M12.5

# MAXIMALLY SPARSE RECONSTRUCTION OF BLURRED STAR FIELD IMAGES

*Brian D. Jeffs and Douglas Elsmore*
Brigham Young University
Provo, UT

### ABSTRACT

In this paper we address the problem of removing blur from, or sharpening, astronomical star field intensity images. A new image restoration algorithm is introduced which recovers image detail using a constrained optimization theoretic approach. Ideal star images may be modeled as a few point sources in a uniform background. It is therefore argued that a direct measure of image sparseness is the appropriate optimization criterion for deconvolving the image blurring function. A sparseness criterion based on the $l_p$ quasinorm is presented and an algorithm for sparse reconstruction is described. Synthetic and actual star image reconstruction examples are presented which demonstrate the algorithm's superior performance as compared with the CLEAN algorithm, a standard star field deconvolution method.

## 1. Introduction

Blur in astronomical star images may be due to atmospheric turbulence, misfocus, poor telescope tracking, finite aperture size, or other optical distortion effects. The blurring operation may be modeled as the two dimensional convolution of a point spread function (usually lowpass in spatial frequency) with the uncorrupted true image. If the point spread function is known, or may be estimated, then image resolution may be improved by any of several deconvolution techniques [1,2]. However, due to the low pass nature of the blurring function, observation noise, and the ill posed nature of the problem, a unique solution is not possible in general, and the type of solution obtained from any given algorithm is highly dependent on its underlying (explicit or implicit) objective function.

A common method for achieving high optical resolution in the presence of atmospheric blur is speckle interferometry, where phase information is extracted from the speckle pattern and used to sharpen the image. In many cases of interest however, only intensity information is available, and another approach is indicated. In this paper we address restoration and resolution enhancement of blurred star field intensity images.

We postulate that ideal star images are inherently sparse in nature, that is they are dominated by a constant flat field background intensity level with a small percentage of image pixels containing star intensity information. This image model justifies using a novel maximally sparse optimization criterion in the reconstruction algorithm, which in turn enables dramatic improvement in resolution. We adopt the following linearized image degradation model:

$$\underline{b} = \mathbf{H}\underline{x} + \underline{n} \qquad (1)$$

where $\underline{b}$ is the observation vector obtained by row scanning the sampled 2-D degraded image, $\underline{x}$ is the uncorrupted image vector, $\mathbf{H}$ is the blurring operator matrix representing the point spread function, and $\underline{n}$ is the additive observation noise vector. $\mathbf{H}$ is assumed known, or is estimated from the data by averaging the intensity distributions around several isolated stars. $\mathbf{H}$ need not be spatially invariant. The restoration problem is cast as one of linear inequality constrained nonlinear minimization:

$$\min_{\underline{x}} \; g(\underline{x}) \;\; \textit{such that} \;\; |\mathbf{H}\underline{x} - \underline{b}| \le \underline{\varepsilon}, \; x_i \ge 0 \qquad (2)$$

where the constraint vector $\underline{\varepsilon}$ represents our observation uncertainty due to the additive noise $\underline{n}$ and possible error in our knowledge of the point spread function. The solution vector $\underline{x}$ is constrained to be non-negative since we are dealing with intensity images from an incoherent imaging system.

There are in general an infinite number of admissible solutions which are consistent with the observed degraded image and satisfy the constraints of eqn. (2). The choice of objective function, $g(\underline{x})$, is key to controlling the form of the solution image, $\underline{x}$. The more common objectives, $l_2$ vector norm, entropy, etc. yield unacceptably smooth results which often do not improve image resolution [3]. Our prior knowledge that the desired image is sparse suggests that the appropriate objective function is a direct measure of solution sparseness. The restoration problem may then be restated as, "find the solution $\underline{x}$ which has the fewest possible non-zero elements and satisfies $|\mathbf{H}\underline{x} - \underline{b}| \le \underline{\varepsilon}$." We propose an objective, $g(\underline{x}) = \sum_{i=1}^{N} |x_i|^p$, which is related to the $l_p$ quasinorm, and which will be shown to be an excellent sparseness metric when $0<p<1$. Equation (2) may then be expressed as

