Schedule…

Date	Day	Class No.	Title	Chapters	HW Due date	Lab Due date	Exam
15 Oct	Wed	13	Phasors	4.4			EXAM 1
16 Oct	Thu						
17 Oct	Fri		Recitation				
18 Oct	Sat						
19 Oct	Sun						
20 Oct	Mon	14	AC Circuit Analysis	4.5		NO LAB	
21 Oct	Tue					NO LAB	
22 Oct	Wed	15	Transient Response 1st Order Circuits	5.4			

Imaginary

<u>JS-H 1: 16</u>

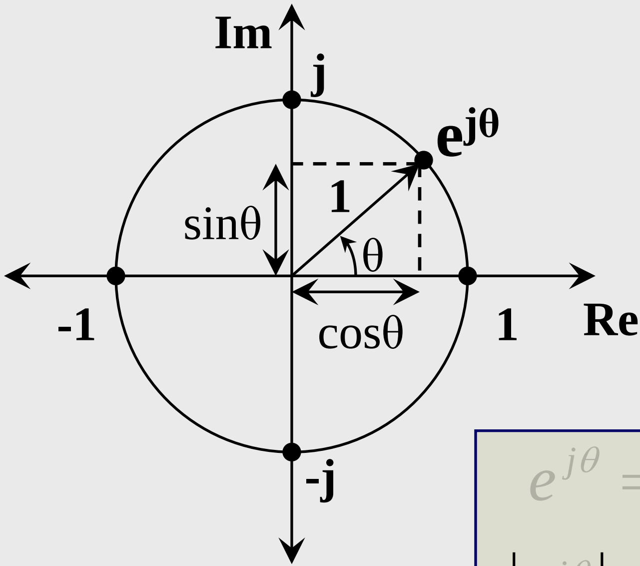
16 But, exerting all my powers to call upon God to deliver me out of the power of this enemy which had seized upon me, and at the very moment when I was ready to sink into despair and abandon myself to destruction—not to an **imaginary** ruin, but to the power of some actual being from the unseen world, who had such marvelous power as I had never before felt in any being—just at this moment of great alarm, I saw a pillar of light exactly over my head, above the brightness of the sun, which descended gradually until it fell upon me.

Lecture 13 – Network Analysis with Capacitors and Inductors

Phasors

Euler's Identity

- **Appendix A** reviews complex numbers



Complex exponential ($\mathbf{e^{j\theta}}$) is a point on the complex plane

$$e^{j\theta} = \cos\theta + j\sin\theta$$

$$\left|e^{j\theta}\right| = 1 \rightarrow \left|\cos\theta + j\sin\theta\right| = \sqrt{\cos^2\theta + \sin^2\theta} = 1$$

$$Ae^{j\theta} = A\cos\theta + jA\sin\theta$$
$$= A\angle\theta$$

Phasors

- Rewrite the expression for a **general sinusoid signal**:

$$A\cos(\omega t+\theta)=\mathrm{Re}\{Ae^{j(\omega t+\theta)}\}$$

magnitude

Angle (or argument)

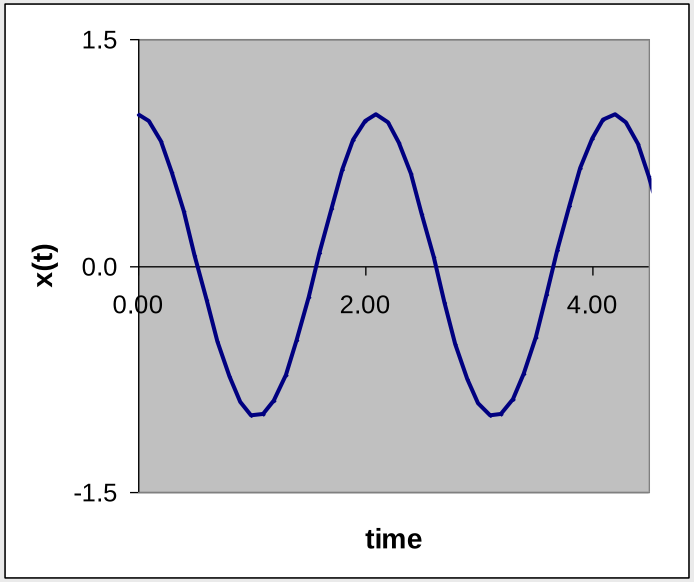
Complex phasor notation for the **simplification**:

