RANK ENHANCEMENT FOR EIGENSTRUCTURE BASED DIRECTION FINDING USING ARRAYS WITH NON-UNIFORM ELEMENT RESPONSES

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ABSTRACT

This paper presents a new spatial smoothing approach for rank enhancement in sensor array direction finding applications. The method works with coherent scene data when non-uniform array element gains would make conventional smoothing ineffective. Sub-array data is pre-weighted before smoothing to regularize the problem, and so that a consistent phase and gain structure is maintained across the sub-arrays.

1. INTRODUCTION

Many of the popular high resolution direction finding (DF) algorithms exploit the eigenstructure of the sensor array covariance matrix to estimate directions to signal sources. Unless rank enhancement is performed, these algorithms fail when multiple sources in the observed scene are coherently related. In this coherent scene environment, the signal subspace component of the autocovariance matrix has rank less than the number of arriving signals. The usual solution is to increase signal subspace rank with a spatial smoothing operation prior to applying the DF algorithm [1]. However, spatial smoothing algorithms require that sensor elements have identical gain and phase responses, as is the case, for example, with the uniform line array (ULA).

This paper addresses the problem of direction finding with a non-uniformly weighted array in the coherent scene environment. Non- uniform element response can be part of the array design, as in the case of physically shaded beamforming arrays, or unintended, as for example when there are element failures, gain errors, or when there is undesired baffling. We will assume that the non-uniform element responses are known, or can be estimated by an array calibration procedure.

We will fist review conventional spatial smoothing as introduced by Shan et al for the (ULA) [1]. Assuming an array of M elements with wavefronts from P narrowband sources impinging upon it, the output from the array at time t may be modeled as

$$\mathbf{x}(t) = \mathbf{A}\mathbf{u}(t) + \eta(t) \tag{1}$$

where $\mathbf{x}(t)$ is the observed array data vector, $\mathbf{u}(t)$ is the length P vector of source amplitudes, the columns of $\mathbf{A}$ are the complex array response vectors corresponding to each of the P sources, and $\eta(t)$ is the additive observation noise. $\mathbf{A} = [\mathbf{a}(\theta_1)|\mathbf{a}(\theta_2)|\cdots|\mathbf{a}(\theta_p)]$, where θ_p is the direction of arrival of the p^{th} source. For the case of a uniform line array and far-field sources, $\mathbf{A}$ is Vandermonde, with columns

$$\begin{aligned} \mathbf{a}(\theta_p) &= [1, e^{-j\varphi(\theta_p)}, e^{-j2\varphi(\theta_p)}, \cdots, e^{-j(M-1)\varphi(\theta_p)}]^T \quad (2) \\ \varphi(\theta) &= \frac{\omega_0 d}{c} sin(\theta) \end{aligned}$$

where ω_0 is the signal center frequency, d the inter-element spacing, and c the wave propagation speed. The autocovariance of $\mathbf{x}(t)$ (assuming zero mean random processes and i.i.d. noise) is given by

$$\mathbf{R} = E\{\mathbf{x}(t)\mathbf{x}^H(t)\} = \mathbf{A}\mathbf{R}_u\mathbf{A}^H + \sigma_\eta^2\mathbf{I} \tag{3}$$

where $\mathbf{R}_u$ and $\sigma_\eta^2\mathbf{I}$ are the autocovariance matrices of $\mathbf{u}(t)$ and $\eta(t)$ respectively.

For the coherent source case of interest, $\mathbf{R}_u$ has rank less than P. Conventional smoothing increases the rank of $\mathbf{A}\mathbf{R}_u\mathbf{A}^H$ by averaging over autocovariance matrices of shorter sub-arrays. L distinct (though overlapping) sub-arrays may be defined for an M element ULA as

$$\begin{aligned} \mathbf{x}_{[i]}(t) &= [x_i(t), x_{i+1}(t), \cdots, x_{M-L+i}(t)]^T \\ & \quad 1 \le i \le L \end{aligned} \tag{4}$$