$$\min_{\underline{x}} \; g(\underline{x}) = \sum_{i=1}^{N} (x_i)^p \;\; \textit{such that} \;\; |\mathbf{H}\underline{x} - \underline{b}| \le \underline{\varepsilon}, \qquad (3)$$

$$0<p<1, \;\; x_i \ge 0$$

In the following sections we justify the choice of $g(\underline{x})$ as a sparseness measure, describe an algorithm for solving eqn. (3), and present experimental results on restoring blurred star images.

## 2. Maximally Sparse Optimization With $l_p$ Quasinorms

The most obvious measure of image sparseness is a simple count of the non-zero pixels. This may be accomplished with an objective, $f(\underline{x})$, based on the indicator function:

$$f(\underline{x}) = \sum_{i=1}^{N} \mathbf{1}(x_i), \qquad \mathbf{1}(x_i) = \begin{cases} 1 & x_i \ne 0 \\ 0 & x_i = 0 \end{cases} \qquad (4)$$



CH2977-7/91/0000-2937 $1.00 © 1991 IEEE

$f(\underline{x})$ is not well suited as an objective function in an optimization algorithm and does not permit any control over the degree of sparseness desired [4]. $g(\underline{x})$ as defined above achieves an equivalent measure of sparseness without some of these difficulties. To demonstrate this consider the unit ball surfaces in $R^2$ for the quasi-norm $\|\underline{x}\|_{l_p} = (\sum_{i=1}^{N} |x_i|^p)^{1/p}$ as shown in Figure 1 for values of $p$ in the range $0 \le p \le \infty$. For $p \ge 1$ we have the conventional $l_p$ vector norm. The linear constraints in (3) form a convex set, and it is well known that within convex constraints, a local minimum of $\|\underline{x}\|_{l_p}$, $p \ge 1$, is a global optimum. Many efficient algorithms exist for solving such problems [5]. Of particular interest are the cases for values of $p=1,2$, and $\infty$, corresponding to linear, quadratic, and minimax objective functions, which form the basis of many widely used optimization procedures. However these methods do not achieve sparse results for an underdetermined problem of the form of eqn. (2).

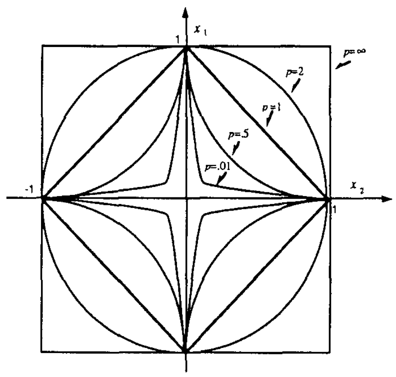


**Figure 1.** Unit Balls of the $l_p$ Norm for Various $p$.
Note that as $p$ approaches 0, the unit ball approaches the axes.

For $0<p<1$, $l_p$ is only a quasi-norm. Over $R^N$, $\|\underline{x}\|_{l_p}$ is neither convex nor concave, containing many strong local minima and presenting a difficult optimization problem. Large values of $p$ result in smooth solutions, however, as $p \to 0$ the solutions tend to become more "spiky," or sparse [6]. The reason for this can be seen in Figure 1. As $p \to 0$, the unit ball curves approach the $x_1, x_2$ axes, which is exactly where the unit ball surface for the indicator function of eqn. (4) lies. We identify minimum order optimization as a special case of generalized $l_p$ optimization. Since $g(\underline{x}) = (\|\underline{x}\|_{l_p})^p$ we have

$$\lim_{p \to 0} \; g(\underline{x}) = \sum_{i=1}^{N} |x_i|^p = \sum_{i=1}^{N} \mathbf{1}(x_i) = f(\underline{x}) \tag{5}$$

