Gain and Aperture Efficiency for a Reflector Antenna With an Array Feed

Karl F. Warnick, *Senior Member, IEEE,* and Brian D. Jeffs

***Abstract*—A figure of merit for array antennas can be defined using the signal processing concept of signal-to-noise ratio (SNR) gain. The resulting array efficiency metric reduces to aperture efficiency for an isolated array and to a combination of aperture and spillover efficiency for an array feed. These results provide a practical way to measure aperture efficiency for active arrays and array feeds.**

***Index Terms*—active arrays, antenna array feeds, aperture efficiency, array gain, gain.**

I. Introduction

THE aperture efficiency of a passive antenna is the ratio of received power to incident power. For an array antenna with an active signal combining network, this simple definition fails, but if the directivity D of the antenna is known, the aperture efficiency can be defined using

$$\eta_{ap} = \frac{\lambda^2 D}{4\pi A_p} \tag{1}$$

where A_p is the projected physical area of the antenna. The directivity can be obtained by considering an equivalent transmitting array and finding the radiation pattern, but this can be difficult for systems with complex network topologies. The directivity can also be obtained directly from the receiving pattern of the array, but the pattern must be known at all angles in order to properly normalize the received power. For some applications, especially array-fed reflector antennas [1]–[5], a more practical way to measure aperture efficiency would facilitate performance comparisons with single-port antennas in terms that are familiar to the antenna engineer.

Much of the existing literature on array efficiency has focused on the problem of the aperture efficiency of a single element embedded in an array. Stein gave an upper bound on the aperture efficiency of an embedded element in terms of pattern overlap with other elements based on conservation of energy [6]. Hannan defined the element aperture efficiency in terms of active reflection coefficient [7]. More recent treatments of this problem include [8]. Beamformer performance is also commonly quantified using taper efficiency [9].

In this letter, we consider an alternate definition of array efficiency in terms of signal-to-noise ratio (SNR) gain. For an array, the SNR gain is the ratio of SNR at the output of the beamforming network to the SNR obtained with a reference antenna:

$$G_a = \frac{\mathrm{SNR}_{\mathrm{out}}}{\mathrm{SNR}_{\mathrm{in}}}. \tag{2}$$

In the signal processing literature, G_a is referred to as array gain [10]. Despite the overlap in the terminology, array gain is different from the standard definition of gain as directivity reduced by radiation efficiency, although as we will see the two quantities are closely related and can be equal under certain conditions.

By analogy with (1), an array efficiency measure can be defined with the directivity replaced by the array gain G_a:

$$\eta_a = \frac{\lambda^2 G_a}{4\pi A_p}. \tag{3}$$

This figure of merit was proposed for array feeds by Ernest Jacobs in 1985 [11], but it appears to have attracted no further attention in the literature. The reason for this may be that no connection between Jacobs' figure of merit and other measures of antenna performance was known at that time. We will show that η_a is equal to the aperture efficiency η_{ap} for an array antenna in a spatially isotropic noise environment, and for an array feed, η_a is a combination of the aperture efficiency and spillover efficiency. These results provide a practical method for measuring aperture efficiency for array antennas and array-fed reflectors.

For single-port antennas, it is standard practice to refer signal and noise powers to the output port of the antenna. For an array feed, there is no simple way to refer powers to the feed, because the feed has multiple ports. The results of this letter provide a way to refer received power levels to an equivalent passive feed which has the same gain as the array.

In this letter, we assume that the system processing bandwidth is narrow enough to allow for a single-frequency (ω) propagation analysis, and all field and signal quantities are represented as phasors with the frequency dependence suppressed. Electric field vectors are denoted by an overbar, whereas vectors of length N associated with array output voltages are in boldface.

II. Transmit Array

We begin by establishing notation for an array operating in the transmit mode. For convenience, we assume that the array consists of N identical elements with input impedance Z_{in}. Let $\overline{E}_m(\mathbf{r})$ denote the radiated electric field of the mth array element in the presence of the reflector, and $\overline{E}^i_m(\mathbf{r})$ the radiated field in the absence of a reflector, with an input current amplitude of I_0. Neglecting mutual coupling, the embedded patterns

Manuscript received April 14, 2006; revised September 19, 2006. This work was supported by the National Science Foundation by Grant 0352705.

