Theoretical Performance Bounds for LOFAR Calibration

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- Array calibration is fundamentally a statistical parameter estimation problem.
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- Provides an absolute frame of reference.
- No algorithm can do better than the CRB.
- The BIG question: Can LOFAR be reliably calibrated?

CRB Definition

- Notation

  $\mathbf{x}$: a vector of random samples with joint probability density $p(\mathbf{x}\,|\,\theta)$.

  $\hat{\theta}$: a<u>ny</u> unbiased estimator for θ.

  $\mathbf{C}_{\hat{\theta}}$: covariance matrix for $\hat{\theta}$

  $\mathbf{M}$: Fisher information matrix.

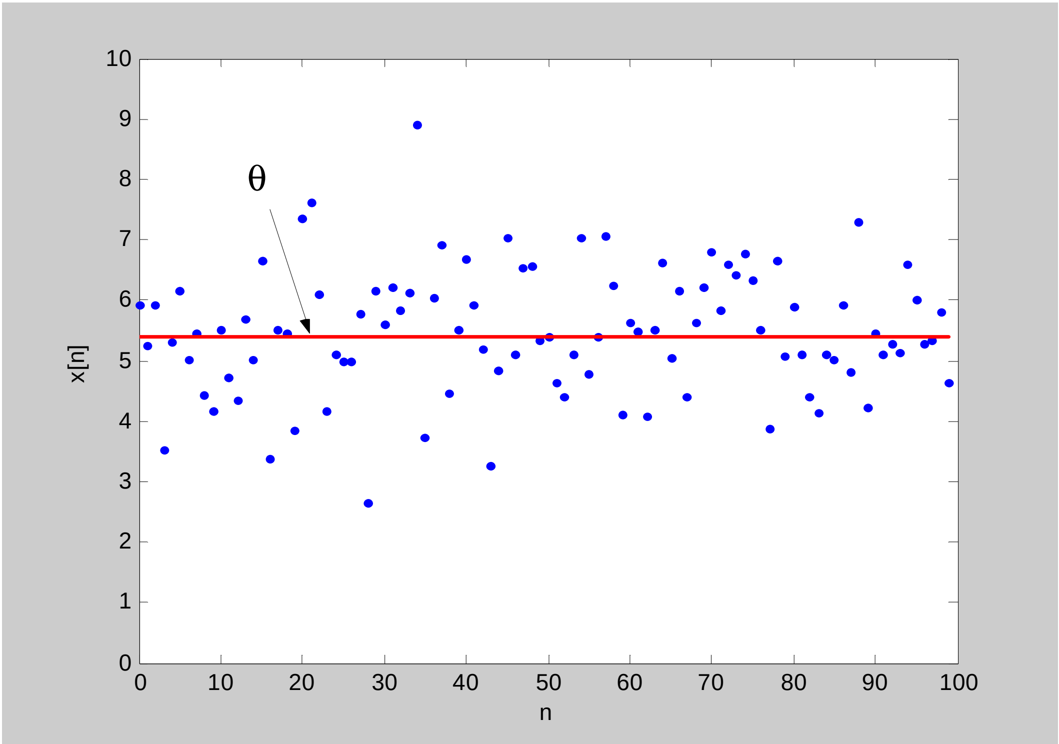
- The Cramer-Rao theorem:

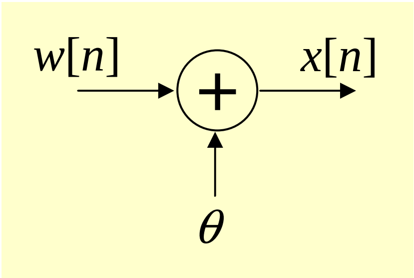
$$\mathbf{C}_{\hat{\theta}} \geq \mathbf{M}^{-1} = -\left(E\left\{ \frac{\partial^2 \ln p(\mathbf{x}\,|\,\boldsymbol{\theta})}{\partial \boldsymbol{\theta}\, \partial \boldsymbol{\theta}^{\mathrm{T}}} \right\} \right)^{-1}$$

- Error variance is lower bounded by $\mathrm{diag}\{\mathbf{C}_{\hat{\theta}}\}$!

A Simple Example

- Estimate a constant in additive white Gaussian noise:





$$x[n] = \theta + w[n]$$

$$\mathbf{x} = [x[0], \cdots, x[N-1]]^T$$

A Simple Example

- Estimate a constant in additive white Gaussian noise:

$$p(\mathbf{x}\,|\,\theta) = \frac{1}{(2\pi\sigma^2)^{\frac{N}{2}}} \exp\left[-\frac{1}{2\sigma^2}\sum_{n=0}^{N-1}(x[n]-\theta)^2\right]$$

$$\frac{\partial \ln p(\mathbf{x}\,|\,\theta)}{\partial\theta} = \frac{1}{\sigma^2}\sum_{n=0}^{N-1}(x[n]-\theta), \qquad \frac{\partial^2 \ln p(\mathbf{x}\,|\,\theta)}{(\partial\theta)^2} = -\frac{N}{\sigma^2}$$

$$\text{Thus} \quad \operatorname{var}(\hat{\theta}) \geq -\left(E\left\{-\frac{N}{\sigma^2}\right\}\right)^{-1} = \frac{\sigma^2}{N}.$$

A Simple Example

- Estimate a constant in additive white Gaussian noise:

$$p(\mathbf{x}\,|\,\theta) = \frac{1}{(2\pi\sigma^2)^{\frac{N}{2}}} \exp\left[-\frac{1}{2\sigma^2}\sum_{n=0}^{N-1}(x[n]-\theta)^2\right]$$

$$\frac{\partial \ln p(\mathbf{x}\,|\,\theta)}{\partial\theta} = \frac{1}{\sigma^2}\sum_{n=0}^{N-1}(x[n]-\theta), \qquad \frac{\partial^2 \ln p(\mathbf{x}\,|\,\theta)}{(\partial\theta)^2} = -\frac{N}{\sigma^2}$$

$$\text{Thus}\quad \operatorname{var}(\hat{\theta}) \geq -\left(E\left\{-\frac{N}{\sigma^2}\right\}\right)^{-1} = \frac{\sigma^2}{N}.$$

This looks very familiar!

A Simple Example

- This proves what we already knew:
  you can't beat the sample mean estimator

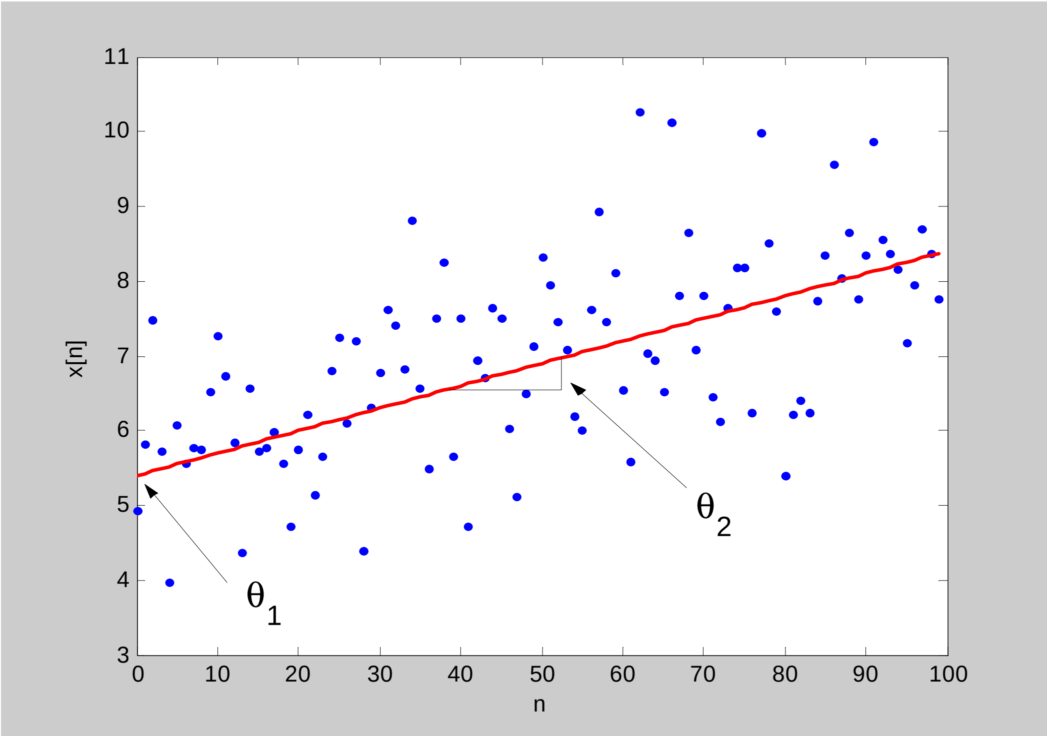
$$\hat{\theta}_{\mathrm{SM}} = \frac{1}{N}\sum_{n=0}^{N-1} x[n], \qquad \mathrm{var}(\hat{\theta}_{\mathrm{SM}}) = E\left\{\left(\frac{1}{N}\sum_{n=0}^{N-1} x[n]\right)^2\right\} - \theta^2 = \frac{\sigma^2}{N}$$

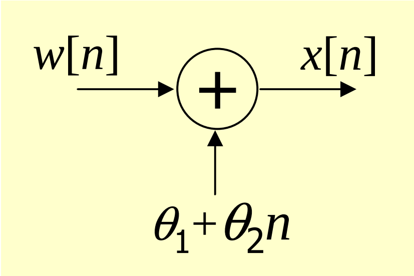
- In general, relationships to parameters of interest are complex, and we do not know $\mathrm{var}(\hat{\theta})$ analytically.

- CRB must be evaluated to bound the problem.

A Second Simple Example

- Line fitting in additive white Gaussian noise:





$$x[n] = \theta_1 + \theta_2 n + w[n]$$

We now have no intuition on estimation error for θ_1 and θ_2!

A Second Simple Example

$$p(\mathbf{x}\,|\,\theta) = \frac{1}{(2\pi\sigma^2)^{\frac{N}{2}}} \exp\left[-\frac{1}{2\sigma^2} \sum_{n=0}^{N-1} (x[n] - \theta_1 + \theta_2 n)^2 \right]$$

$$\frac{\partial \ln p(\mathbf{x}\,|\,\theta)}{\partial \theta_1} = \frac{1}{\sigma^2} \sum_{n=0}^{N-1} (x[n] - \theta_1 + \theta_2 n), \quad \frac{\partial \ln p(\mathbf{x}\,|\,\theta)}{\partial \theta_2} = \frac{1}{\sigma^2} \sum_{n=0}^{N-1} (x[n] - \theta_1 + \theta_2 n) n$$

$$\frac{\partial^2 \ln p(\mathbf{x}\,|\,\theta)}{\partial \theta_1 \partial \theta_2} = -\frac{1}{\sigma^2} \sum_{n=0}^{N-1} n, \quad \frac{\partial^2 \ln p(\mathbf{x}\,|\,\theta)}{\partial \theta_2^2} = -\frac{1}{\sigma^2} \sum_{n=0}^{N-1} n^2, \quad \frac{\partial^2 \ln p(\mathbf{x}\,|\,\theta)}{\partial \theta_1^2} = -\frac{N}{\sigma^2}$$

A Second Simple Example: Insights

$$\mathbf{M} = \frac{1}{\sigma^2}\begin{bmatrix} N & \frac{N(N-1)}{2} \\ \frac{N(N-1)}{2} & \frac{N(N-1)(2N-1)}{6} \end{bmatrix},$$

$$\text{var}(\theta_1) = \left[\mathbf{M}^{-1}\right]_{1,1} = \frac{2(2N-1)\sigma^2}{N(N+1)}, \quad \text{var}(\theta_2) = \left[\mathbf{M}^{-1}\right]_{2,2} = \frac{12\sigma^2}{N(N^2-1)}$$

$$\lim_{N\to\infty} \text{var}(\theta_1) \geq \frac{4\sigma^2}{N} \qquad \lim_{N\to\infty} \text{var}(\theta_2) \geq \frac{12\sigma^2}{N^3}$$

- Variance on constant term θ_1 is now higher.
  $\rightarrow$ estimating more parameters increases error.
- Variance of slope term, θ_2, drops more rapidly with N.
  $\rightarrow$ θ_2 is easier to estimate.
  $\rightarrow$ $x[n]$ is more sensitive to θ_2 due to multiplication by n.