This suggests that we may use (at least in the limiting case) $g(\underline{x})$ from eqn. (3) for sparse optimization. In [4] it is proved that if the set of feasible solutions to the constraint equation of (3) are bounded in magnitude, then there is a finite $p_o>0$ such that for all $0<p\le p_o$ any solution to eqn (3) is in fact maximally sparse. $p$ need not approach zero to achieve sparse results. The utility of this observation is that for $p$ finite, $g(\underline{x})$ eliminates some of the handicaps of using $f(\underline{x})$ in an optimization algorithm while still yielding a maximally sparse solution. $g(\underline{x})$ is continuous everywhere and differentiable except at the axes. We may also adjust the desired sparseness of the solution by varying $p$ in the range $p_o \le p \le 1$.

## 3. The $l_p$ Simplex Search Algorithm

In [4] and [7] we introduced three theorems which together prove that solving eqn (3) is equivalent to solving the following problem involving linear *equality* constraints.

$$\min_{x} \; g(x) = \sum_{i=1}^{N} (x_i)^p \;\; such\ that \;\; \mathcal{H}x = b \,, \tag{6}$$

$$0<p<1, \quad x_i \ge 0$$

where

$$\mathcal{H} = \begin{bmatrix} \mathbf{H} & \mathbf{I} & \mathbf{0} \\ \mathbf{H} & \mathbf{0} & \mathbf{-I} \end{bmatrix}, \; x = \begin{bmatrix} \underline{x} \\ \underline{s}^+ \\ \underline{s}^- \end{bmatrix}, \; b = \begin{bmatrix} \underline{b} + \underline{\varepsilon} \\ \underline{b} - \underline{\varepsilon} \end{bmatrix}$$

$$\underline{x} \in R^N, \; \underline{s}^+, \underline{s}^- \in R^M, \; \underline{b} \in R^M, \; \mathcal{H} : (2M)\times(N+2M)$$

$\underline{s}^+$ and $\underline{s}^-$ are respectively slack and surplus variables as commonly employed in linear programming [5]. Note these variables are not included in the computation of $g(x)$ and are discarded in the final solution leaving only the maximally sparse $\underline{x}$.

It was also shown in [4,7] that the globally optimum solution to eqn. (6) must lie at an extreme point of the constraint set. These extreme points are called *basic feasible solutions*, and are defined as any $x$ which satisfies $\mathcal{H}x = b$, $x_i \ge 0$, and contains at most $M$ non-zero elements. This observation has tremendous significance since there are only a finite number of basic feasible solutions thus limiting the search space for the global optimum. Unfortunately, this set is prohibitively large. The restriction to basic feasible solutions does however suggest that an algorithm similar to the simplex method used for linear programming (LP) may be effective [5]. This is the basis for the algorithm presented here, which we have called the $l_p$ simplex search.

We may compute a basic feasible solution by partitioning $\mathcal{H}$

$$[\, \mathbf{A} \;|\mathbf{D}\, ] = \mathcal{H}, \; where\ \mathbf{A} \in R^{M \times M} \tag{7}$$

multiplying by $\mathbf{A}^{-1}$ leads directly to a basic solution $x_B$

$$[\, \mathbf{I} \,|\, \mathbf{A}^{-1}\mathbf{D}\, ]x = \mathbf{A}^{-1}b \,, \;\; x_B = [\, \mathbf{A}^{-1}b \,, 0, \cdots 0]^{\mathrm{T}} \tag{8}$$

Any choice of $M$ columns from $\mathcal{H}$ for which $\mathbf{A}^{-1}$ is non-singular is acceptable for the initial solution $x_B$. The variables $x_i$ associated with columns of $\mathbf{A}$ are termed basic variables, and $\mathbf{A}$ the basis. As in the standard LP simplex algorithm, program iterations consist of "pivoting" a column from $\mathbf{D}$ into the basis, and the appropriate column from $\mathbf{A}$ out of the basis into $\mathbf{D}$. Each pivot represents a move from one basic solution to an adjacent basic solution, which may be read directly from the right hand side of eqn. (8). The utility of the simplex algorithm is that we need not recompute $\mathbf{A}^{-1}$ explicitly with each pivot. Once the columns to enter and leave the basis are selected, a simple algebraic operation is performed on each element of the system (7) to implement the pivot move [5].

