A Subspace Decomposition Method for Point Source Localization in Blurred Images

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Abstract-*In this paper we address the problem of resolving blurred point sources in intensity images. A new approach to image restoration is introduced which is a 2-D generalization of techniques originating from the field of direction of arrival estimation (DOA). In the 2-D frequency domain, algorithms, such as MUSIC, may be adapted to search for these point sources. A generalization of array smoothing based on a regularization operator is introduced for 2-D arrays in order to achieve rank enhancement in the signal space of the covariance matrix.*

1. Introduction

Point source localization (PSL) is the problem of resolving individual points in an image that have been corrupted by convolution with some finite support blurring function. It has been addressed in various forms [1]-[3]. PSL is pertinent to the fields of astronomical image restoration, biomedical imaging and echo resolution. Deblurring of star fields is one of the major applications of PSL in images. Atmospheric turbulence can cause closely spaced stars to become blurred beyond resolution so as to appear as a single star. We will address the 2-D case, but the algorithm may be extended to 1-D and higher dimensional data.

The PSL problem for a blurred $M_1 \times M_2$ image can be formulated in the following linear model for a single "snap shot" at time t:

$$\boldsymbol{g}(t) = \boldsymbol{Df} + \boldsymbol{n}(t) \tag{1}$$

where $\boldsymbol{g}$ is the colexigraphically ordered (row-scanned) observation vector of length $M_1 \times M_2 = M$, $\boldsymbol{D}$ is an $M \times M$ circulant convolution matrix whose columns are formed from spatially shifted copies of the point spread function (PSF), $\boldsymbol{f}$ is the desired uncorrupted deterministic image vector of length M, and $\boldsymbol{n}$ is the additive observation noise vector. It is assumed that the noise is uncorrelated among the elements. We may make the assumption that the columns of $\boldsymbol{D}$ are circularly shifted copies of one another if we assume that points that we are trying to locate are away from boundaries of our image.

The new algorithm casts the PSL problem into a new form which can be viewed as a coherent source direction of arrival estimation problem. Because our algorithm is based on eigenvector techniques it naturally inherits the property of super-resolution or inter-pixel resolution of points. We do not assume that the source amplitudes are time varying in order to build up rank in the signal space of our covariance matrix as other authors have assumed [2][3], but introduce a new method of 2-D rank enhancement which is a generalization of the array smoothing technique [6]. Our method has an added benefit of allowing regularization of the spatial smoothing.

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2. Theoretical Development

In this section we transform the problem expressed by (1) into a problem which can be solved with traditional DOA estimation techniques. For a background in DOA, [5] is an excellent tutorial.

2.1. Frequency Domain Signal Model

Since $\boldsymbol{f}$ is sparse we may replace $\boldsymbol{Df}$ with $A\boldsymbol{u}$, where $\boldsymbol{u}$ contains the intensity coefficients of all the non-zero elements of $\boldsymbol{f}$, and A contains only those columns of $\boldsymbol{D}$ which correspond to the non-zero elements of $\boldsymbol{u}$:

$$\boldsymbol{g}(t) = \boldsymbol{Au} + \boldsymbol{n}(t). \tag{2}$$

This differs from the usual DOA problem in two ways: 1) $\boldsymbol{u}$ which contains the magnitudes of the sources is not time varying, and 2) the columns of A are not cisoids but are shifted versions of the blur function which is of finite support. In order to map spatial shifts in source position to phase shifts, (2) can be transformed into the frequency domain by multiplying $\boldsymbol{g}(t)$ by $\mathcal{F}$, a truncated version of the 2-D DFT matrix. The observed image is real and its 2-D DFT has conjugate symmetry, thus half of the elements contain redundant information, and the order of the system can be reduced by half without loss of resolution. The truncated 2-D DFT matrix is formed from the upper $N = M/2$ rows of the unitary DFT matrix $\mathcal{F}'$ which is defined as