What algorithm do we use?

- What estimator achieves the CRB?
  - Maximum likelihood (ML) does asymptotically ($N \to \infty$).

$$\hat{\theta}_{\mathrm{ML}} = \arg\max_{\theta} \ln p(\mathbf{x} \mid \theta)$$

- Consider our second example:

$$\frac{\partial \ln p(\mathbf{x} \mid \boldsymbol{\theta})}{\partial \theta_1} = \frac{1}{\sigma^2} \sum_{n=0}^{N-1} (x[n] - \theta_1 + \theta_2 n) = 0, \quad \frac{\partial \ln p(\mathbf{x} \mid \boldsymbol{\theta})}{\partial \theta_2} = \frac{1}{\sigma^2} \sum_{n=0}^{N-1} (x[n] - \theta_1 + \theta_2 n) n = 0$$

$$\hat{\boldsymbol{\theta}}_{\mathrm{ML}} = \begin{bmatrix} \hat{\theta}_1 \\ \hat{\theta}_2 \end{bmatrix} = \begin{bmatrix} N & -\sum_{n=0}^{N-1} n \\ \sum_{n=0}^{N-1} n & -\sum_{n=0}^{N-1} n^2 \end{bmatrix}^{-1} \begin{bmatrix} \sum_{n=0}^{N-1} x[n] \\ \sum_{n=0}^{N-1} n x[n] \end{bmatrix}$$

Unfortunately ML is not practical for LOFAR calibration

CRB Uses in Radio Astronomy

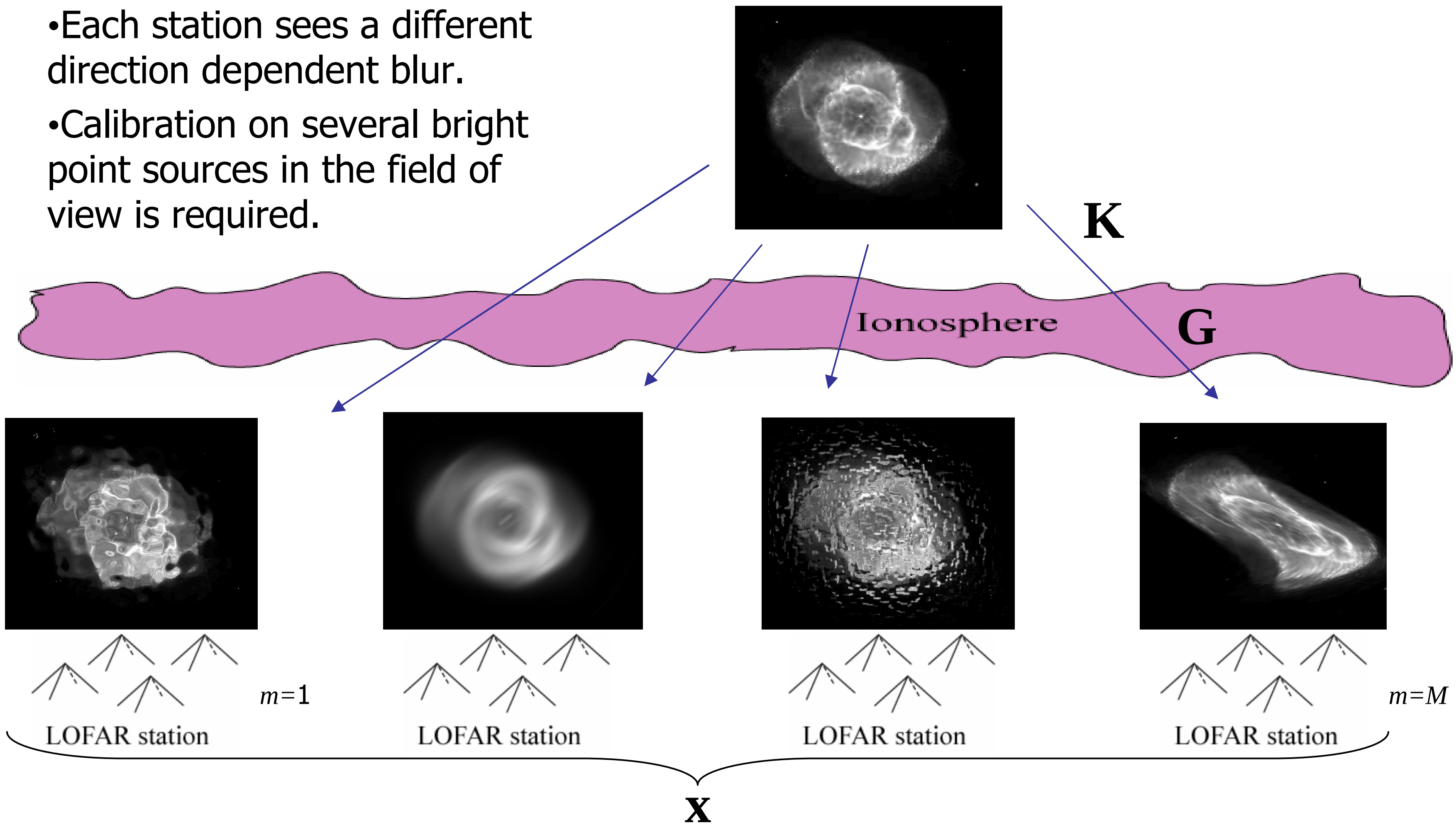
1. Calibration algorithm development and assessment
   - Is the existing algorithm adequate?
   - Is there hope for finding a better solution?
   - Permits trading off performance and computational burden. Answers “How close are we?”
   - Faster than simulation, more flexible than direct observation.
   - Can be computed even if no algorithm exists yet.

CRB Uses in Radio Astronomy

2. Performance Prediction in real observations
   - The bound is specific to signal conditions.
   - Can forewarn an astronomer of poor calibration conditions.
   - A real-time tool could be developed. "Will this observation work?"

Direction Dependent Calibration

- •Each station sees a different direction dependent blur.
- •Calibration on several bright point sources in the field of view is required.



Direction Dependent Calibration
Data Model

V: visibility matrix, computed over a series of time-frequency intervals. Observed.

G: calibration complex gain matrix. One column per calibrator source. Unknown.

K: Fourier kernel, geometric array response. $\mathbf{s}_q$ is source direction vector. $\mathbf{r}_m$ is station location. Known.

B: Calibrator source intensity. Known.

D: Noise covariance. Unknown.

$$\mathbf{V} = E\{\mathbf{x}[n]\mathbf{x}^{\mathrm{H}}[n]\}$$

$$= (\mathbf{G} \odot \mathbf{K})\mathbf{B}(\mathbf{G} \odot \mathbf{K})^{\mathrm{H}} + \mathbf{D}$$

$$\mathbf{G} = \begin{bmatrix} g_{1,1} & \cdots & g_{1,Q} \\ \vdots & & \vdots \\ g_{M,1} & \cdots & g_{M,Q} \end{bmatrix}$$

$$\mathbf{K} = \begin{bmatrix} k_{1,1} & \cdots & k_{1,Q} \\ \vdots & & \vdots \\ k_{M,1} & \cdots & k_{M,Q} \end{bmatrix}, \quad k_{m,q} = \exp\{i \tfrac{2\pi f}{c} \mathbf{s}_q \cdot \mathbf{r}_m\}$$

$$\mathbf{B} = \begin{bmatrix} b_1 & & \\ & \ddots & \\ & & b_Q \end{bmatrix}, \quad \mathbf{D} = \begin{bmatrix} d_1 & & \\ & \ddots & \\ & & d_M \end{bmatrix}$$

CRB for Calibration Parameters

- For the general multivariate Gaussian case the Fisher information has a nice closed form:

$$\mathbf{M} = \mathbf{F}^{\mathrm{H}}(\theta_0)\left[(\overline{\mathbf{V}}(\theta_0))^{-1} \otimes (\mathbf{V}(\theta_0))^{-1}\right]\mathbf{F}(\theta_0), \qquad \mathbf{F}(\boldsymbol{\theta}) = \frac{\partial\, \mathrm{vec}(\mathbf{V}(\theta))}{\partial \boldsymbol{\theta}^{\mathrm{T}}}$$

- For LOFAR $\mathbf{V}(\theta)$ is the visibility (covariance) matrix for full array sample vector $\mathbf{x}$.

- θ contains unknown gains and phases for Q calibrator sources, and noise powers for each station:

$$\theta = [\mathrm{vec}\{|\mathbf{G}|\}^{\mathrm{T}}, \mathrm{vec}\{\angle\mathbf{G}\}^{\mathrm{T}}, \mathrm{vec}\{\mathbf{D}\}^{\mathrm{T}}]^{\mathrm{T}}$$

$$= [\gamma_1, \cdots \gamma_Q, \varphi_1, \cdots \varphi_Q, \mathbf{d}]^{\mathrm{T}}$$

Applying Parameter Model Constraints

- Performance improves by estimating a smaller set of constrained parameters, $\mathbf{p}$. $\theta = f(\mathbf{p})$.
- Constraint examples:
  - Compact core sees coherent scene.
  - Phase is a deterministic function of frequency.
  - Smoothing polynomial over time-frequency-space.
- $\mathbf{V}_{k,n}$ are statistically independent over time-frequency. Each bin (k,n) has distinct $\mathbf{M}_{k,n}$ and $\theta_{k,n}$.

$$\mathbf{M}_{\mathbf{p}} = \sum_{k=1}^{K}\sum_{n=1}^{N} \mathbf{J}_{k,n}^{\mathrm{H}} \mathbf{M}_{k,n} \mathbf{J}_{k,n}, \qquad \mathbf{J}_{k,n} = \frac{\partial \theta_{k,n}(\mathbf{p})}{\partial \mathbf{p}^{\mathrm{T}}}$$

Enforces constraint

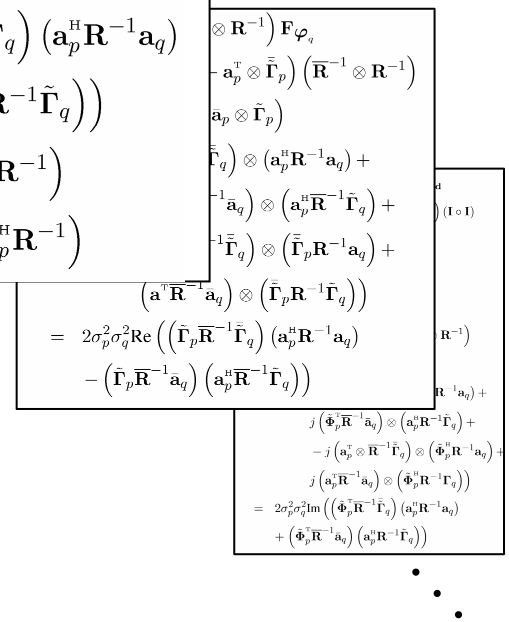
Now it Gets a Little Messy

$$\mathbf{M}_{k,n} = \begin{bmatrix} \mathbf{M}_{\gamma_1\gamma_1} \cdots \mathbf{M}_{\gamma_1\gamma_Q} & \mathbf{M}_{\gamma_1\varphi_1} \cdots \mathbf{M}_{\gamma_1\varphi_Q} & \mathbf{M}_{\gamma_1 d} \\ \ddots & \ddots & \vdots \\ \mathbf{M}_{\gamma_Q\gamma_1} \cdots \mathbf{M}_{\gamma_Q\gamma_Q} & \mathbf{M}_{\gamma_Q\varphi_1} \cdots \mathbf{M}_{\gamma_Q\varphi_Q} & \mathbf{M}_{\gamma_Q d} \\ \mathbf{M}_{\varphi_1\gamma_1} \cdots \mathbf{M}_{\varphi_1\gamma_Q} & \mathbf{M}_{\varphi_1\varphi_1} \cdots \mathbf{M}_{\varphi_1\varphi_Q} & \mathbf{M}_{\varphi_1 d} \\ \ddots & \ddots & \vdots \\ \mathbf{M}_{\varphi_Q\gamma_1} \cdots \mathbf{M}_{\varphi_Q\gamma_Q} & \mathbf{M}_{\varphi_Q\varphi_1} \cdots \mathbf{M}_{\varphi_Q\varphi_Q} & \mathbf{M}_{\varphi_Q d} \\ \mathbf{M}_{d\gamma_1} \cdots \mathbf{M}_{d\gamma_Q} & \mathbf{M}_{d\varphi_1} \cdots \mathbf{M}_{d\varphi_Q} & \mathbf{M}_{dd} \end{bmatrix}$$