Locating the optimum basic feasible solution is accomplished by appropriate selection of entering and leaving columns at each pivot, thus choosing only those moves to adjacent basic solutions which reduce the cost $g(x)$. The sequence of iterations produces a series of solutions with monotonically decreasing cost. The algorithm terminates when all surrounding adjacent solutions are of higher cost than the current one.

With the linear cost function used in LP, the normal pivot computations can be augmented to lead to a direct indication of the best pivot column to select. however in the $l_p$ simplex search the nonlinear cost $g(x)$ must be explicitly computed for each candidate adjacent basic solution, and one which leads to a cost reduction is selected. The steps of the $l_p$ simplex search algorithm are as follows:

1) $i = 0$, find any initial basic feasible solution to eqn. (6) $x_{B_0} = [ A^{-1}b , 0, \cdots 0]^T$.
2) Compute the cost of the bounded basic feasible solutions adjacent to $x_{B_i}$
3) If no adjacent solution is of equal of lower cost, terminate, optimum found. Otherwise, if any lower cost solutions exist, select one and pivot to it. Otherwise, perform an anti-cycling procedure [5] to pivot to an equal cost solution. Increment $i$ and set $x_{B_i}$ to the new solution.
4) Repeat 2) and 3) to termination.

This algorithm has been demonstrated in several different applications to produce excellent approximations to the optimally sparse solution, but due to the nonlinearity of the cost function, $g(x)$, it may terminate before a true global optimum is located. Results do however compare favorably with existing algorithms for sparse star field image restoration and such other diverse applications as sparse beamforming array design, seismic deconvolution, and neuromagnetic image reconstruction [4,7].

When global optimality is of paramount concern, the $l_p$ simplex search is easily adapted to a simulated annealing algorithm [4]. Here the choice of which adjacent solution to pivot to (step 3 above) is not always based on strict reduction of the cost. A degree of randomness is introduced in the choice so that less sparse solutions may occasionally be chosen in a given iteration. The degree of randomness is gradually reduced according to an "annealing schedule" until the algorithm stabilizes at the globally optimum solution. Selection of an appropriate annealing schedule is a trade-off between the required computation time to reach termination, and the degree of solution sparseness.

## 4. Comparison with the CLEAN Algorithm

Perhaps the most widely accepted algorithm for incoherent star field deblurring is CLEAN, introduced in 1974 by Högbom [1]. It has been used successfully for deconvolving atmospheric blur, imaging system point spread functions, and other distortion sources from both optical and radio telescopic star images. A CLEAN iteration consists essentially of locating the peak blurred image intensity value, then subtracting a scaled copy of the known point spread function, centered on the peak, from the image. This process is repeated until the peak in the residual image is below a predetermined error tolerance limit. The "cleaned" image consists of nonzero values only in the locations corresponding the peaks which were processed [1,8].

This procedure has similarities with the $l_p$ simplex search. Both methods are "designed" to operate on sparse sources, that is when the true image consists largely of blank sky [1,8]. In fact, if this model is inappropriate, neither method will yield acceptable results. Since CLEAN iterations terminate when restored image error drops below a specified limit, it may also be cast in the form of the constrained optimization of eqn. (2). CLEAN however was proposed as an ad-hoc procedure, with no optimization theoretic basis. Although it clearly sharpened processed images, it was unclear what underlying implicit objective function, $g(\underline{x})$, was active.