$$\mathcal{F}' = \boldsymbol{F}^{row} \otimes \boldsymbol{F}^{col},$$

where $\otimes$ is a Kronecker product, and $\boldsymbol{F}^{row}$ and $\boldsymbol{F}^{col}$ are the 1-D DFT matrices whose elements are $F^{row}_{i,k} = e^{-j2\pi mn/M_1}$, $M_1/2 \le i,k \le M_1/2-1$ and $F^{col}_{i,k} = e^{-j2\pi k/M_2}$, $M_2/2 \le i,k \le M_2/2-1$, respectfully. Thus $\mathcal{F}$ is $M \times N$ and is defined as:

$$\mathcal{F}_{i,k} = \mathcal{F}'_{i,k} \tag{3}$$

for $0 \le i \le N$ and $0 \le k \le M$. Multiplying a vector containing a row scanned image by $\mathcal{F}$ results in a frequency unwrapped image, a condition which is necessary for fractional pixel resolution. The frequency domain version of (2) now becomes

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$$\mathcal{F}g(t) = \mathcal{F}Au + \mathcal{F}\eta(t)$$
$$\tilde{g}(t) = \tilde{A}u + \tilde{\eta}(t), \tag{4}$$

where ~ designates the 2-D DFT of the row scanned images contained in $g(t)$, $n(t)$ and each of the columns of the matrix A. The vector $\tilde{g}(t)$ has complex elements and length N. Since u is independent of the time index, equation (4) is in the form of a coherent DOA problem. All the columns of $\tilde{A}$ have the same magnitude, but are phase shifted versions of one another because the columns of A are spatially shifted versions of the blurring function. We can express $\tilde{A}$ as the product of a diagonal matrix H whose elements correspond to the frequency domain PSF, and a matrix V which contains the phase information for each of the columns of $\tilde{A}$:

$$\tilde{A} = H\left[v_{x_1,y_1} | v_{x_2,y_2} | \cdots | v_{x_P,y_P}\right] = HV. \tag{5}$$

The column vectors v_{x_p,y_p} of V correspond to images with a single unblurred point at location (x_p, y_p) which have been transformed into the frequency domain, and are expressed as

$$v_{x,y} = \Bigg[e^{-j2\pi\left(\frac{0x}{M_1}+\frac{0y}{M_2}\right)}, e^{-j2\pi\left(\frac{0x}{M_1}+\frac{1y}{M_2}\right)}, \ldots, e^{-j2\pi\left(\frac{0x}{M_1}+\frac{(M_2-1)y}{M_2}\right)},$$
$$e^{-j2\pi\left(\frac{1x}{M_1}+\frac{0y}{M_2}\right)}, e^{-j2\pi\left(\frac{1x}{M_1}+\frac{1y}{M_2}\right)}, \ldots, e^{-j2\pi\left(\frac{1x}{M_1}+\frac{(M_2-1)y}{M_2}\right)}, \ldots,$$
$$e^{-j2\pi\left(\frac{(M_1-1)x}{M_1}+\frac{0y}{M_2}\right)}, \ldots, e^{-j2\pi\left(\frac{(M_1-1)x}{M_1}+\frac{(M_2-1)y}{M_2}\right)} \Bigg]^T \tag{6}$$

Substituting (5) into (4),

$$\tilde{g}(t) = HVu + \tilde{\eta}(t) \tag{7}$$

yielding an autocovariance matrix of

$$R_{\tilde{g}} = HVR_uV^HH^H + I\sigma_\eta^2, \tag{8}$$

which is clearly of the same form as the DOA problem. The above formulation assumes that each of the blurred point sources in the observed image have integer pixel shifts in relation to the PSF.

2.2. Generalized 2-D Array Smoothing

The signal space of $R_{\tilde{g}}$ is of rank one because the magnitudes of the point sources are not time varying, resulting in a coherent problem. In addition, the elements in the data array are non-uniformly weighted by H. The rank of the signal space must be increased to P for eigenvector based methods to work and this can be accomplished by a new technique which is a generalization of spatial smoothing [6]. Rank enhancement is accomplished by averaging in the frequency domain over the autocovariance matrices of subimages. If we average vertically across an image by L_1 shifts and horizontally by L_2 shifts, then a total of $L=L_1L_2$ subimages may be averaged together to build the sample covariance matrix.