The authors are with the Department of Electrical and Computer Engineering, Brigham Young University, Provo, UT 84602 USA (e-mail: warnick@byu.edu).

Digital Object Identifier 10.1109/LAWP.2006.886308

will be identical to the isolated element patterns. For later convenience, we introduce the normalized radiated fields

$$\overline{e}^i_m(\Omega) = \frac{4\pi j r e^{jkr}}{k\eta I_0}\overline{E}^i_m(\mathbf{r}) \tag{4}$$

with an analogous definition for $\overline{e}_m$. In this expression, Ω and r are the spherical angle and magnitude of the vector $\mathbf{r}$, respectively, k is the wavenumber, and η is the intrinsic impedance of space. $\overline{e}_m$ and $\overline{e}^i_m$ have units of length and are the effective receiving length of the mth element with and without the reflector, respectively.

The array element pattern overlap matrix is defined to be

$$A_{mn} = \frac{1}{2\eta\sqrt{P_m P_n}}\int \overline{E}^i_m(\mathbf{r})\cdot\overline{E}^{i\,*}_n(\mathbf{r})r^2 d\Omega \tag{5}$$

$$= \frac{\eta}{4\lambda^2 R_{\rm in}}\int \overline{e}^i_m(\Omega)\cdot\overline{e}^{i\,*}_n(\Omega)d\Omega \tag{6}$$

where P_m is the total power radiated by the mth element. For identical elements and neglecting mutual coupling, $P_m = P_{el}$ for all m. The normalization is such that the diagonal elements of $\mathbf{A}$ are unity.

We denote the current amplitudes at the element inputs relative to I_0 as $\mathbf{w} = [w_1, w_2, \ldots, w_N]^T$. The total radiated power is

$$P_{\rm rad} = P_{el}\mathbf{w}^T\mathbf{A}\mathbf{w}^*. \tag{7}$$

The radiated power density into the polarization $\hat{p}$ is

$$S(\mathbf{r}) = \frac{1}{2\eta}\left|w_1\hat{p}\cdot\overline{E}_1(\mathbf{r}) + \cdots + w_N\hat{p}\cdot\overline{E}_N(\mathbf{r})\right|^2$$

$$= \frac{\eta P_{el}}{4\lambda^2 r^2 R_{\rm in}}\mathbf{w}^T\mathbf{B}(\Omega)\mathbf{w}^* \tag{8}$$

where

$$B_{mn}(\Omega) = \hat{p}\cdot\overline{e}_n(\Omega)\hat{p}\cdot\overline{e}^*_m(\Omega). \tag{9}$$

From (7) and (8), the partial directivity of the array with respect to $\hat{p}$ is

$$D(\Omega) = \frac{4\pi r^2 S(\mathbf{r})}{P_{\rm rad}} = \frac{\pi\eta}{R_{\rm in}\lambda^2}\frac{\mathbf{w}^T\mathbf{B}(\Omega)\mathbf{w}^*}{\mathbf{w}^T\mathbf{A}\mathbf{w}^*}. \tag{10}$$

Because the fields $\overline{e}_m$ are used in (9) rather than $\overline{e}^i_m$, the directivity includes the focusing effect of the reflector. For an isolated array, $\overline{e}_m = \overline{e}^i_m$.

III. Receive Array Gain

For an isolated phased array in a spatially isotropic noise environment, directivity and array gain are equal [10]. The goal in this section is to extend this result to an array-fed reflector antenna. For a receive array, the signals from each array element are combined using beamformer weights w^*_n. The conjugate follows the signal processing convention that the beamformer output is $\mathbf{w}^\dagger\mathbf{v}$, where † denotes the conjugate transpose and $\mathbf{v}$ is a vector of phasor voltages from each array element after amplification, basebanding, and sampling.

As a model for the receive array, we consider each element to be loaded by an impedance Z_L which represents the input impedance of a low-noise amplifier or an impedance matching structure connected to the element output port. Let g_r be the complex voltage gain from the load impedance to the receiver output voltage after basebanding and sampling (in practice, the gains of the element signal paths may be unequal, but the extension to that case is straightforward). From classical antenna theory, the open-circuit phasor voltage at the antenna terminals for an incoming wave with angular distribution $\overline{E}(\Omega)$ is

$$v_{oc,m} = \int \overline{E}(\Omega)\cdot\overline{e}_m(\Omega)d\Omega. \tag{11}$$

This provides a Thévenin equivalent for the receive elements with open-circuit voltages $v_{oc,m}$ and source impedance $Z_{\rm in}$. We assume a conjugate field match at each element, so that $Z_L = Z^*_{\rm in}$.