Block Fisher information

$$\begin{aligned}
\mathbf{M}_{\boldsymbol{\gamma}_p\boldsymbol{\gamma}_q} &= 2\sigma_p^2\sigma_q^2 \mathrm{Re}\left(\left(\tilde{\boldsymbol{\Phi}}_p^{\mathrm{T}}\overline{\mathbf{R}}^{-1}\bar{\tilde{\boldsymbol{\Phi}}}_q\right)\left(\mathbf{a}_p^{\mathrm{H}}\mathbf{R}^{-1}\mathbf{a}_q\right) + \left(\tilde{\boldsymbol{\Phi}}_p^{\mathrm{T}}\overline{\mathbf{R}}^{-1}\bar{\mathbf{a}}_q\right)\left(\mathbf{a}_p^{\mathrm{H}}\mathbf{R}^{-1}\tilde{\boldsymbol{\Phi}}_q\right)\right) \\
\mathbf{M}_{\boldsymbol{\varphi}_p\boldsymbol{\varphi}_q} &= 2\sigma_p^2\sigma_q^2 \mathrm{Re}\left(\left(\tilde{\boldsymbol{\Gamma}}_p\overline{\mathbf{R}}^{-1}\bar{\tilde{\boldsymbol{\Gamma}}}_q\right)\left(\mathbf{a}_p^{\mathrm{H}}\mathbf{R}^{-1}\mathbf{a}_q\right) - \left(\tilde{\boldsymbol{\Gamma}}_p\overline{\mathbf{R}}^{-1}\bar{\mathbf{a}}_q\right)\left(\mathbf{a}_p^{\mathrm{H}}\mathbf{R}^{-1}\tilde{\boldsymbol{\Gamma}}_q\right)\right) \\
\mathbf{M}_{\mathbf{dd}} &= \overline{\mathbf{R}}^{-1} \odot \mathbf{R}^{-1} \\
\mathbf{M}_{\boldsymbol{\gamma}_p\boldsymbol{\varphi}_q} &= 2\sigma_p^2\sigma_q^2 \mathrm{Im}\left(\left(\tilde{\boldsymbol{\Phi}}_p^{\mathrm{T}}\overline{\mathbf{R}}^{-1}\bar{\tilde{\boldsymbol{\Gamma}}}_q\right)\left(\mathbf{a}_p^{\mathrm{H}}\mathbf{R}^{-1}\mathbf{a}_q\right) + \left(\tilde{\boldsymbol{\Phi}}_p^{\mathrm{T}}\overline{\mathbf{R}}^{-1}\bar{\mathbf{a}}_q\right)\left(\mathbf{a}_p^{\mathrm{H}}\mathbf{R}^{-1}\tilde{\boldsymbol{\Gamma}}_q\right)\right) \\
\mathbf{M}_{\boldsymbol{\gamma}_p\mathbf{d}} &= 2\sigma_p^2 \mathrm{Re}\left(\tilde{\boldsymbol{\Phi}}_p\overline{\mathbf{R}}^{-1} \circ \mathbf{a}_p^{\mathrm{H}}\mathbf{R}^{-1}\right) \\
\mathbf{M}_{\boldsymbol{\varphi}_p\mathbf{d}} &= -2\sigma_p^2 \mathrm{Im}\left(\tilde{\boldsymbol{\Gamma}}_p\overline{\mathbf{R}}^{-1} \circ \mathbf{a}_p^{\mathrm{H}}\mathbf{R}^{-1}\right)
\end{aligned}$$

Block closed forms



Constraint Jacobian
for packed central core

$$\mathbf{J}_{k,n} = \begin{bmatrix} \mathbf{1}_Q \otimes \begin{bmatrix} \mathbf{I}_{M_c} \\ \mathbf{0}_{M_r,M_c} \end{bmatrix} & \mathbf{I}_Q \otimes \begin{bmatrix} \mathbf{0}_{M_c,M_r} \\ \mathbf{I}_{M_r} \end{bmatrix} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{1}_Q \otimes \begin{bmatrix} \mathbf{I}_{M_c} \\ \mathbf{0}_{M_r,M_c} \end{bmatrix} & \mathbf{I}_Q \otimes \begin{bmatrix} \mathbf{0}_{M_c,M_r-1} \\ \mathbf{I}^s_{M_r} \end{bmatrix} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{I}_M \end{bmatrix}$$

Now it Gets a Little Messy

The important points:

- Closed form CRB expressions have been derived for most important LOFAR calibration models.
- Though expressions are complex, computer codes have been developed to evaluate them.
- These solutions exist now and could be made available for astronomers to predict calibration performance for a given observations.

The Single Snapshot LOFAR Calibration Ambiguity

- For conventional arrays without direction dependent ionospheric phase perturbation calibration is possible with one $\mathbf{V}_{k,n}$ observation.
- Not so for LOFAR, there is an essential ambiguity.

$$\mathbf{V} = (\mathbf{G} \odot \mathbf{K})\mathbf{B}^{\frac{1}{2}}\mathbf{U}\mathbf{U}^{\mathrm{H}}\mathbf{B}^{\frac{1}{2}}(\mathbf{G} \odot \mathbf{K})^{\mathrm{H}} + \mathbf{D}$$

Rotation by $\mathbf{U}$ is invisible in $\mathbf{V}$

$$\mathbf{U}\mathbf{U}^{\mathrm{H}} = \mathbf{I}, \qquad \text{a unitary matrix}$$

$$\tilde{\mathbf{G}}(\mathbf{U}) = ((\mathbf{G} \odot \mathbf{K})\mathbf{B}^{\frac{1}{2}}\mathbf{U}\mathbf{B}^{-\frac{1}{2}}) \odot \mathbf{K}^{\bullet -1}$$

Each $\mathbf{U}$ produces a different calibration

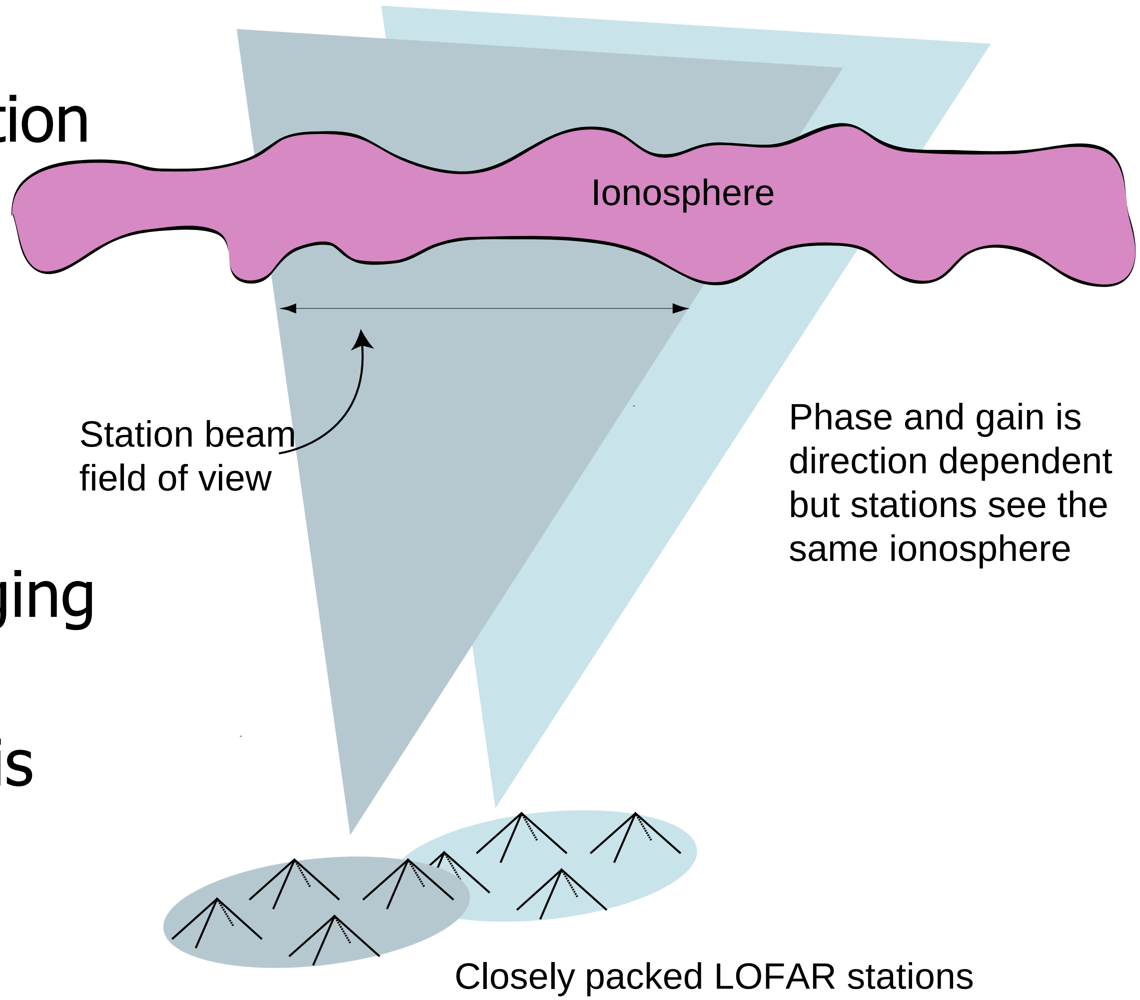
- CRB blows up, $\mathbf{M}$ is singular!

Solutions to Calibration Ambiguity

- Time-frequency diversity
  - Fringe rotation over time and across bands changes visibility structure while calibration gains are relatively constant.
  - Cells, Snippets, Polc's, UV Bricks, Peeling.
  - Low order polynomial fitting.
  - Peeling.
- Single snapshot calibration
  - Compact core.
  - Deterministic Frequency dependence.
  - Known gain magnitudes.
- CRB analysis is completed for most of these scenarios.

LOFAR Calibration with Compact Central Core

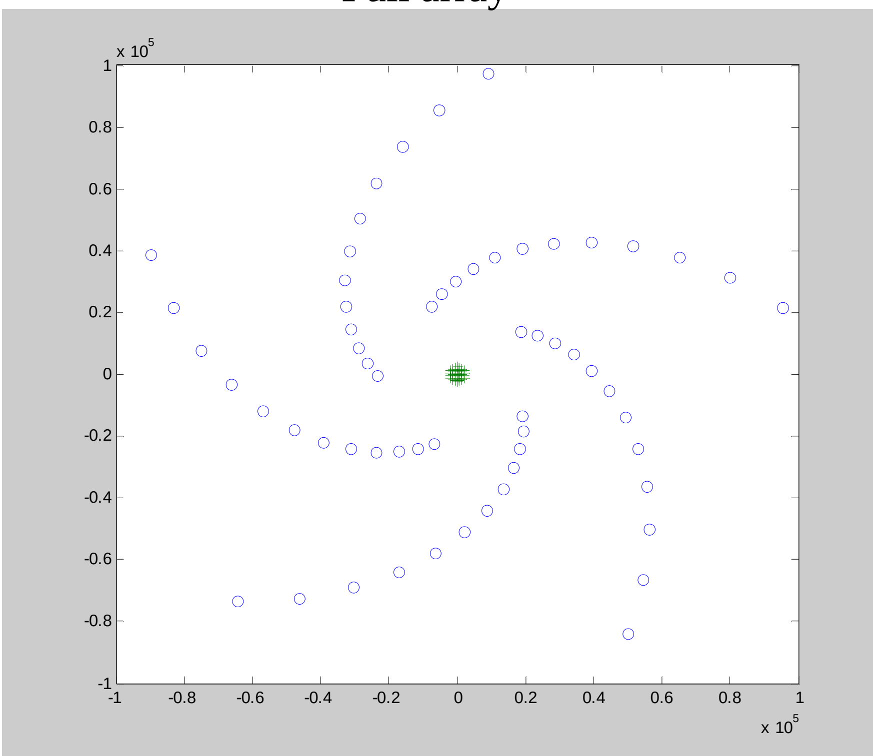
- Though phase distortion is direction dependent, each station sees the same ionosphere.
- Central core sees a coherent scene, imaging is possible
- One calibration gain is estimated for each station.