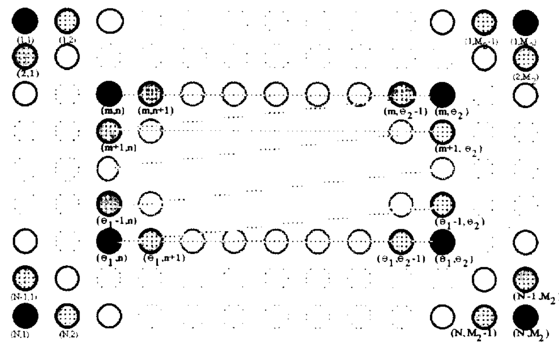


Figure 1. Row scanned subimage at (m,n).

Figure 1 shows how the subimages are extracted from a larger image and how they are arranged into a vector. We will now introduce some notation that will aid us in annotating the colexigraphically ordered subimages. Note that in (7) each of the columns of V, the diagonal of H, and the column vectors g and n are colexigraphically ordered images. The subscript $[m,n]$ will be applied to these variables to refer to the subimage whose upper left corner is at the position (m,n) in the corresponding 2-D images. The subimages are of size $\Theta_1 \times \Theta_2$ where $\Theta_1 = N - L_1 + 1$ and $\Theta_2 = M_2 - L_2 + 1$.

For example, in the case of the image contained in $\tilde{g}(t)$ of (7) the subimage vectors are then defined as:

$$\tilde{g}_{[m,n]}(t) = \Big[\tilde{g}_{m,n}, \tilde{g}_{m,n+1}, \cdots, \tilde{g}_{m,\Theta_2},$$
$$\tilde{g}_{m-1,n}, \tilde{g}_{m+1,n+1}, \cdots, \tilde{g}_{m+1,\Theta_2}, \cdots,$$
$$\tilde{g}_{\Theta_1,n}, \tilde{g}_{\Theta_1,n+1}, \cdots, \tilde{g}_{\Theta_1,\Theta_2}\Big]^T$$

Using this notation we can express any subimage as

$$\tilde{g}_{[m,n]}(t) = H_{[m,n]}V_{[m,n]} + \tilde{\eta}_{[m,n]}(t), \tag{9}$$

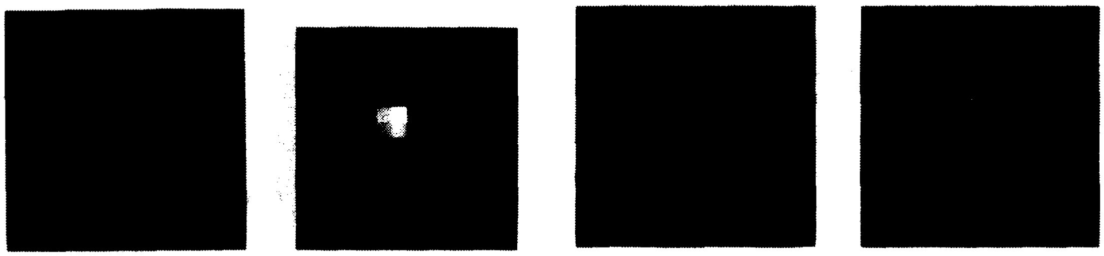
V is not Vandermonde as it is in the formulation of array smoothing, but fortunately there exists a simple relationship between the all of the $V_{[m,n]}$. We can express the $V_{[m,n]}$ in terms of the product of $V_{[1,1]}$ and the diagonal matrices $C^{m,n}$ whose elements are

$$\mathrm{diag}\left(C^{m,n}\right) = \Bigg[e^{-j2\pi\left(\frac{x_1(m-1)}{M_1}+\frac{y_1(n-1)}{M_2}\right)},$$
$$e^{-j2\pi\left(\frac{x_2(m-1)}{M_1}+\frac{y_2(n-1)}{M_2}\right)}, \ldots, e^{-j2\pi\left(\frac{x_P(m-1)}{M_1}+\frac{y_P(n-1)}{M_2}\right)} \Bigg].$$