Due to the Vandermonde structure of $\mathbf{A}$ for the ULA, equation (1) can be modified as follows to represent any sub-array:

$$\begin{aligned} \mathbf{x}_{[i]}(t) &= \mathbf{A}\mathbf{C}^{i-1}\mathbf{u}(t) + \eta_{[i]}(t) \\ \mathbf{C} &= diag\left\{[e^{-j\varphi(\theta_1)}, e^{-j\varphi(\theta_2)}, \cdots, e^{-j\varphi(\theta_P)}]\right\} \end{aligned} \tag{5}$$

where columns of $\mathbf{A}$ are truncated to the sub-array length. The smoothed covariance matrix is then computed as

$$\bar{\mathbf{R}} = \frac{1}{L}\sum_{i=1}^{L} E\{\mathbf{x}_{[i]}(t)\mathbf{x}_{[i]}^H(t)\} \tag{6}$$

$$\begin{aligned} \bar{\mathbf{R}} &= \mathbf{A}\left[\frac{1}{L}\sum_{i=1}^{L}\mathbf{C}^{i-1}\mathbf{R}_u(\mathbf{C}^{i-1})^H\right]\mathbf{A}^H + \sigma_\eta^2\mathbf{I} \\ &= \mathbf{A}\bar{\mathbf{R}}_u\mathbf{A}^H + \sigma_\eta^2\mathbf{I}. \end{aligned} \tag{7}$$

It has been shown that if $L > P$, then $\bar{\mathbf{R}}_u$ has rank P and retains the desired phase information for each source [1]. These are the conditions necessary for successfully applying eigenstructure based DF algorithms, like MUSIC [2], directly to $\bar{\mathbf{R}}$.

2. DEVELOPMENT

With non-uniform element responses, equation (1) is modified as follows

$$\mathbf{x}(t) = \mathbf{SAu}(t) + \eta_s(t) \tag{8}$$

where $\mathbf{S}$ is a diagonal shading matrix containing (possibly complex) element gains with respect to the signal. If the noise component at each element undergoes the same scaling as the signal, then $\eta_s(t) = \mathbf{S}\eta(t)$, and the obvious approach would be to multiply equation (8) by $\mathbf{S}^{-1}$, which yields the same smoothing problem as equation (7). However, since observed noise may arise from effects other than linear propagation in the medium (e.g. electronic noise in the analog front-end circuits, quantization, structural vibrations, etc.), this simple inverse shading approach is often unacceptable. It is not unusual, for example, in a SONAR hydrophone array to simultaneously experience a decrease in a channel's signal sensitivity while its noise level goes up due to some mechanical or acoustic failure mode. In such cases, the direct inverse shading approach will introduce noise amplification which degrades DF performance. The regularization offered by the following proposed method is needed to stabilize the problem.

The covariance matrix of the i^{th} shaded sub-array is given by

$$\mathbf{R}_{[i]} = \mathbf{S}_{[i]}\mathbf{AC}^{i-1}\mathbf{R}_u(\mathbf{C}^{i-1})^H\mathbf{A}^H\mathbf{S}_{[i]}^H + \Sigma_{[i]} \tag{9}$$

where the diagonal of $\mathbf{S}_{[i]}$ contains the i through $(M-L+i)^{th}$ diagonal elements of $\mathbf{S}$, and $\Sigma_{[i]}$ is the autocovariance of $\eta_{s[i]}$. Note that due to shading, we have had to drop the i.i.d. noise assumption. Since $\mathbf{S}_{[i]}$ is dependent on the sub-array index, a smoothed covariance computed as in equation (6) can not be factored as in equation (7) to produce a rank enhanced $\bar{\mathbf{R}}_u$. To overcome this difficulty, a method first introduced by the author for point source image restoration is adopted [3]. We define a "smoothing regularization matrix," $\mathbf{Q}$, which is chosen to be an arbitrary, constant, diagonal matrix. Given a choice for $\mathbf{Q}$, data pre-weighting matrices, $\mathbf{W}_{[i]}$, are computed as solutions to

$$\mathbf{Q} = \mathbf{W}_{[i]}\mathbf{S}_{[i]} \;\; \forall\{i|1 \le i \le M-L+1\}. \tag{10}$$

The weighted smoothed covariance matrix is then

$$\bar{\mathbf{R}}_w = \frac{1}{L}\sum_{i=1}^{L} E\{\mathbf{W}_{[i]}\mathbf{x}_{[i]}(t)\left(\mathbf{W}_{[i]}\mathbf{x}_{[i]}(t)\right)^H\} \tag{11}$$

$$\begin{aligned}\bar{\mathbf{R}}_w &= \mathbf{QA}\left[\frac{1}{L}\sum_{i=1}^{L}\mathbf{C}^{i-1}\mathbf{R}_u(\mathbf{C}^{i-1})^H\right]\mathbf{A}^H\mathbf{Q}^H \\ &\quad + \frac{1}{L}\sum_{i=1}^{L}\mathbf{W}_{[i]}\Sigma_{[i]}\mathbf{W}_{[i]}^H \\ &= \mathbf{QA}\bar{\mathbf{R}}_u\mathbf{A}^H\mathbf{Q}^H + \bar{\Sigma}.\end{aligned} \tag{12}$$