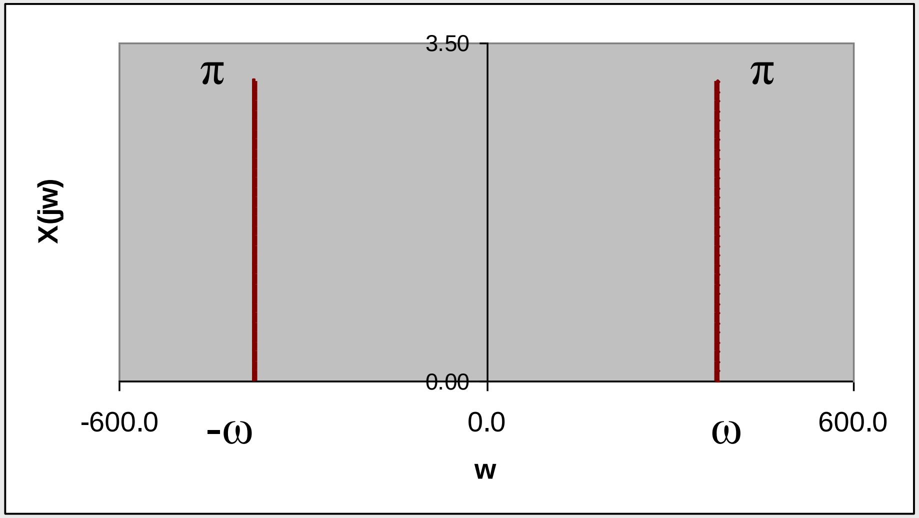
$$A\cos(\omega t+\theta)\rightarrow A\angle\theta = Ae^{j\theta}$$

NB: The $\mathbf{e^{jwt}}$ term is **implicit** (it is there but not written)