Thus $V_{[m,n]} = V_{[1,1]}C^{m,n}$. Equation (9) becomes

$$\tilde{g}_{[m,n]}(t) = H_{[m,n]}V_{[1,1]}C^{m,n}u + \tilde{\eta}_{[m,n]}(t), \tag{10}$$

and assuming the noise is uncorrelated from pixel to pixel its autocovariance matrix is



(a) (b) (c) (d)

Figure 2. Synthetic star example. a) unblurred image. b) blurred image with 40dB SNR. c) MUSIC spectrum. d) peaks of spectrum.

$$\boldsymbol{R}_{[m,n]} = \boldsymbol{H}_{[m,n]}\boldsymbol{V}_{[1,1]}\boldsymbol{C}^{m,n}\boldsymbol{R}_u\left(\boldsymbol{C}^{m,n}\right)^H\boldsymbol{V}_{[1,1]}^H\boldsymbol{H}_{[m,n]}^H + I\sigma_\eta^2 \quad (11)$$

Before these subimages can be averaged, the modulation introduced by $\boldsymbol{H}$ must be removed. A single arbitrary diagonal regularization matrix, $\boldsymbol{Q}$, may be chosen and corresponding weighting matrices, $\boldsymbol{S}_{[m,n]}$, are computed such that $\boldsymbol{Q}=\boldsymbol{S}_{[m,n]}\boldsymbol{H}_{[m,n]}\ \forall\ [m,n]$, then

$$\begin{aligned}
\boldsymbol{R} &= \frac{1}{L}\sum_{\forall m,n}\boldsymbol{S}_{[m,n]}\boldsymbol{R}_{[m,n]}\boldsymbol{S}_{[m,n]}^{\mathrm{H}} \\
&= \boldsymbol{Q}\boldsymbol{V}_{[1,1]}\left[\frac{1}{L}\sum_{\forall m,n}\boldsymbol{C}^{m,n}\boldsymbol{R}_u\left(\boldsymbol{C}^{m,n}\right)^H\right]\boldsymbol{V}_{[1,1]}^{\mathrm{H}}\boldsymbol{Q}^{\mathrm{H}} + \frac{\sigma_\eta^2}{L}\sum_{\forall m,n}\left|\boldsymbol{S}_{[m,n]}\right|^2 \\
&= \boldsymbol{Q}\boldsymbol{V}_{[1,1]}\overline{\boldsymbol{R}}_u\boldsymbol{V}_{[1,1]}^{\mathrm{H}}\boldsymbol{Q}^{\mathrm{H}} + \sigma_\eta^2\overline{\boldsymbol{S}}. \\
&= \boldsymbol{Q}\boldsymbol{V}_{[1,1]}\overline{\boldsymbol{R}}_u\boldsymbol{V}_{[1,1]}^{\mathrm{H}}\boldsymbol{Q}^{\mathrm{H}} + \sigma_\eta^2\boldsymbol{Q}\boldsymbol{\Sigma}\boldsymbol{Q}^{\mathrm{H}},
\end{aligned} \quad (12)$$

where $\boldsymbol{\Sigma}$ is a diagonal matrix

$$\boldsymbol{\Sigma}_{i,i} = \sum_{\forall m,n}\left|\frac{1}{\left(\boldsymbol{H}_{[m,n]}\right)_{i,i}}\right|^2.$$

The weighting matrices $\boldsymbol{S}_{[m,n]}$ allow the corresponding $\boldsymbol{H}_{[m,n]}$ to be pulled out of the summation. As in the traditional DOA problem it can be shown that the rank $\overline{\boldsymbol{R}}_u$ will increase to a maximum of P for every additional subarray which is averaged in, if subimages from the diagonal (i.e. L_1=L_2) are used in the averaging. We shall refer to $\boldsymbol{Q}$ as the smoothing regularization matrix.