Pre-weighting the sub-array data replaces $\mathbf{S}_{[i]}$ of equation (9) with the constant matrix $\mathbf{Q}$, which can be factored out of all summation terms, thus providing the desired form for $\bar{\mathbf{R}}_u$ in (12). With proper design of $\mathbf{Q}$, it can be shown [3] that $\bar{\mathbf{R}}_w$ will have a signal subspace with rank equal to P, and will be suitable for eigenstructure based DF algorithms, such as MUSIC.

A MUSIC based DF algorithm using the proposed weighted smoothing includes the following steps:

1. Select a $\mathbf{Q}$ matrix and compute the associated $\mathbf{W}_{[i]}$ as solutions to (10).

2. Compute a sample estimate of $\bar{\mathbf{R}}_w$ by pre-weighting sub-array data samples and averaging over time

$$\hat{\mathbf{R}}_w = \frac{1}{LN}\sum_{i=1}^{L}\sum_{n=1}^{N-1}\{\mathbf{W}_{[i]}\mathbf{x}_{[i]}(nT)\left(\mathbf{W}_{[i]}\mathbf{x}_{[i]}(nT)\right)^H\} \tag{13}$$

where T is the sample spacing and N is the total number of samples.

3. Solve the generalized eigenvector problem,

$$\hat{\mathbf{R}}_w[\hat{\mathbf{E}}_s|\hat{\mathbf{E}}_\eta] = \bar{\Sigma}[\hat{\mathbf{E}}_s|\hat{\mathbf{E}}_\eta]\Lambda \tag{14}$$

to partition the signal ($\hat{\mathbf{E}}_s$) and noise ($\hat{\mathbf{E}}_\eta$) subspaces.

4. Scan the modified MUSIC spectrum, $D(\theta)$, to estimate the directions of arrival

$$D(\theta) = \frac{\mathbf{a}^H(\theta)\mathbf{Q}^H\mathbf{Q}\mathbf{a}(\theta)}{\mathbf{a}^H(\theta)\mathbf{Q}^H\hat{\mathbf{E}}_\eta\hat{\mathbf{E}}_\eta^H\mathbf{Q}\mathbf{a}(\theta)}. \tag{15}$$

5. Local maxima in $D(\theta)$ correspond to source directions

$$\hat{\theta}_p = \arg\left\{\max_{\theta}{}_p\{D(\theta)\}\right\} \tag{16}$$

where $\hat{\theta}_p$ is the estimated bearing for the p^{th} source and $\max_p$ indicates the p^{th} local extremum.

Though $\mathbf{Q}$ is arbitrary, its design does affect DF performance. We will discuss two possible choices. First, if $\mathbf{Q} = \mathbf{I}$, then $\mathbf{W}_{[i]} = \mathbf{S}_{[i]}^{-1}$. This simple choice for $\mathbf{Q}$ is only acceptable if $\mathbf{S}$ is invertible and noise amplification is not an extreme problem. A second formulation for $\mathbf{Q}$ which provides improved regularization, eliminates noise amplification, and is always computable, is given by

$$q_m = \min_i \left|s_m^{[i]}\right|, \;\; \forall m, \; 1 \le m \le M-L+1 \tag{17}$$

$$w_m^{[i]} = \begin{cases} \frac{q_m}{s_m^{[i]}} & |q_m| > \epsilon \\ 0 & |q_m| = \epsilon \end{cases} \tag{18}$$

where $diag\{\mathbf{Q}\} = [q_1, \cdots, q_L]^T$, $diag\{\mathbf{S}_{[i]}\} = [s_1^{[i]}, \cdots, s_L^{[i]}]^T$ and $diag\{\mathbf{W}_{[i]}\} = [w_1^{[i]}, \cdots, w_L^{[i]}]^T$ ϵ is chosen close to zero (rather than at zero) to avoid numerical evaluation problems in computing the ratio $\frac{q_m}{s_m^{[i]}}$. Note however that since we avoid division by zero, (18) is well defined, and the weighting matrix is bounded by $0 \le w_m^{[i]} \le 1$.