Frequency Domain

<u>Graphing in the frequency domain</u>: helpful in order to understand Phasors





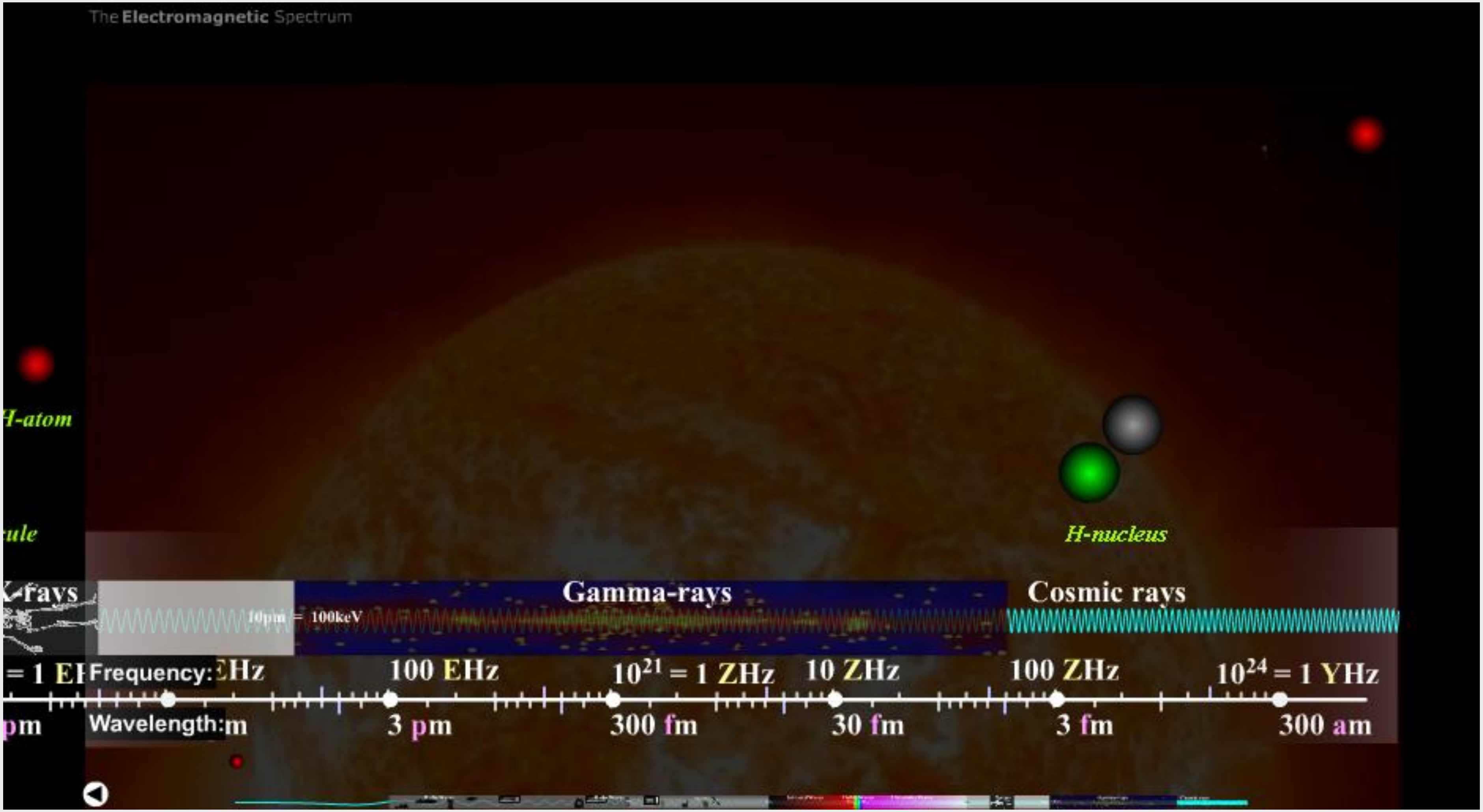
$\cos(\omega_0 t)$

$\pi[\delta(\omega - \omega_0) + \delta(\omega - \omega_0)]$

Time domain

Frequency domain

Electromagnetic Spectrum

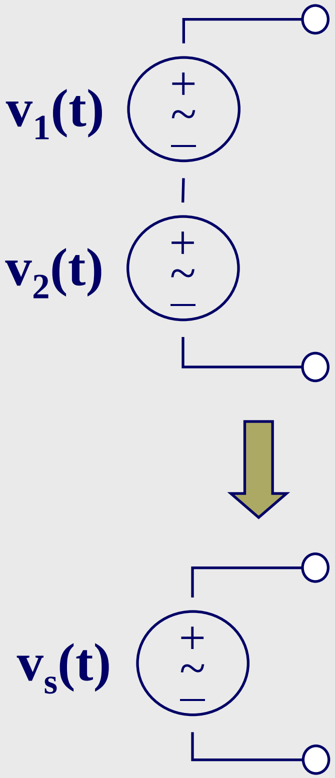


Phasors

1. Any sinusoidal signal can be represented by either:
   - **<u>Time-domain form</u>: $v(t) = A\cos(\omega t+\theta)$**
   - **<u>Frequency-domain form</u>: $V(j\omega) = Ae^{j\theta} = A\angle\theta$**
2. **<u>Phasor</u>**: a complex number expressed in polar form consisting of:
   - **Magnitude (A)**
   - **Phase angle (θ)**
3. Phasors do not explicitly include the sinusoidal frequency (ω) but this information is still important