2.3. Restoration Algorithm

$\boldsymbol{Q}$ is selected depending upon the type of regularization desired, and the appropriate weighting matrices $\boldsymbol{S}_{[m,n]}$ are computed. The sample covariance matrix is computed as an estimate of $\boldsymbol{R}$

$$\hat{\boldsymbol{R}} = \frac{1}{L\Omega}\sum_{\forall[m,n]}\boldsymbol{S}_{[m,n]}\left(\sum_{l=1}^{\Omega}\tilde{\boldsymbol{g}}_{[m,n]}(lT)\tilde{\boldsymbol{g}}_{[m,n]}^H(lT)\right)\boldsymbol{S}_{[m,n]}^H \quad (13)$$

where T is the sample interval between snapshots. The algorithm is suitable for the single or multiple snapshot cases. If multiple snapshots are available, then a more accurate estimate of $\boldsymbol{R}$ is obtained. If only one is available and the image is of sufficient size so that the number of subimages is greater than the number of sources, then (19) will provide a estimate of $\boldsymbol{R}$ with necessary rank to render a signal subspace of full rank.

Estimates of the signal and noise subspaces, $\hat{\boldsymbol{E}}_s$ and $\hat{\boldsymbol{E}}_n$, are found by solving the generalized eigenvector problem:

$$\hat{\boldsymbol{R}}\left[\hat{\boldsymbol{E}}_s \mid \hat{\boldsymbol{E}}_n\right] = \hat{\boldsymbol{\Lambda}}\overline{\boldsymbol{S}}\left[\hat{\boldsymbol{E}}_s \mid \hat{\boldsymbol{E}}_n\right] \quad (14)$$

and the MUSIC spectrum is defined as:

$$\mathrm{P}(x,y) = \frac{\left(\boldsymbol{Q}\boldsymbol{v}_{x,y}\right)^H\left(\boldsymbol{Q}\boldsymbol{v}_{x,y}\right)}{\left(\boldsymbol{Q}\boldsymbol{v}_{x,y}\right)^H\hat{\boldsymbol{E}}_n\hat{\boldsymbol{E}}_n^{\,H}\left(\boldsymbol{Q}\boldsymbol{v}_{x,y}\right)}, \quad (15)$$

where $\boldsymbol{v}_{x,y}$ is the position vector corresponding to a point at location (x,y). The peaks of $\mathrm{P}(x,y)$ are the estimated locations of the stars. $\mathrm{P}(x,y)$ may be scanned at any desired resolution scale. Once source locations are found, a simple least squares fit can be made for the amplitudes of the peaks, because solving for the location and amplitude are separable problems [3].

2.4. Smoothing Regularization Matrix Selection

Noise amplification and reduced effective aperture are two conflicting problems that reduce resolution in $\mathrm{P}(x,y)$ and must be traded off against each other in the design of the smoothing regularization matrix $\boldsymbol{Q}$. If maximum aperture is desired, then it is best to weight $\boldsymbol{Q}$ equally among its diagonal elements, $\boldsymbol{Q}=\boldsymbol{I}$. This selection of $\boldsymbol{Q}$ produces corresponding weighting matrices $\boldsymbol{S}_{[m,n]}$ that inverse filter each of the subimages $\tilde{g}_{[m,n]}(t)$ and introduce noise amplification. This may be reduced if we sacrifice aperture by weighting more lightly or setting to zero elements of $\boldsymbol{Q}$ which correspond to small values in the $\boldsymbol{H}_{[m,n]}$ matrices. However, this introduces regularization error due to reduction in aperture.

With these trade-offs in mind, there are a number ways in which $\boldsymbol{Q}$ may selected. One heuristic method of accomplishing this is to let

$$\boldsymbol{Q} = \boldsymbol{\Sigma}^{-1/2} \tag{16}$$

which causes (12) to become,

$$\boldsymbol{R} = \boldsymbol{\Sigma}^{-1/2}\boldsymbol{V}_{[1,1]}\overline{\boldsymbol{R}}_u\boldsymbol{V}_{[1,1]}^{\mathrm{H}}\boldsymbol{\Sigma}^{-1/2} + \sigma_\eta^2\boldsymbol{I}. \tag{17}$$

Since $\boldsymbol{Q}$ has caused the noise covariance matrix to have equal energy, it is easy to see how it has de-emphasized those elements of the signal covariance matrix which correspond to small values in the PSF and therefore reduced the noise amplification.