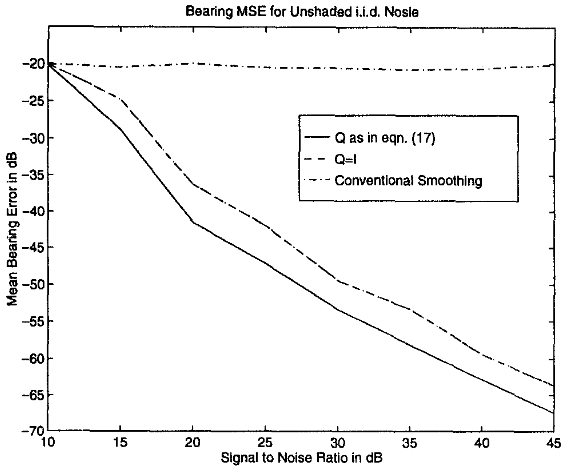

Figure 1. Bearing estimation error as a function of SNR. Solid line is for $\mathbf{Q}$ designed as in equation (17), dashed line is for $\mathbf{Q} = \mathbf{I}$. Dash-dot line is for conventional smoothing, and is limited to -20 dB only because the MUSIC scan window was limited and did not allow more error.

3. PERFORMANCE ANALYSIS

In this section we address the issue of how various algorithm parameters affect weighted spatial smoothing performance. Mean squared error in the bearing estimate for the p^{th} source, $\epsilon^2_{\theta_p} = E\{(\hat{\theta_p} - \theta_p)^2\}$, will be used as the performance metric. Among the factors which can influence estimation error are the choice for $\mathbf{Q}$, both full array and sub-array sizes, the number of sub-arrays used in smoothing, SNR, and the number of time samples available. Monte-Carlo simulations will be presented to quantify error as a function of some of these parameters.

Figure 1 illustrates how the choice of $\mathbf{Q}$ can affect performance. Mean squared bearing error was estimated by averaging over 200 random trials for each graph entry, with each trial including an array data set of 50 time samples. A Hamming shaded 12 element line array, with the fifth element response set to 0.01, was used. Noise was i.i.d. (at the processor), Gaussian, and equal amplitude sources were located at zero and 17 degrees. Error was computed for the 17 degree source based on the distance to the nearest peak in the MUSIC spectrum. The solid curve corresponds to $\mathbf{Q}$ computed as in equation (17), and shows that for this choice, position error can be more than 5 dB less than when $\mathbf{Q} = \mathbf{I}$. It is believed this is due to noise amplification at the fifth sensor element when $\mathbf{Q} = \mathbf{I}$. Note that for conventional smoothing $\epsilon^2_{\theta_p}$ does not drop as SNR increases. This is because the two sources were not resolved at any SNR, and -20 dB error corresponds to a uniform distribution of bearing estimates over the limited bearing window used in the MUSIC scan.

Figure 2 shows how bearing error depends on the number of sub-arrays, L used in smoothing for a fixed full array size. Sub-array size was the largest possible for the given full array, i.e. $M - L + 1$. The same array shading, source configuration, and data parameters used in Figure 1 were

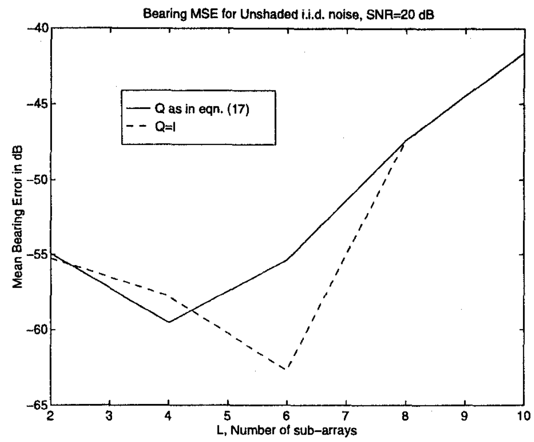

Figure 2. Bearing estimation error as a function of sub-array size, L, for two choices of $\mathbf{Q}$. Solid line is for $\mathbf{Q}$ designed as in equation (17), dashed line is for $\mathbf{Q} = \mathbf{I}$.

used in this analysis. Note that the best choice of the number of sub-arrays to use can also depend on the choice for $\mathbf{Q}$.