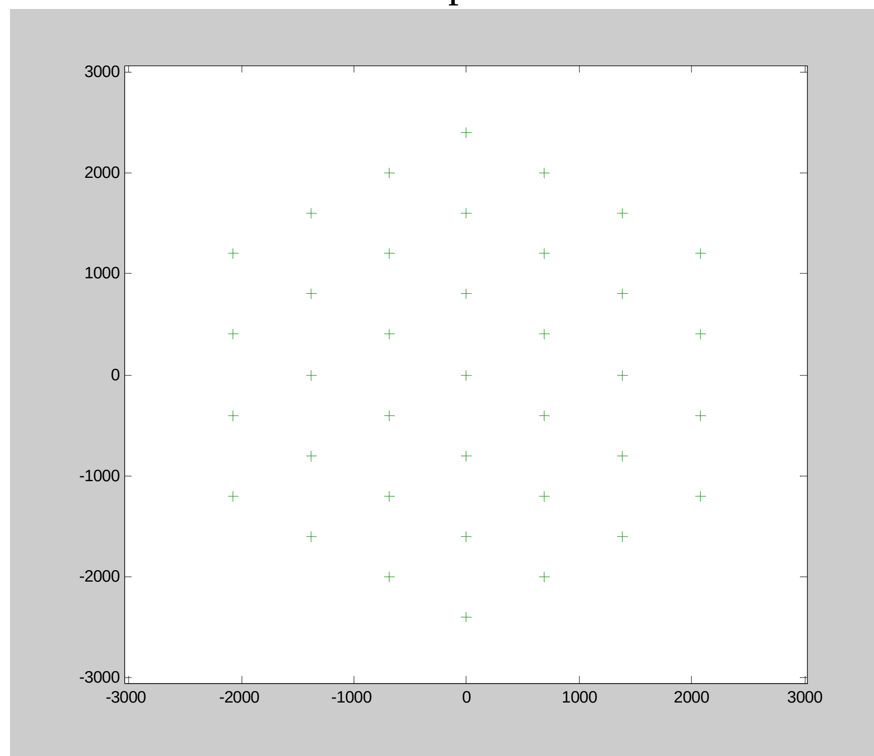
LOFAR Calibration with Compact Central Core

- The full array can be calibrated given a compact central core.
- The wisdom of the LOFAR design is confirmed by CRB analysis

Full array

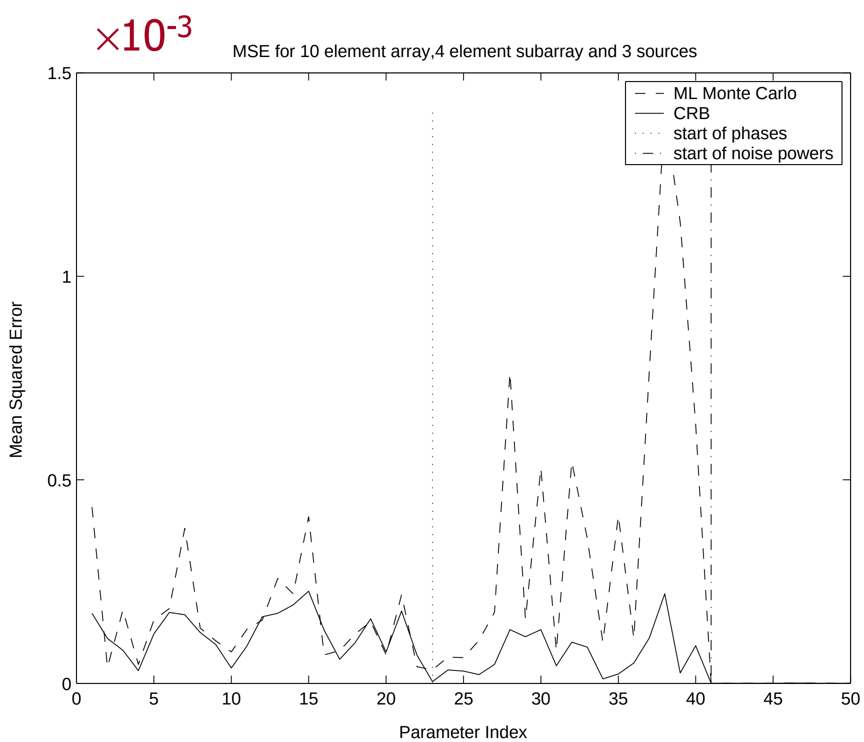


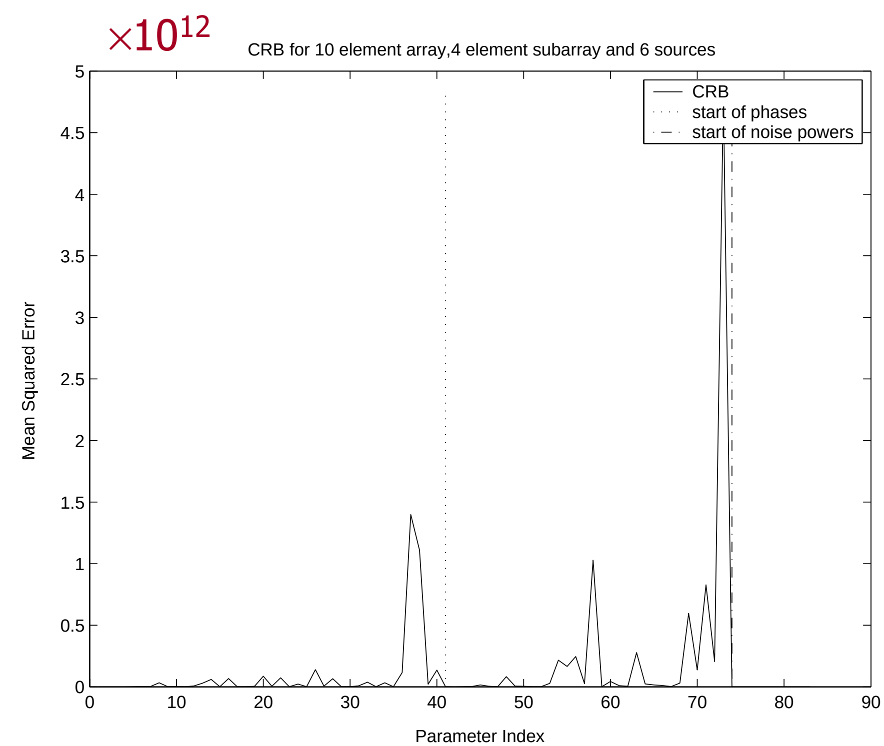
Compact core



LOFAR Calibration with Compact Central Core (Optional slide instead of last)

- Calibration succeeds for $Q \leq M_c + 1$.
  Q calibrator sources and an M_c element core.





2-D Polynomial Model over time-frequency for Ionospheric Variation

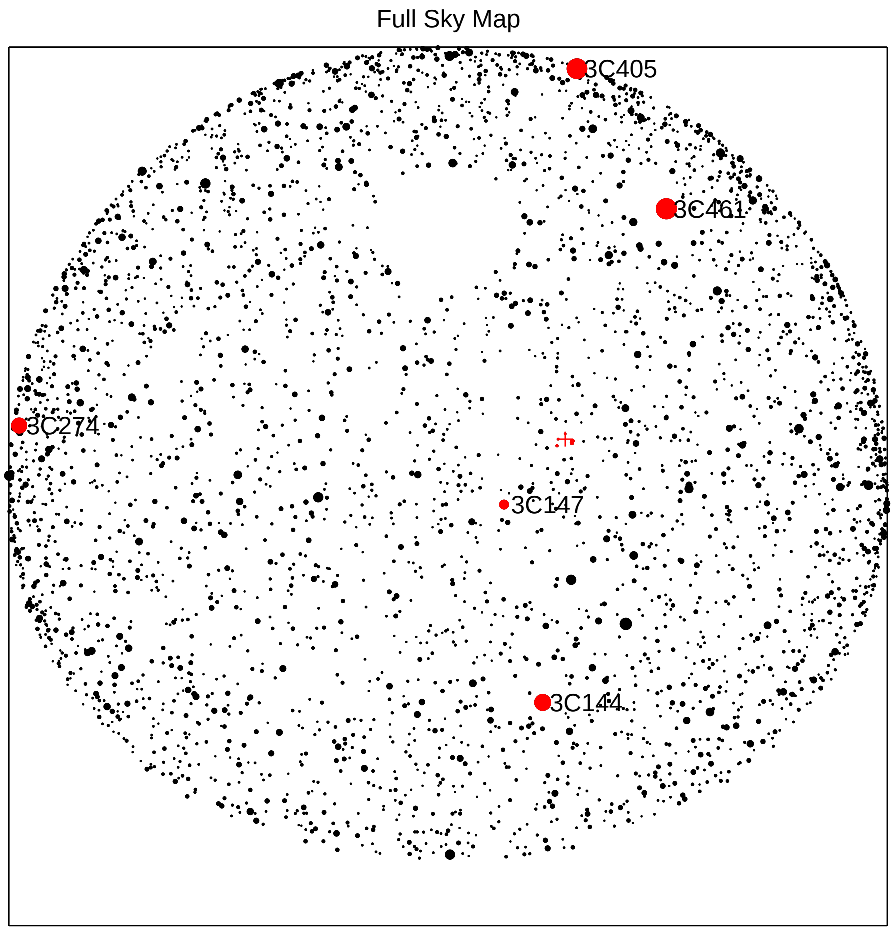
- Variations in $\mathbf{G}$ are smooth over time and frequency

$$\mathbf{G}_{k,n} = (\Gamma_{00} + \Gamma_{10} f_k + \Gamma_{20} f_k^2 + \Gamma_{01} t_n + \Gamma_{02} t_n^2 + \Gamma_{11} f_k t_n)$$
$$\odot \exp\left\{ i(\Phi_{00} + \Phi_{10} f_k + \Phi_{20} f_k^2 + \Phi_{01} t_n + \Phi_{02} t_n^2 + \Phi_{11} f_k t_n) \right\}$$

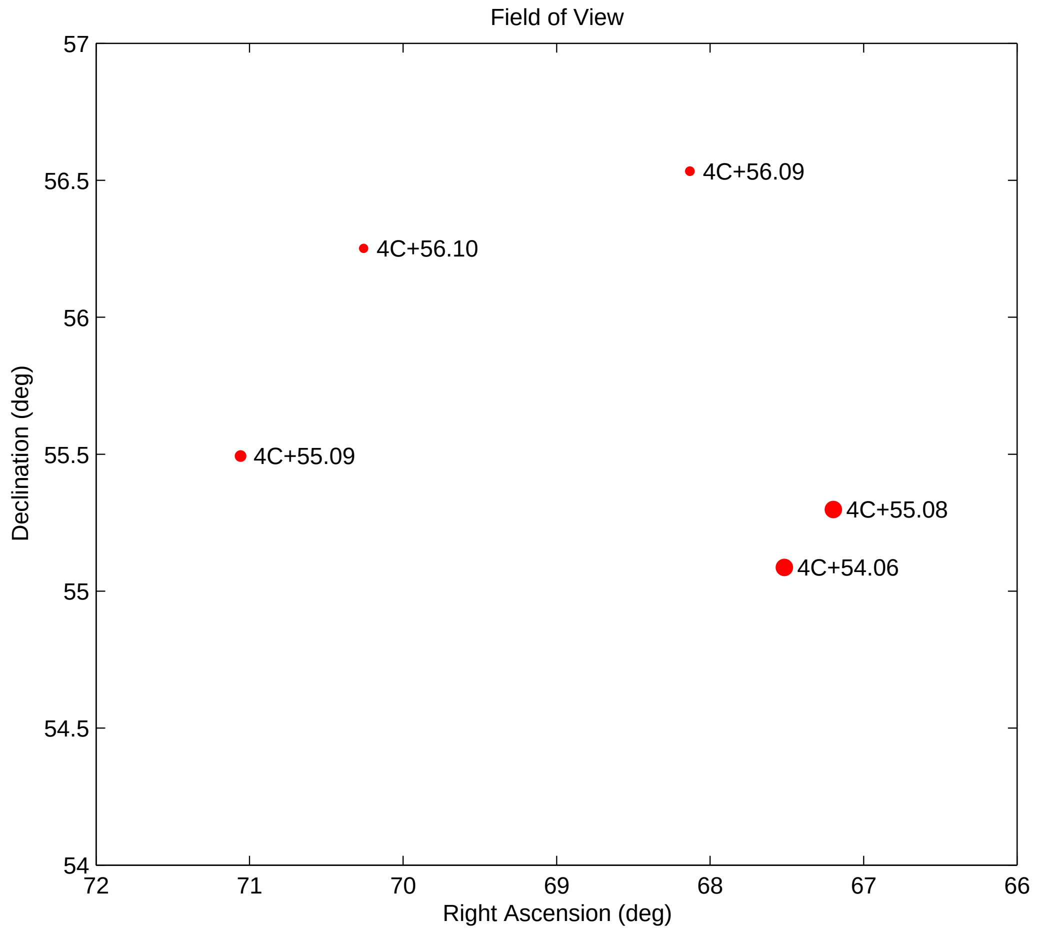
$$\mathbf{p} = \left[\mathrm{vec}\{\Gamma_{00}\}^{\mathrm{T}}, \cdots, \mathrm{vec}\{\Gamma_{11}\}^{\mathrm{T}}, \mathrm{vec}\{\Phi_{00}\}^{\mathrm{T}}, \cdots, \mathrm{vec}\{\Phi_{11}\}^{\mathrm{T}}, \mathrm{diag}\{\mathbf{D}\} \right]^{\mathrm{T}}$$