Values of $\boldsymbol{Q}$ can be found which optimize various metrics. For example, a reasonable criterion based on a constrained maximization of signal to noise ratio is

$$\breve{\boldsymbol{Q}} = \arg\left\{\max_{Q}\left(\frac{\mathrm{tr}\left(\boldsymbol{Q}\boldsymbol{\mathcal{R}}\boldsymbol{Q}^{\mathrm{H}}\right)}{\sigma^2\,\mathrm{tr}\left(\overline{\boldsymbol{S}}\right)}\right)\right\},\ \sum_{i=1}^{r} Q_{i,i}^{\xi} = 1, \tag{18}$$

where $Q_{i,i} \geq 0$, and $\boldsymbol{\mathcal{R}} = \boldsymbol{V}_1\overline{\boldsymbol{R}}_u\boldsymbol{V}_1^{\mathrm{H}}$. Letting $\boldsymbol{q} = \mathrm{diag}(\boldsymbol{Q})$, (18) can be expressed as a minimization problem of the form

$$\breve{\boldsymbol{Q}} = \arg\left(\min_{Q}\left(\frac{\boldsymbol{q}^H\boldsymbol{\Sigma}\boldsymbol{q}}{\boldsymbol{q}^H\boldsymbol{q}}\right)\right),\ \sum_{i=1}^{r} q_i^{\xi} = 1,\ q_i \geq 0 \tag{19}$$

because $\mathcal{R}_{i,i} = \mathcal{R}_{k,k}$. This equality constrained minimization problem has no known closed form solution, but may be solved for a given value ξ of using a standard constrained minimization routine.

The positive power ξ determines the degree of regularization performed. As ξ tends toward zero, $\boldsymbol{Q}$ has maximum aperture and weights the elements of $\boldsymbol{R}$ equally. This corresponds to the inverse filter which has no regularization error, but has error due to noise amplification. As ξ tends to 2 and beyond, a $\boldsymbol{Q}$ is formed such that the aperture is reduced to a single element which introduces large regularization error but small error due to noise amplification.

3. Results

Figure 1 demonstrates the case of deblurring a synthetic star cluster using eigenvector based methods. The star image is blurred by a Gaussian shaped PSF and i.i.d. Gaussian noise is added for a SNR of 40 dB. The upper-left cluster contains two stars which are separated by only one half of a pixel. Figure 1.c is the processed image showing the MUSIC spectrum at a resolution of 4 times the original image. Figure 1.d shows the locations of the peaks of the MUSIC spectrum. Note that the original source positions are located correctly and that high resolution is obtained. Equation (16) was selected to generate $\boldsymbol{Q}$. In addition, the values of $\boldsymbol{Q}$ corresponding to certain high frequency regions of the $\boldsymbol{H}$ which have large modeling error were set to zero.

4. Conclusions

We have shown that there exists a duality between the PSL and the coherent DOA problems. Indeed, if the point sources are not blurred the two problems are frequency domain duals of one another. In the coherent DOA problem the aim is to determine the linear combination of cisoids that are superimposed on the linear array, while in PSL it is to determine the linear combination of shifts of a particular finite support function.

Since the resulting frequency domain problem is coherent, a means of rank enhancement of the covariance function was introduced. This technique is a generalization of smoothing because it introduces a regularization operator $\boldsymbol{Q}$ with its associated weighting matrices $\boldsymbol{S}$. These weighting matrices compensate for the affects of the modulation introduced by the blurring function, and allow regularization to reduce noise amplification in the system which is caused by the division of small values of the frequency domain blurring function.

Our algorithm could also be particularly useful in the 1-D case of finding the location of overlapping echoes. There is no requirement that the sources be time varying, and the case of data consisting of a single time sample is easily dealt with.

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