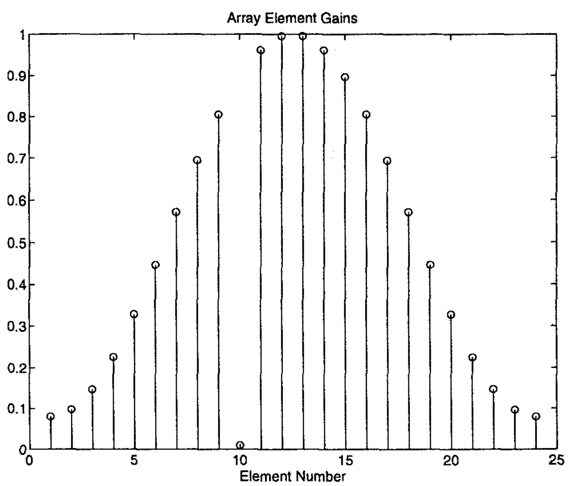

Figure 3. Array shading, $diag\{\mathbf{S}\}$, used for the 24 element array. Overall shading is from a Hamming window, with element 10 set to 0.01.

4. APPLICATION EXAMPLES

The weighted spatial smoothing algorithm is demonstrated in the following computer simulated experiments with a 24 element line array. As shown in Figure 3, the array shading, $\mathbf{S}$, was based on a Hamming window, with the tenth element set to an amplitude response of 0.01 to simulate a sensor failure. Array elements were separated by 0.3 wavelength, and equal amplitude coherent sources were located in the far field at bearings of -40, -31, 15 and 25 degrees relative to array broadside. The data set consisted of 100 time samples per channel.

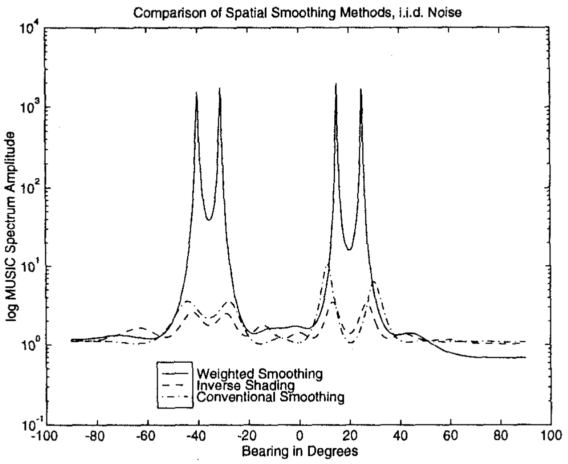

Figure 4. MUSIC spectrum plots for i.i.d. Gaussian noise, not scaled by element shading. True source locations are -40, -31, 15 and 25 degrees.

Figure 4 shows comparative results for the case of unshaded i.i.d. Gaussian noise, i.e. when array signal response is as given by Figure 3, but noise arises from sources unaffected by shading. The SNR at the center array elements was 20 dB, and $\Sigma_{[i]} = \sigma_\eta^2 \mathbf{I}$. The solid curve plots the MUSIC spectrum, $P(\theta)$, obtained using the proposed weighted smoothing method. Equation (17) was used to specify $\mathbf{Q}$. This response clearly shows a higher resolution, more accurate estimate of source locations than inverse shading with conventional smoothing, or simple conventional smoothing.

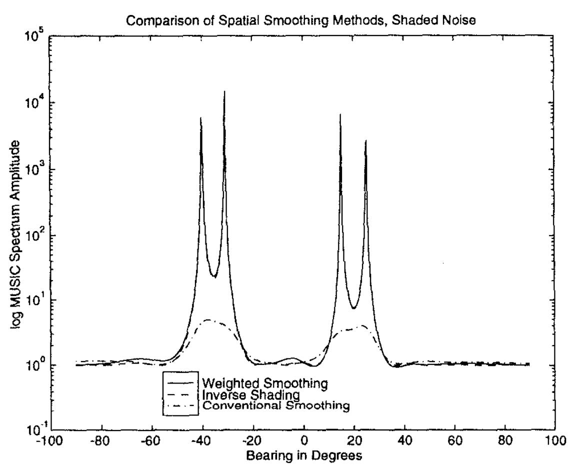

Figure 5. MUSIC spectrum plots for shaded noise case. Noise seen by the processor is i.i.d. Gaussian noise which has been scaled by the element gains in Figure 3.

Figure 5 shows performance for shaded noise, where non-uniform element responses affected signal and noise levels equally. The SNR at each element was 20 dB, and $\eta_s(t) = \mathbf{S}\eta(t)$, with $\eta(t)$ distributed i.i.d across the array. In this case the weighted smoothing and inverse shading performances are essentially equivalent, while conventional smoothing fails to resolve the sources because the non-uniform element responses were ignored.

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