A. Noise Model

Blackbody radiation is assumed to arrive at the array feed from warm background in the scene around the antenna. The thermal noise voltage covariance matrix at the sampled receiver outputs has elements given by

$$R_{t,mn} = \frac{|g_r Z_{\rm in}|^2}{4R^2_{\rm in}}R^{oc}_{t,mn}$$

$$R^{oc}_{t,mn} = \int\int \overline{e}_m(\Omega)\cdot\left\langle\overline{E}(\Omega)\overline{E}^*(\Omega')\right\rangle\cdot\overline{e}^*_n(\Omega')d\Omega\, d\Omega'$$

where $\langle\cdot\rangle$ denotes temporal expectation. Assuming independent sources with angular brightness temperature distribution $T(\Omega)$, we have

$$\left\langle\overline{E}(\Omega)\overline{E}^*(\Omega')\right\rangle = \frac{2\eta k_b B T(\Omega)}{\lambda^2}\left(\overline{\overline{I}} - \hat{r}\hat{r}\right)\delta(\Omega-\Omega') \tag{12}$$

where $\hat{r} = \mathbf{r}/r$, $\overline{\overline{I}}$ is the identity dyad, k_b is Boltzman's constant and B is the system bandwidth. This leads to the result

$$R_{t,mn} = \frac{|g_r Z_{\rm in}|^2}{2R^2_{\rm in}}\frac{\eta k_b B}{\lambda^2}\int T(\Omega)\overline{e}_m(\Omega)\cdot\overline{e}^*_n(\Omega)d\Omega. \tag{13}$$

If the ground around the antenna is at a uniform temperature and we ignore for convenience the small contribution of thermal noise from the sky, $T(\Omega)$ is a constant and the region of integration in (13) extends over ground visible to the array and not obscured by the reflector. For this noise model, we have

$$\mathbf{R}_t = \frac{|g_r Z_{\rm in}|^2}{R_{\rm in}}2k_b T B\mathbf{A}_{sp} \tag{14}$$

where $\mathbf{A}_{sp}$ is defined analogously to (6) as

$$A_{sp,mn} = \frac{\eta}{4\lambda^2 R_{\rm in}}\int_{\Omega_{sp}} \overline{e}^i_m(\Omega)\cdot\overline{e}^{i\,*}_n(\Omega)d\Omega \tag{15}$$

where Ω_{sp} denotes the solid angle over which blackbody radiation arrives at the array. In terms of the overlap matrices, the spillover efficiency of the array feed can be expressed as

$$\eta_{sp} = 1 - \frac{\mathbf{w}^\dagger \mathbf{A}_{sp} \mathbf{w}}{\mathbf{w}^\dagger \mathbf{A} \mathbf{w}}. \tag{16}$$

B. Signal Model

We assume that the signal $\overline{E}_s(\mathbf{r})$ is radiated by a point source located at the spherical angle Ω. The open-circuit signal voltage at the mth array element is

$$v_{oc,m} = \overline{E}_s(\mathbf{r}=0) \cdot \overline{e}_m(\Omega). \tag{17}$$

We use $\overline{e}_m$ rather than $\overline{e}_m^i$ to account for scattering of the signal by the reflector. The signal covariance matrix is

$$\mathbf{R}_s = \frac{\eta S_{sig} |g_r Z_{\text{in}}|^2}{4R_{\text{in}}^2} \mathbf{B}(\Omega) \tag{18}$$

where $S_{sig} = |E_s|^2/\eta$ is the incident power density associated with the signal source. Following the radio astronomy convention, the signal is assumed to be randomly polarized, so that only half of the incident power is available to be received by an antenna in a single polarization.