- We have CRB analysis for 2-D polynomial interpolation:

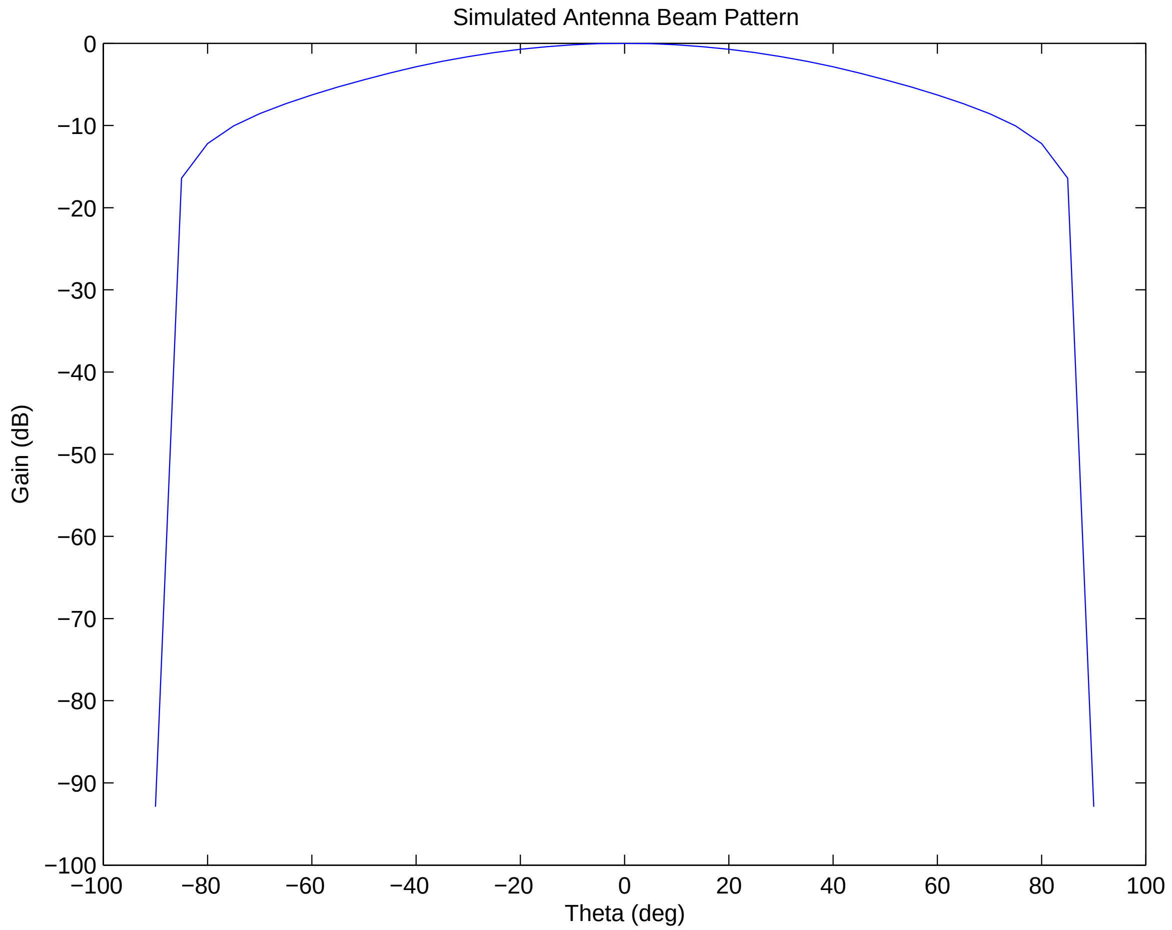
Full Sky Map



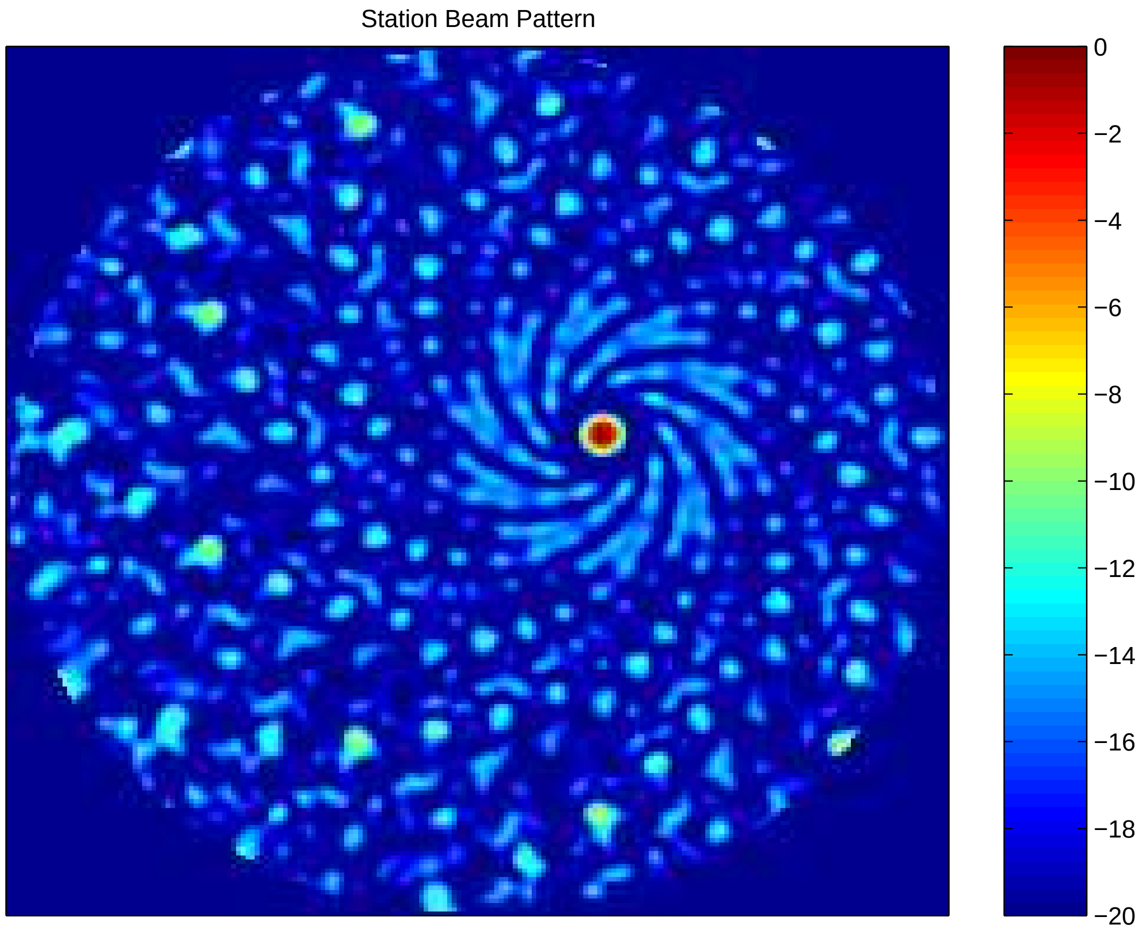
Field of View



Antenna Beam Pattern



Station Beam Pattern



Noise

Assume sky noise limited and 550K @ 90 MHz

$$T_{sky} \sim \lambda^{2.7}$$

Rayleigh-Jeans law

$$B = \frac{2kT}{\lambda^2}$$

Integration over a hemisphere

$$P_{noise} = 2\pi B$$

Source power & SNR

Source powers from 3C & 4C catalogs @ 178MHz.
Assume all sources have spectral index $\alpha = 0.7$, then

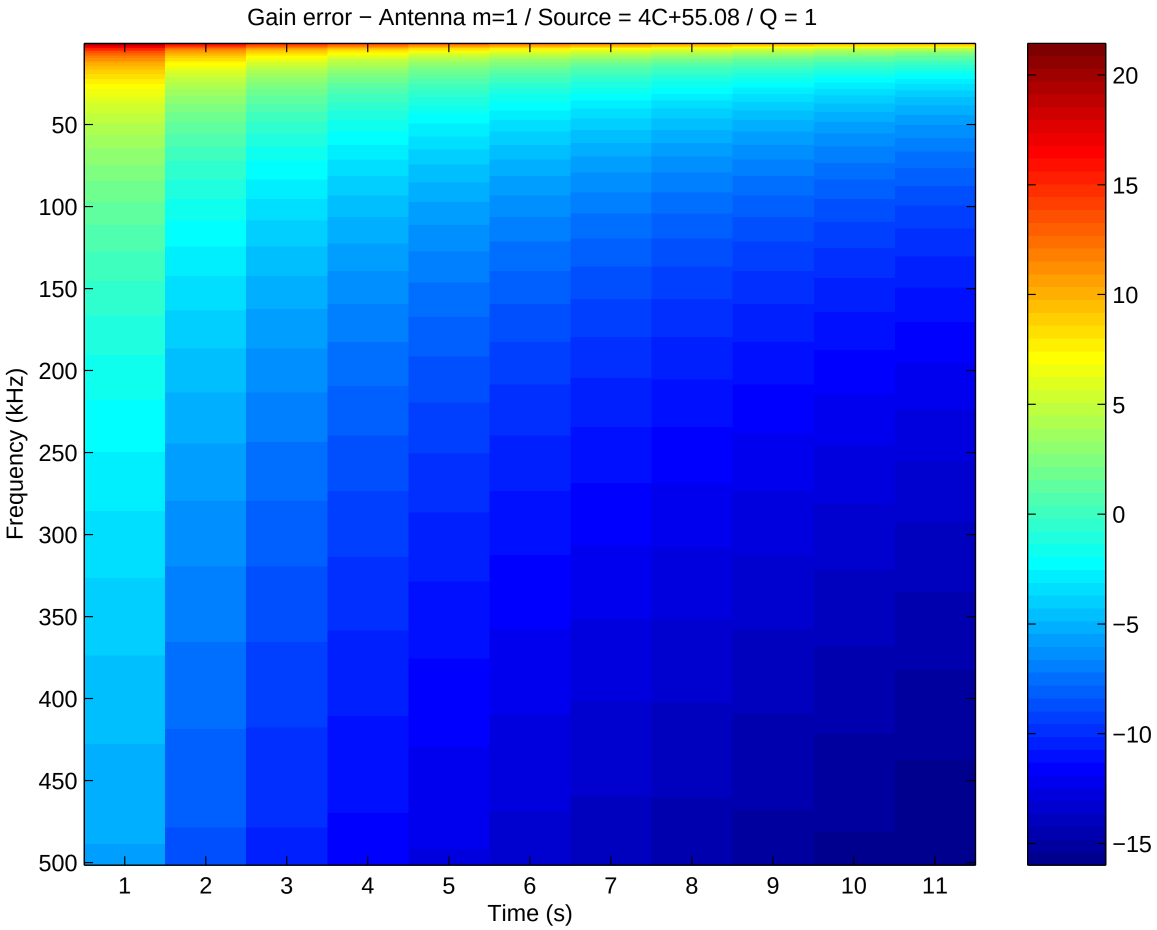
$$P_{source} \sim \lambda^{0.7}$$

The Signal to Noise Ratio (SNR) does not change with frequency.