It has more recently been shown that the underlying objective in CLEAN approximates $l_1$ minimization [8]. A cleaned image minimizes the sum of pixel intensities within the error constraints:

$$\min_{\underline{x}} \sum_{i=1}^{N} x_i \text{ such that } |H\underline{x} - \underline{b}| \leq \underline{\varepsilon},\ x_i \geq 0 \qquad (9)$$

This is readily seen as a special case of the $l_p$ optimization provided by the $l_p$ simplex search. In [2] Stark points out the enhanced resolution and generally sparse results in images restored by solving a system equivalent to eqn. (9). $l_1$ minimization however cannot in general achieve the *maximally* sparse result. As was shown in [4], solutions continue to be more sparse as $p$ decreases from 1 to some $p_o$, at which time eqn. (4) yields the maximally sparse result. Thus CLEAN, having been designed to recover a true image consisting a few stars in a blank sky, can fall short of its stated objective. In the following section we present examples where the $l_p$ simplex search outperforms CLEAN and restores resolution with fewer extraneous artifacts.

## 5. Results

In this section we present experimental results of star field image restoration using the $l_p$ simplex search. Figures 2 through 5 are from a synthetic image case and give a comparison between the $l_p$ simplex search and CLEAN. Figures 6 and 7 are from an actual telescopic star image. All images shown are 20 by 20 pixels in size.

Figure 2 shows the original ideal image of a star pair in a black background. Each star has an intensity of 1.9. Figure 3 is the degraded version of Figure 1. A 2-D Gaussian function with a standard deviation of 2.5 pixels was used to blur the image, and spatially independent Rayleigh distributed noise was added for a signal to noise ratio of 39 dB ($\sigma = .02$). Figure 4 is the the $l_p$ simplex search restoration of Figure 3. For this example values of $p = .14$ and $\varepsilon = .06$ were used. Both of the original stars were resolved, though the intensity of the upper right star was incorrectly estimated lower than in the true image.

Figure 5 is the CLEAN reconstruction of Figure 3. As in Figure 4 the error limit was $\varepsilon = .06$, while the loop gain [8] was set to 0.95. Note that an extraneous star artifact to the lower left was generated. We have noted in our experiments that the more severely blurred images will be correctly resolved by the $l_p$ simplex search while CLEAN will introduce extraneous stars. If the blurring is less severe, so that a peak can be located in the degraded image corresponding to each of the true stars, then $l_p$ simplex search and CLEAN produce equivalent results.

Figure 6 is a star group near globular cluster M15, extracted from an actual telescopic image. This image has a resolution of approximately 0.5 arc seconds per pixel. Figure 7 is the $l_p$ simplex search restoration of Figure 6, using values of $p = .14$ and $\varepsilon = .08$. The blurring point spread function used in the restoration was proposed by King [9], and consists of a Guassian central lobe, an exponential decay outer ring, followed by an inverse square decay on the outer skirts. This function provides a good match to a wide range of observed star intensity point spread functions. The radius of the function was scaled to closely match isolated stars seen in the original image near the extracted window shown in figure 6.

## 6. Conclusions

The above examples demonstrate that maximally sparse restoration is a promising approach to star deblurring. It is ideally suited to the case where only point sources, and no distributed objects, exist in the true image field. The form of a restored image is highly dependent on the objective function (explicit or implicit) underlying the restoration algorithm. We may argue that when a sparse source image model is used, the only justifiable objective is a measure of sparseness. The $l_p$ simplex search is a practical maximally sparse algorithm, and outperforms CLEAN in cases of severe blurring.

