. \quad (2)$$

- The *dwell time* of $\sigma(t)$ is defined by $\tau = \inf_k \delta_k$.

□

Throughout this note, we will need the following standard assumption: H1) The models (A_i, B_i), $i = 1, \ldots, N$, are controllable.

Our first result is for the case when switching functions are explicitly given.

Theorem 2.1: Assume H1) holds for a switched system as in (1). Let $\alpha > 0$ be given. Then, there exist a set of gain matrices $\{K_i : i = 1, \ldots, N\}$ such that for any given switching law σ with a frequency $f \leq \alpha$, the switched linear system (1) under the switched feedback law $u(t) = K_{\sigma(t)}x(t)$ is exponentially stable.