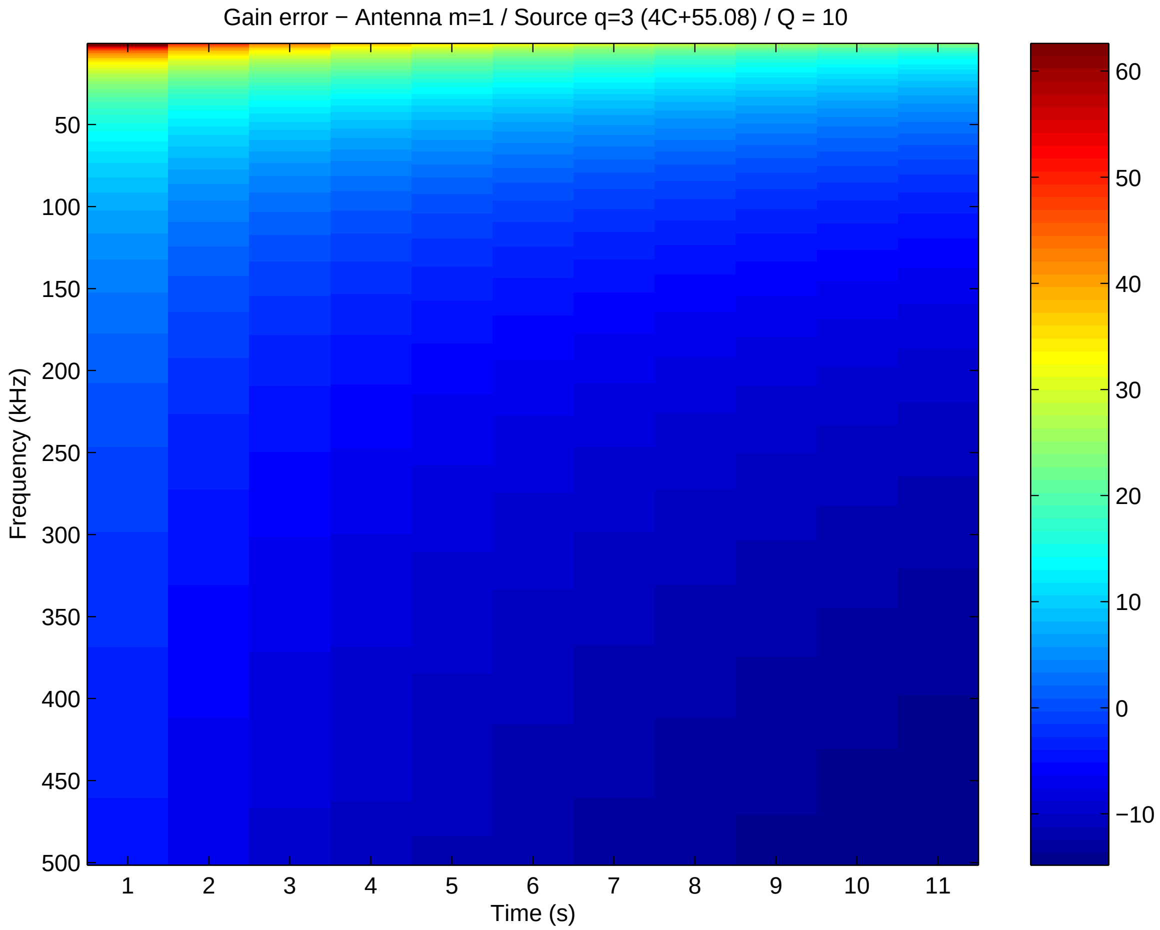
The ten sources

Source	SNR (dB)	Ang. dist.
3C461	-29.3694	39.6633
3C144	-32.3766	35.6677
4C+55.08	-34.4506	1.0305
4C+54.06	-34.6182	1.0479
3C405	-35.8227	74.0794
3C147	-36.7395	11.8521
4C+56.09	-36.9509	0.6951
3C274	-37.4495	95.0506
4C+56.10	-37.6375	0.9507
4C+55.09	-38.2076	1.4073

Single source calibration - constant gain



Multiple sources - constant gain



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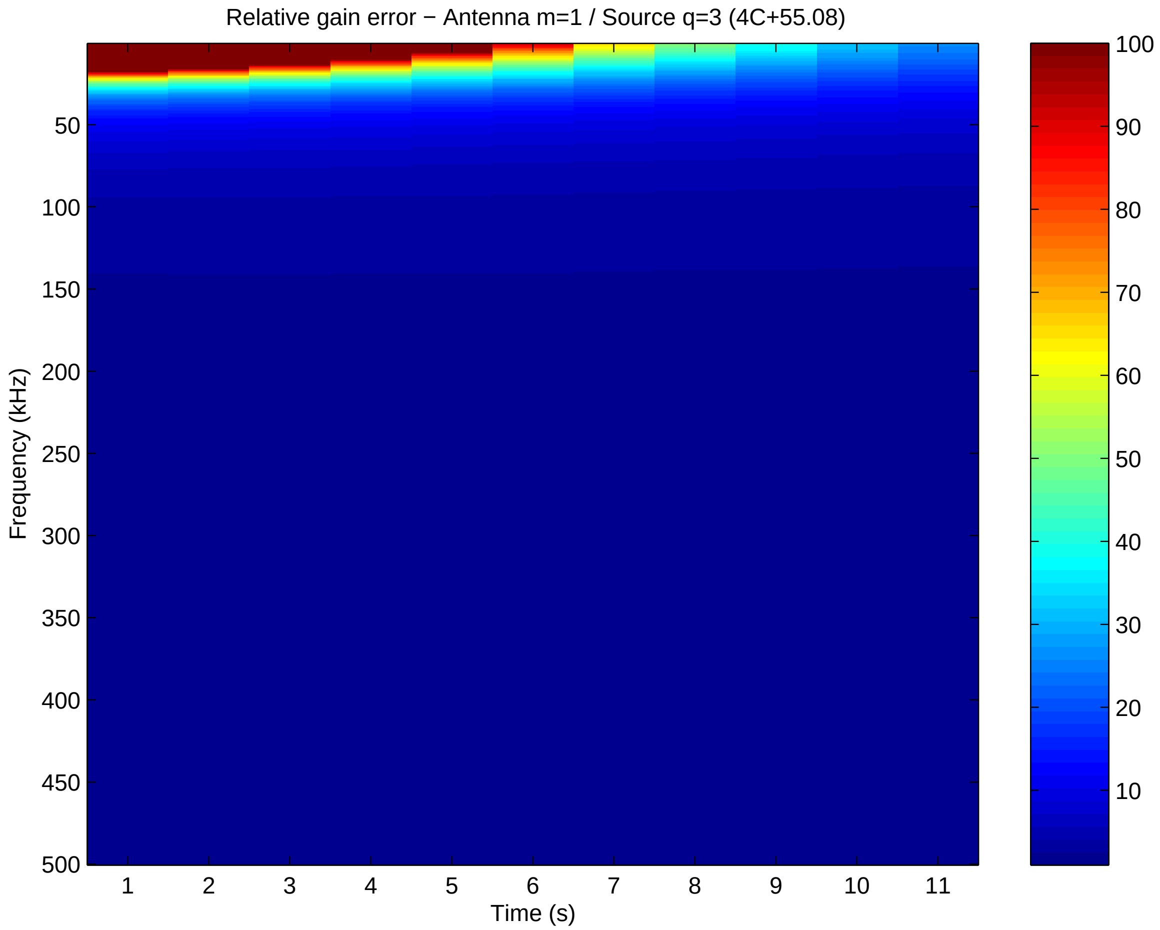
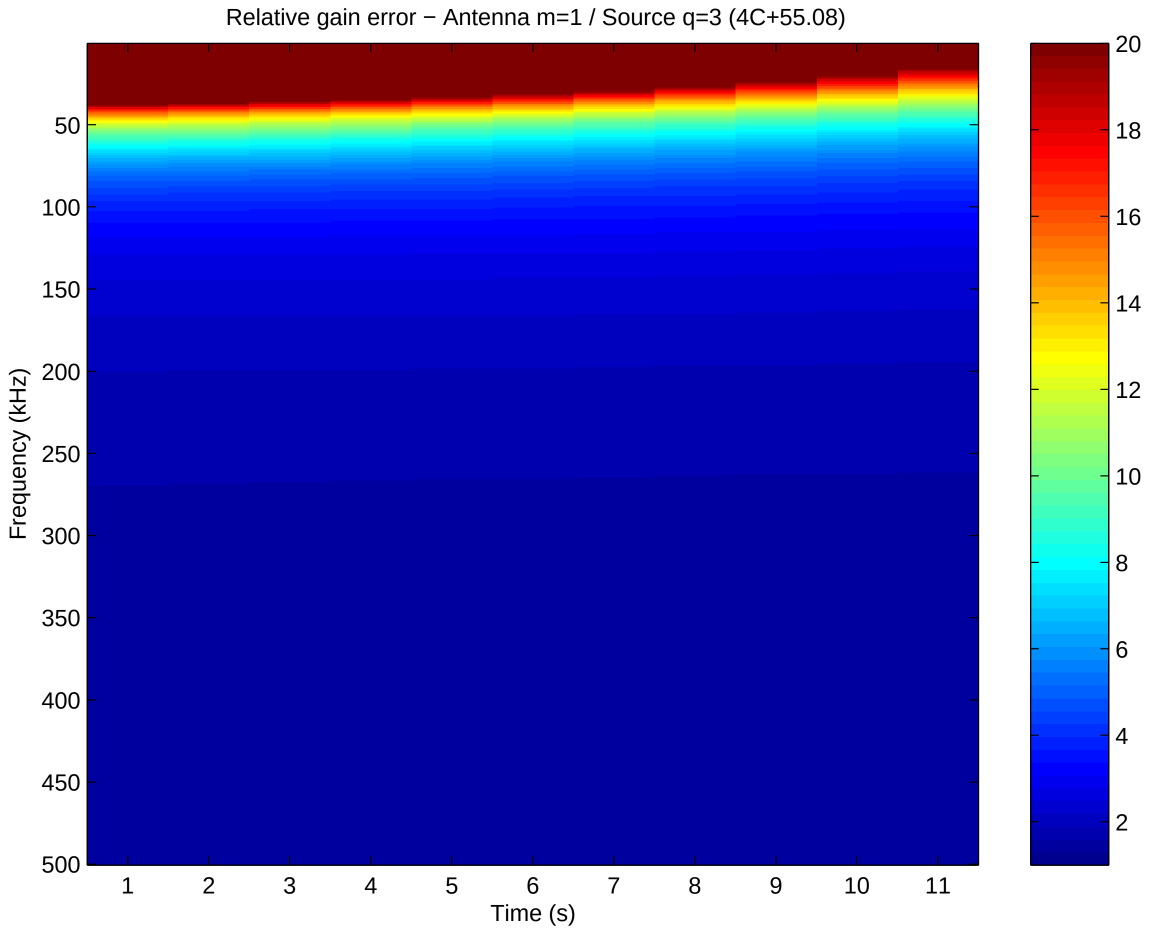
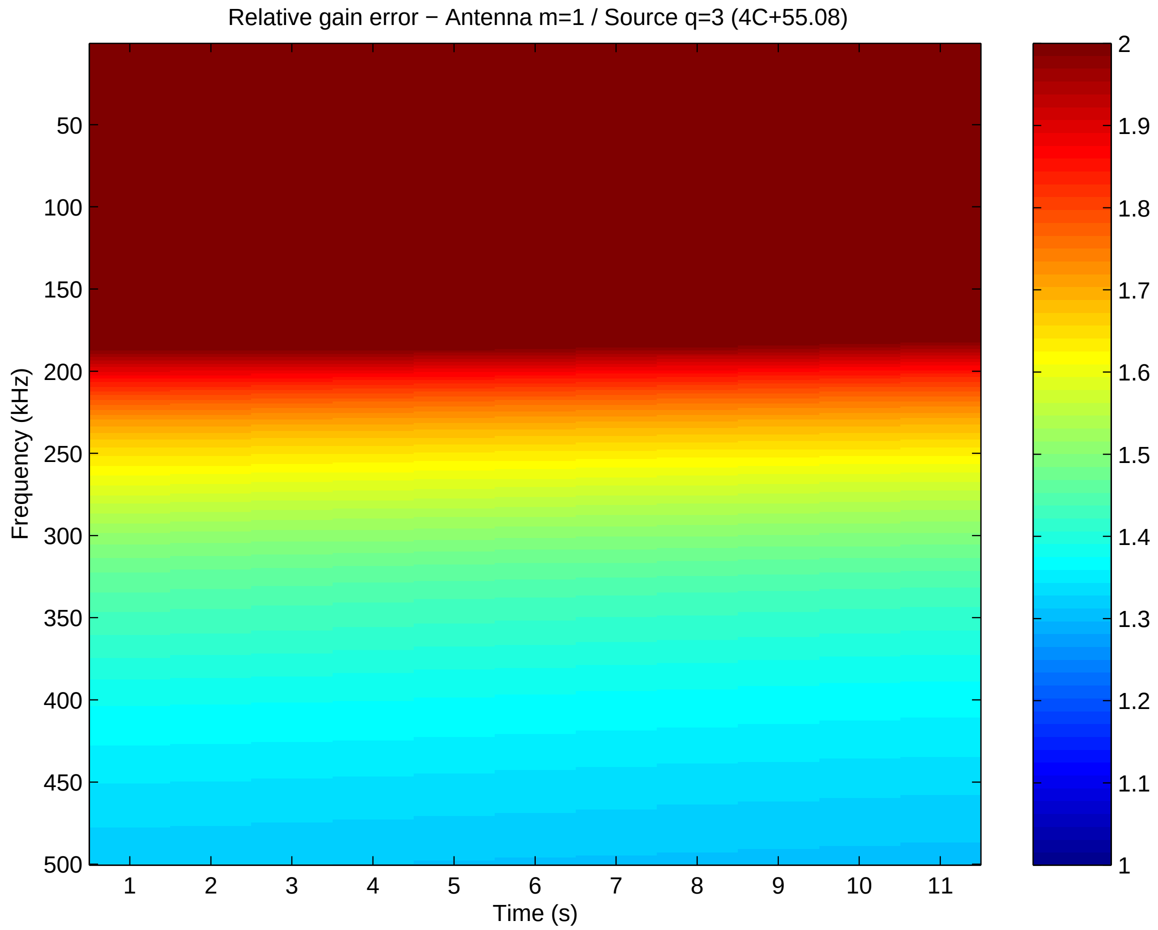