C. Array Gain and Array Efficiency

Array gain as defined in (2) requires a reference SNR. A natural choice is an isotropic antenna, for which the SNR at a conjugate matched load is

$$\text{SNR}_0 = \frac{\lambda^2 \frac{S_{sig}}{2}}{4\pi k_b T B}. \tag{19}$$

By making use of (14) and (18), the SNR at the array beamformer output is

$$\text{SNR}_{\text{out}} = \frac{\mathbf{w}^\dagger \mathbf{R}_s \mathbf{w}}{\mathbf{w}^\dagger \mathbf{R}_t \mathbf{w}} = \frac{\eta S_{sig}}{8k_b B T R_{\text{in}}} \frac{\mathbf{w}^\dagger \mathbf{B}(\Omega) \mathbf{w}}{\mathbf{w}^\dagger \mathbf{A}_{sp} \mathbf{w}}. \tag{20}$$

The array gain is

$$G_a = \frac{\pi \eta}{R_{\text{in}} \lambda^2} \frac{\mathbf{w}^\dagger \mathbf{B}(\Omega) \mathbf{w}}{\mathbf{w}^\dagger \mathbf{A}_{sp} \mathbf{w}}. \tag{21}$$

In order for the receiving pattern to be the same as the radiation pattern of the array configured as a transmitter, the beamformer weights of the receiving array must be related to the excitation currents of the transmit array according to $w_n^* = I_n$. With this choice of weights, comparing (21) and (10) in view of (16) shows that

$$G_a = \frac{D}{1-\eta_{sp}} = \frac{4\pi A_p}{\lambda^2} \frac{\eta_{ap}}{1-\eta_{sp}}. \tag{22}$$

From this result, we can see that the array efficiency defined in (3) is

$$\eta_a = \frac{\eta_{ap}}{1-\eta_{sp}}. \tag{23}$$

The factor of $1/(1-\eta_{sp})$ arises because directivity is inversely proportional to total radiated power, whereas array gain is inversely proportional to total received noise power. The presence of a reflector breaks the mathematical symmetry between these quantities. In the transmit case, power radiated by an antenna in all directions contributes to the total radiated power, whereas in the receiving case thermal noise is blocked by the reflector surface and arrives at the feed only outside the angular extent of the reflector.

These relationships are based on the assumption that only noise due to isotropic thermal radiation is included in SNR_{out}. Amplifiers, ohmic losses, and other noise sources decrease the output SNR and consequently lower the array gain. If each receiver channel has an equivalent noise temperature T_{rec}, for example, then (23) becomes

$$\eta_a = \frac{\eta_{ap}}{1-\eta_{sp} + \frac{T_{\text{rec}}}{T} \frac{\mathbf{w}^\dagger \mathbf{w}}{\mathbf{w}^\dagger \mathbf{A} \mathbf{w}}} \tag{24}$$

D. Measuring Aperture Efficiency

Array efficiency as defined by (3) can be determined by measuring the system output SNR, but if the aperture efficiency is desired, the spillover efficiency and receiver noise contribution must be known in order to use (24) to find η_{ap}. Alternately, a strong signal measurement can be used to obtain the aperture efficiency. Using (22), (21), (16), and (18),

$$\eta_{ap} = \frac{R_{\text{in}}}{S_{sig} A_p |g_r Z_{\text{in}}|^2} \frac{\mathbf{w}^\dagger \mathbf{R}_s \mathbf{w}}{\mathbf{w}^\dagger \mathbf{A} \mathbf{w}}. \tag{25}$$

The overlap matrix $\mathbf{A}$ for the isolated array can be found from simulations or antenna range measurements, and the other quantities can be readily measured as well. Alternately, if the covariance matrix $\mathbf{R}_{\text{iso}}$ of the response of the isolated array to spatially isotropic noise is available, by noting that $\mathbf{R}_{\text{iso}}$ and $\mathbf{A}$ are related in the same way as $\mathbf{R}_t$ and $\mathbf{A}_{sp}$ in (14), the aperture efficiency can be expressed as

$$\eta_{ap} = \frac{2k_b T B}{S_{sig} A_p} \frac{\mathbf{w}^\dagger \mathbf{R}_s \mathbf{w}}{\mathbf{w}^\dagger \mathbf{R}_{\text{iso}} \mathbf{w}}. \tag{26}$$

In many cases, unlike (16), $\mathbf{w}^\dagger \mathbf{A} \mathbf{w}$ in (25) is relatively insensitive to the off-diagonal elements of $\mathbf{A}$. Since $\mathbf{A}$ is normalized to have diagonal elements equal to unity, it may be reasonably accurate to choose $\mathbf{A} = \mathbf{I}$ if the overlap matrix (or isotropic noise response) is unavailable.