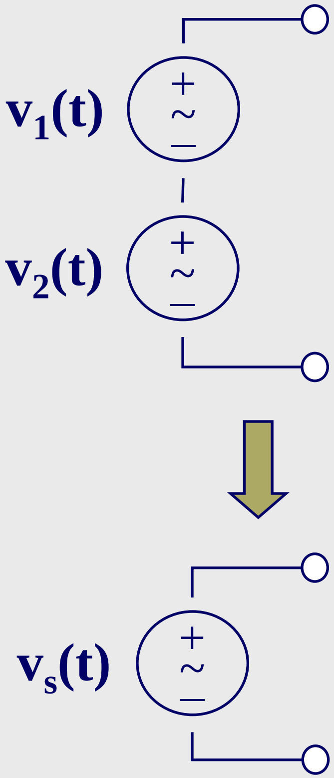
Phasors

- **Example1**: compute the phasor voltage for the equivalent voltage $\mathbf{v_s(t)}$
  - $\mathbf{v_1(t) = 15\cos(377t+\pi/4)}$
  - $\mathbf{v_2(t) = 15\cos(377t+\pi/12)}$



Phasors

- **Example1**: compute the phasor voltage for the equivalent voltage $\mathbf{v_s(t)}$
  - $v_1(t) = 15\cos(377t+\pi/4)$
  - $v_2(t) = 15\cos(377t+\pi/12)$



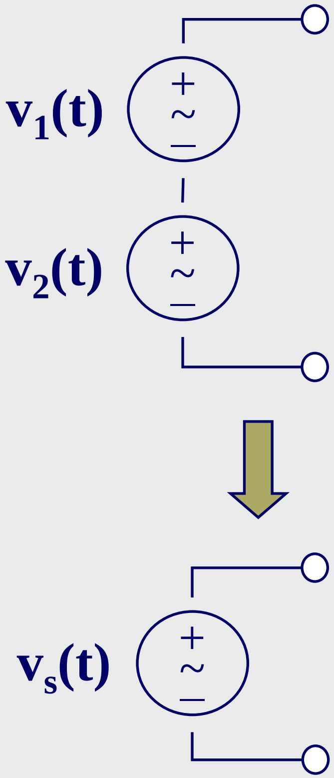
1. Write voltages in phasor notation

$$V_1(j\omega) = 15e^{j\pi/4} = 15\angle\frac{\pi}{4}\ V$$

$$V_2(j\omega) = 15e^{j\pi/12} = 15\angle\frac{\pi}{12}\ V$$

Phasors

- **Example1**: compute the phasor voltage for the equivalent voltage $\mathbf{v_s(t)}$
  - $\mathbf{v_1(t) = 15\cos(377t+\pi/4)}$
  - $\mathbf{v_2(t) = 15\cos(377t+\pi/12)}$



1. Write voltages in phasor notation
2. Convert phasor voltages from polar to rectangular form (see Appendix A)

$$V_1(j\omega) = 15\angle\frac{\pi}{4}\ V$$

Convert to rectangular:

$$V_1(j\omega) = 15\cos\left(\frac{\pi}{4}\right) + j15\sin\left(\frac{\pi}{4}\right)$$
$$= 10.61 + j10.61\ V$$

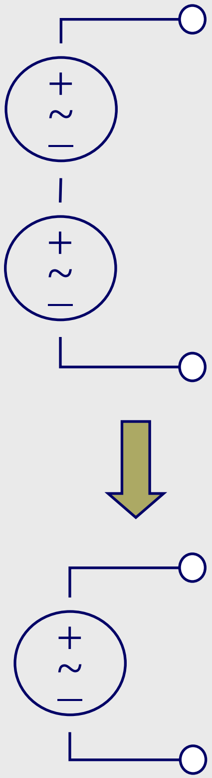
$$V_2(j\omega) = 15\angle\frac{\pi}{12}\ V$$