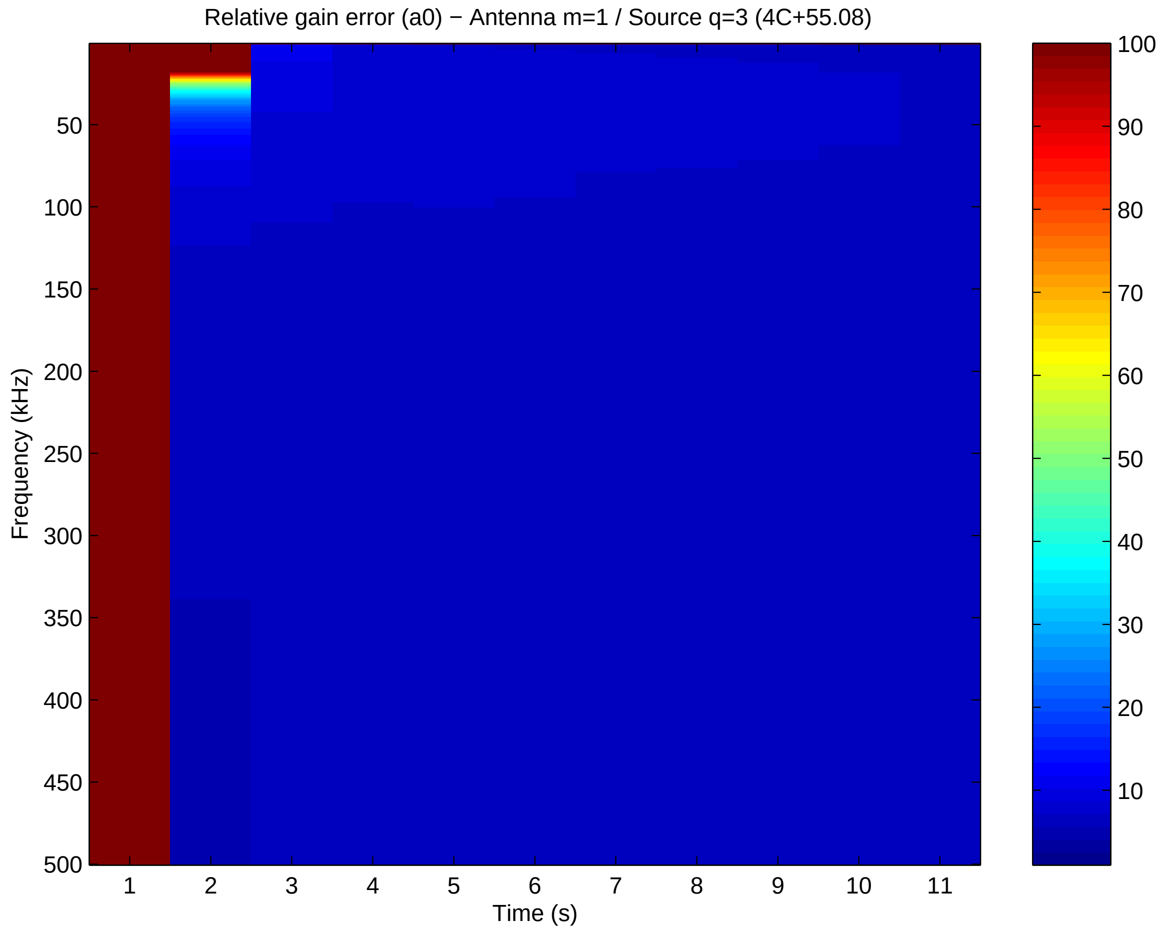
Multiple sources - constant gain



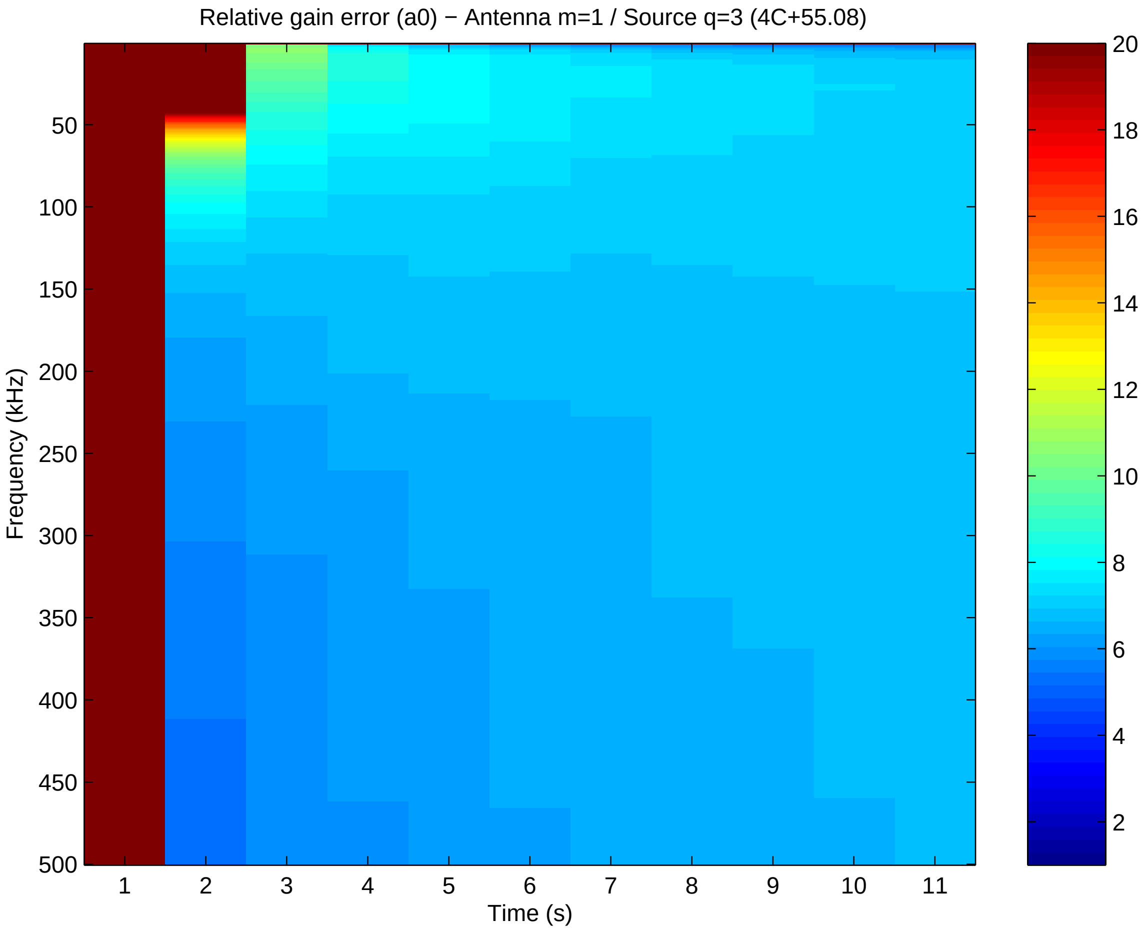
Multiple sources - constant gain



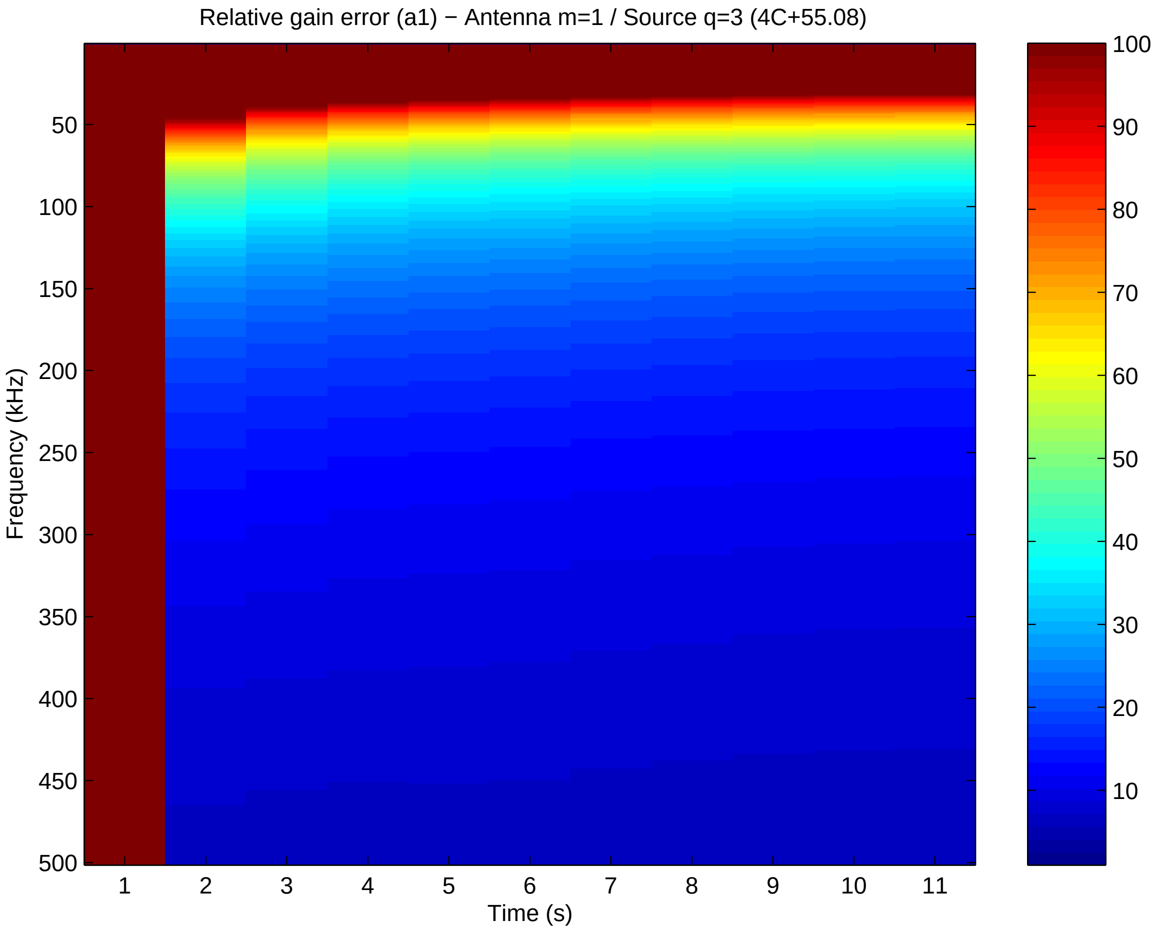
Multiple sources - 2D polynomial



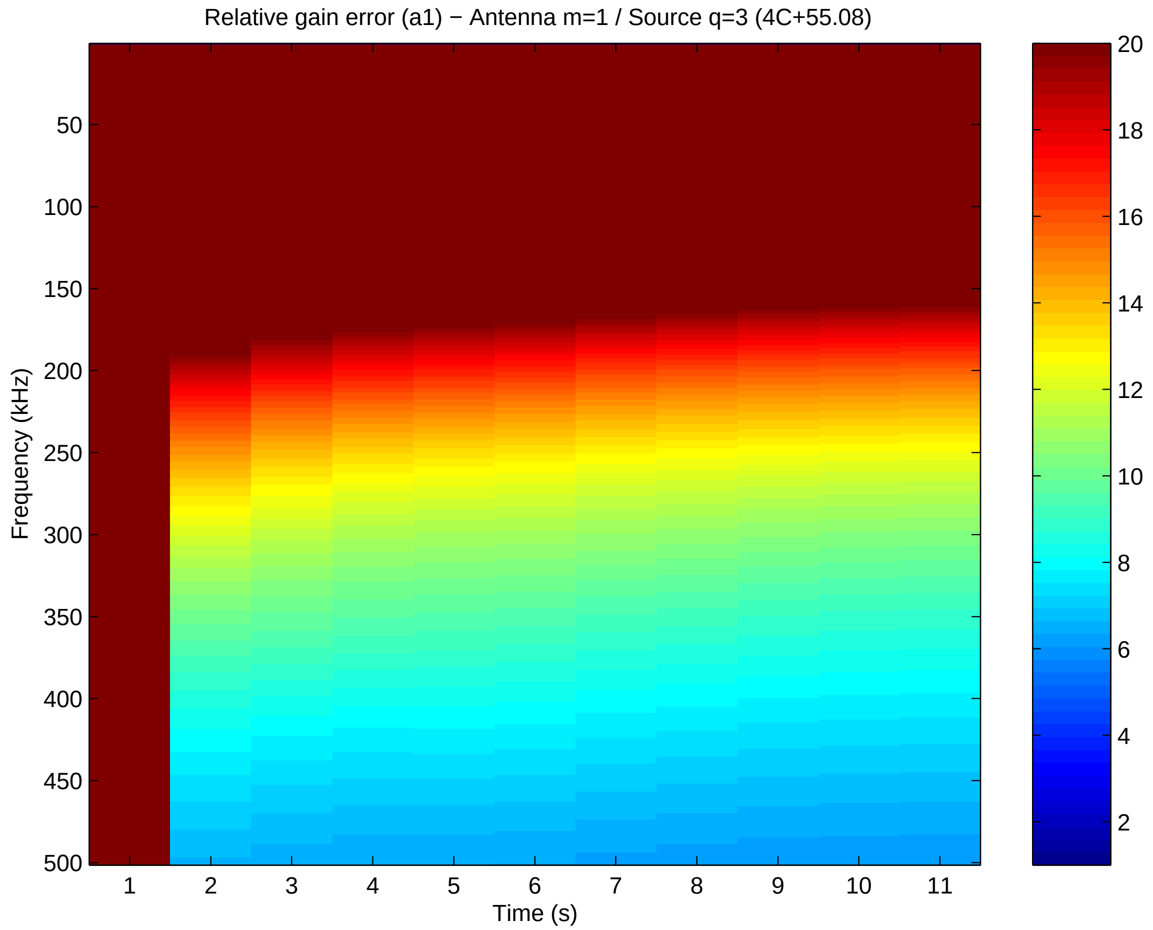
Multiple sources - 2D polynomial



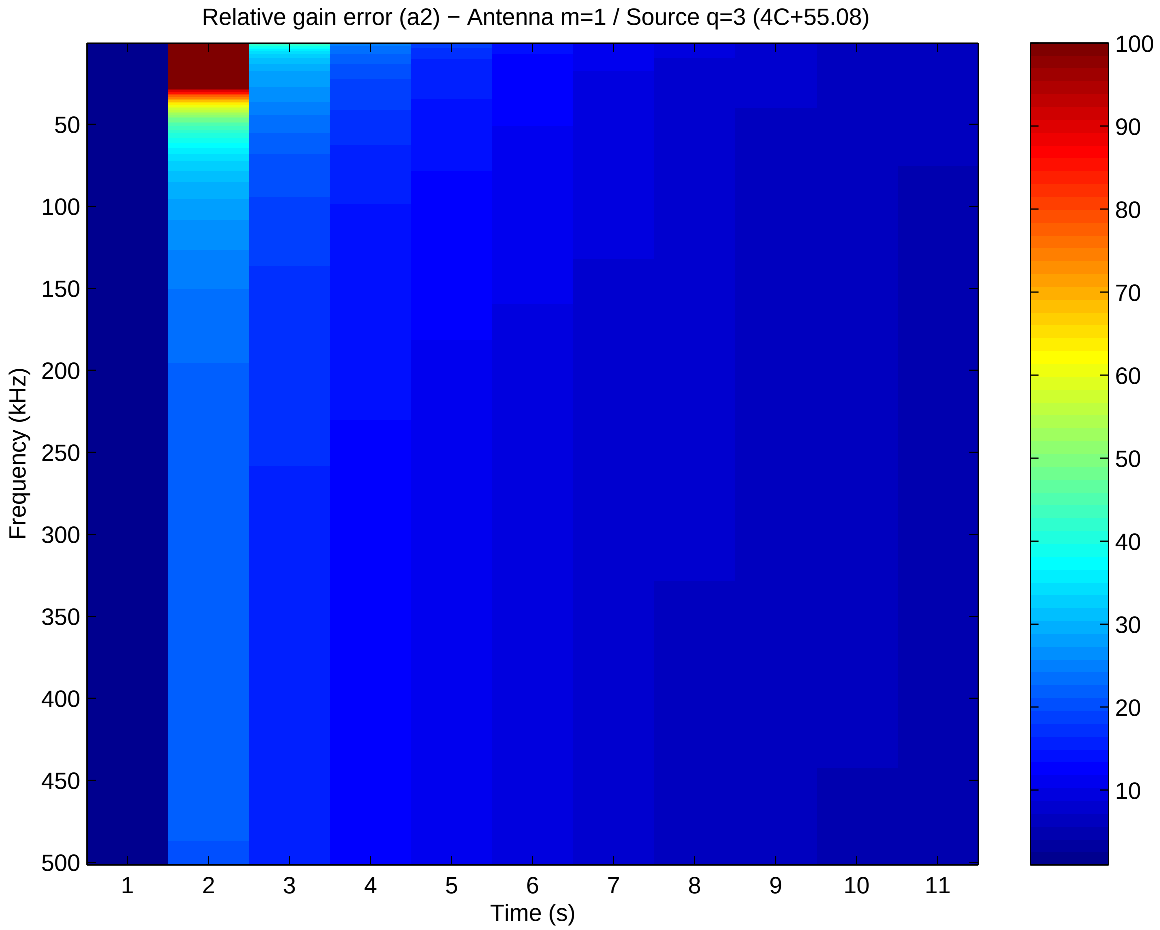
Multiple sources - 2D polynomial



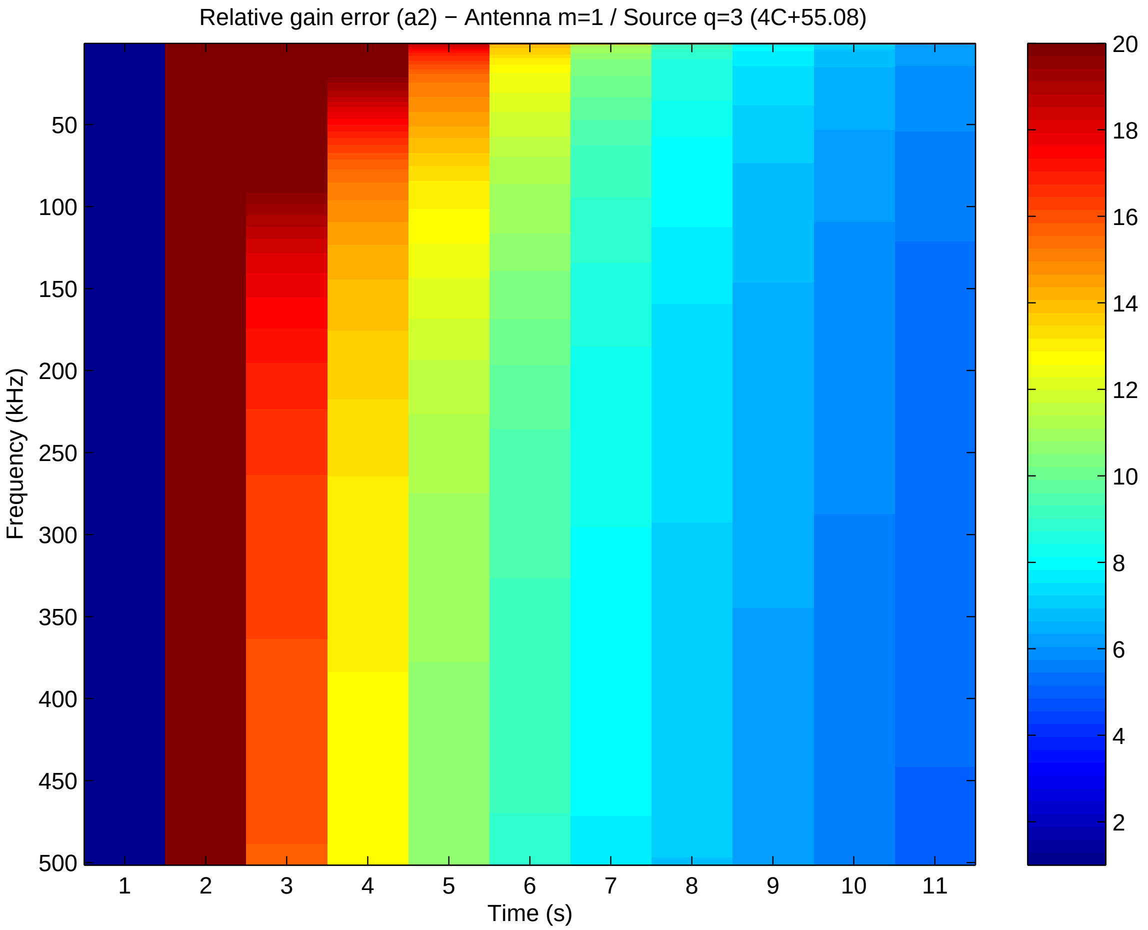
Multiple sources - 2D polynomial



Multiple sources - 2D polynomial



Multiple sources - 2D polynomial



Limitations to Current CRB Analysis

- So far it includes only error effects due to noise and sample covariance estimation.
- There can also be modeling errors, e.g. maybe a polynomial fits time-frequency variation poorly.
- Errors in tabulated source location and brightness are not considered.
- Array station location is assumed to be exact.
- Station calibration errors are lumped in with ionospheric gains.

Conclusions

- The BIG answer: Yes, LOFAR can be calibrated
- Given a range of time-frequency observations and compact core geometry: there are no theoretical roadblocks to achieving useful calibration estimates.
- If an algorithm can be developed to approach the CRB calibration error should be acceptable.