Known aperture efficiency allows signal and noise powers to be referred to an equivalent passive feed by choosing a power scaling constant such that

$$P_{\text{out}} = \frac{1}{c_1} \mathbf{w}^\dagger \mathbf{R}_s \mathbf{w} = \eta_{ap} A_p \frac{S_{sig}}{2}. \tag{27}$$

From (25) and (26),

$$c_1 = \frac{2|g_r|^2 |Z_{\text{in}}|^2}{R_{\text{in}}} \mathbf{w}^\dagger \mathbf{A} \mathbf{w} = \frac{\mathbf{w}^\dagger \mathbf{R}_{\text{iso}} \mathbf{w}}{k_b T B}. \tag{28}$$

This factor be viewed as normalizing the net gain of the amplifiers, receivers, and beamformer weights to unity.

E. Array Without Reflector

For an isolated array in an arbitrary noise field, it can be shown that (23) becomes

$$\eta_a = \eta_{ap} \frac{\mathbf{w}^\dagger \mathbf{R}_{\rm iso} \mathbf{w}}{\mathbf{w}^\dagger \mathbf{R}_n \mathbf{w}} \tag{29}$$

where $\mathbf{R}_n$ is the noise covariance at the receiver outputs (including amplifier noise and any other source of system noise). If the noise model consists only of spatially isotropic noise, then the ratio of quadratic forms becomes unity and $\eta_a = \eta_{ap}$. This reflects the fact that while array gain can be defined for an arbitrary noise model, it is the array gain for spatially isotropic noise, or isotropic noise gain, that is equivalent to directivity.

IV. Conclusion

We have shown that array efficiency defined in terms of SNR gain is related in a simple way to aperture and spillover efficiency. This array efficiency can be obtained without measuring the full array radiation or receiving pattern, because the output noise power level is determined by the integral of the receiving pattern over the noise field around the array. As a consequence, array efficiency is defined relative to a particular noise model. When the noise model consists solely of spatially isotropic thermal radiation, array efficiency coincides with the standard definition of aperture efficiency.

These tools will permit antenna engineers and users such as radio astronomers to make informed decisions as they consider moving from single-port feeds to array feed implementations. Since major investments have been made in large reflector instruments to achieve good performance metrics, it is important to know how array feeds will impact effective aperture and spillover efficiencies in comparison to single-feed implementations.

References

[1] P. Shelton, "Multiple-feed systems for objectives," *IEEE Trans. Antennas Propag.*, vol. 13, pp. 992–994, Nov. 1965.
[2] A. W. Rudge and M. J. Withers, "New techniques for beam steering with fixed parabolic reflector," *Proc. Inst. Elect. Eng.*, vol. 118, pp. 857–863, Jul. 1971.
[3] T. S. Bird, J. L. Boomars, and P. J. B. Clarricoats, "Multiple-beam dual-offset reflector antenna with an array feed," *Electron. Lett.*, vol. 14, pp. 439–441, Jul. 1978.
[4] S. J. Blank and W. A. Imbriale, "Array feed synthesis for correction of reflector distortion and vernier beamsteering," *IEEE Trans. Antennas Propag.*, vol. AP-36, pp. 1351–1358, Oct. 1988.
[5] C. K. Hansen, K. F. Warnick, B. D. Jeffs, J. R. Fisher, and R. Bradley, "Interference mitigation using a focal plane array," *Radio Sci.*, vol. 40, Jun. 2005.
[6] S. Stein, "On cross coupling in multiple-beam antennas," *IRE Trans. Antennas Propag.*, vol. 10, pp. 548–557, Sep. 1962.
[7] P. W. Hannan, "The element-gain paradox for a phased-array antenna," *IEEE Trans. Antennas Propag.*, vol. 12, pp. 423–433, Jul. 1964.
[8] C. Craeye and M. Arts, "On the receiving cross section of an antenna in infinite linear and planar arrays," *Radio Sci.*, vol. 39, no. RS2010, 2004.
[9] E. L. Holzman, "A different perspective on taper efficiency for array antennas," *IEEE Trans. Antennas Propag.*, vol. 51, pp. 2963–2967, Oct. 2003.
[10] H. L. Van Trees, *Optimum Array Processing*. New York: Wiley, 2002.
[11] E. Jacobs, "A figure of merit for signal processing reflector antennas," *IEEE Trans. Antennas Propag.*, vol. 33, pp. 100–101, Jan. 1985.