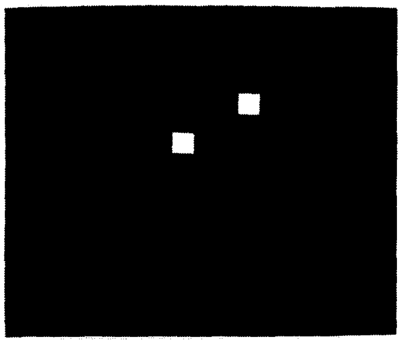

**Figure 2.** Unblurred synthetic star pair image.

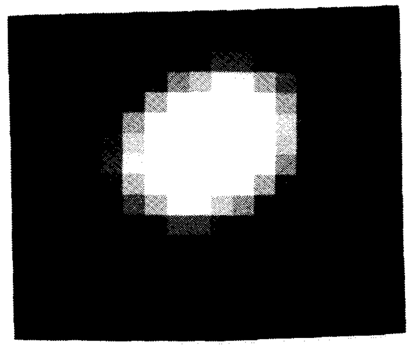

**Figure 3.** Blurred and noise corrupted version of Figure 2. Gaussian blur with $\sigma$ =1.5 pixels. -39 dB noise.

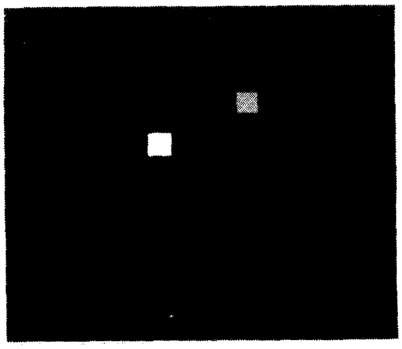

**Figure 4.** $l_p$ simplex search restoration of Figure 3.

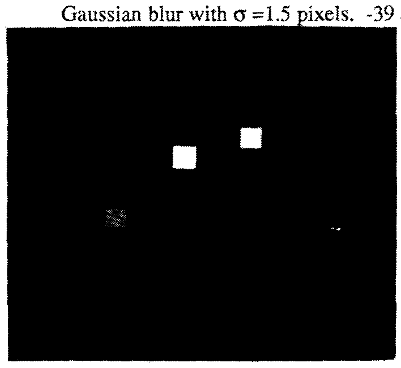

**Figure 5.** CLEAN algorithm restoration of Figure 3.

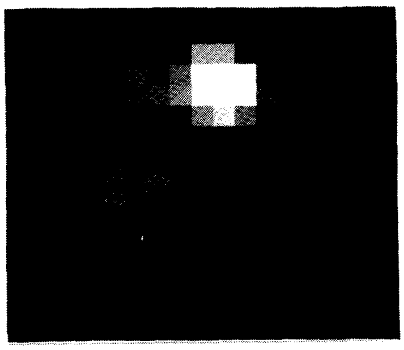

**Figure 6.** Star group near globular cluster M15.

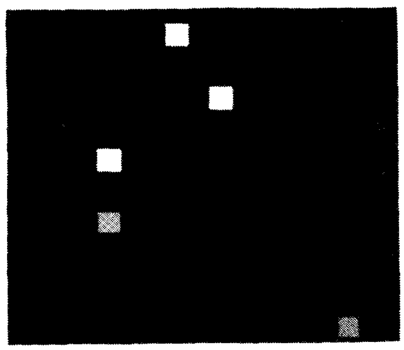

**Figure 7.** $l_p$ simplex search restoration of Figure 6.

## 7. References


[1] J.A. Högbom, *Astronomy and Astrophysics Supplement*, Vol. 15, p. 417, 1974.

[2] H. Stark, *Image Recovery: Theory and Application*, Academic Press, 1987.

[3] B. Jeffs, R. Leahy and M. Singh, "An Evaluation of Methods for Neuromagnetic Image Reconstruction," *IEEE Trans. Biomed. Eng.*, Vol. BME-34, pp. 713-723, 1987.

[4] B.D. Jeffs, "Maximally Sparse Constrained Optimization for Signal Processing Application," Ph.D. dissertation, University of Southern California, January 1989.

[5] D.G. Luenberger, *Linear and Nonlinear Programming*, second edition, Addison-Wesley, Reading, Mass, 1984.

[6] W. Gray, "Variable norm deconvolution", Ph. D. Thesis, Stanford University, 1979.

[7] R. Leahy and B. Jeffs, "On the Design of Maximally Sparse Beamforming Arrays," Submitted to *IEEE Trans. AP*, Aug. 1990.

[8] K.A. Marsh and J.M. Richardson, "The objective function implicit in the clean algorithm," *Astron. Astrophys.*, Vol. 182, pp.17~178, 1987.

[9] I.R. King, "The Profile of a Star Image," Pub. Astron. Soc. Pacific, Vol. 86, April 1971.