Convert to rectangular:

$$V_2(j\omega) = 15\cos\left(\frac{\pi}{12}\right) + j15\sin\left(\frac{\pi}{12}\right)$$
$$= 14.49 + j3.88\ V$$

Phasors

- **<u>Example1</u>**: compute the phasor voltage for the equivalent voltage $\mathbf{v_s(t)}$
  - $\mathbf{v_1(t) = 15\cos(377t+\pi/4)}$
  - $\mathbf{v_2(t) = 15\cos(377t+\pi/12)}$

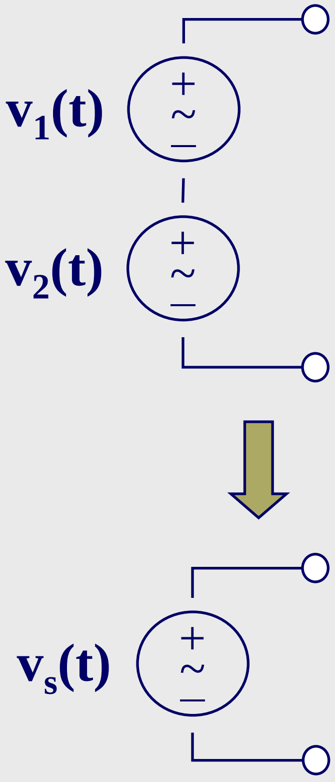


1. Write voltages in phasor notation
2. Convert phasor voltages from polar to rectangular form (see Appendix A)
3. Combine voltages

$$
\begin{aligned}
V_S(j\omega) &= V_1(j\omega) + V_2(j\omega) \\
&= 25.10 + j14.49
\end{aligned}
$$

Phasors

- **Example1**: compute the phasor voltage for the equivalent voltage $\mathbf{v_s(t)}$
  - $\mathbf{v_1(t) = 15\cos(377t+\pi/4)}$
  - $\mathbf{v_2(t) = 15\cos(377t+\pi/12)}$



1. Write voltages in phasor notation
2. Convert phasor voltages from polar to rectangular form (see Appendix A)
3. Combine voltages
4. Convert rectangular back to polar

$$V_S(j\omega) = 25.10 + j14.49$$

Convert to polar :

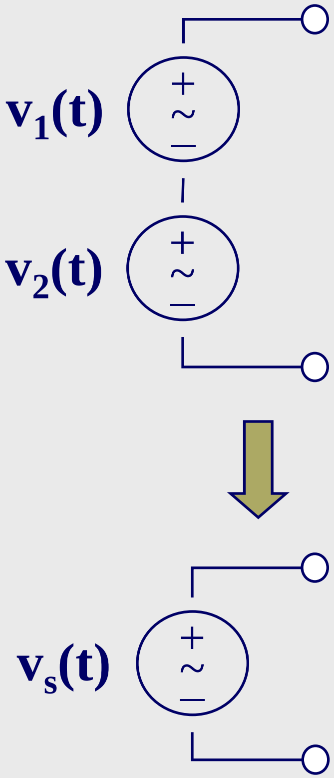
$$r = \sqrt{(25.10)^2 + (14.49)^2} = 28.98$$

$$\theta = \tan^{-1}\left(\frac{14.49}{25.10}\right) = \frac{\pi}{6}$$

$$V_S(j\omega) = 28.98\angle\frac{\pi}{6}$$

Phasors

- **<u>Example1</u>**: compute the phasor voltage for the equivalent voltage $\mathbf{v_s(t)}$
  - $\mathbf{v_1(t) = 15\cos(377t+\pi/4)}$
  - $\mathbf{v_2(t) = 15\cos(377t+\pi/12)}$



1. Write voltages in phasor notation
2. Convert phasor voltages from polar to rectangular form (see Appendix A)
3. Combine voltages
4. Convert rectangular back to polar
5. Convert from phasor to time domain

$$V_S(j\omega) = 28.98\angle\frac{\pi}{6}$$

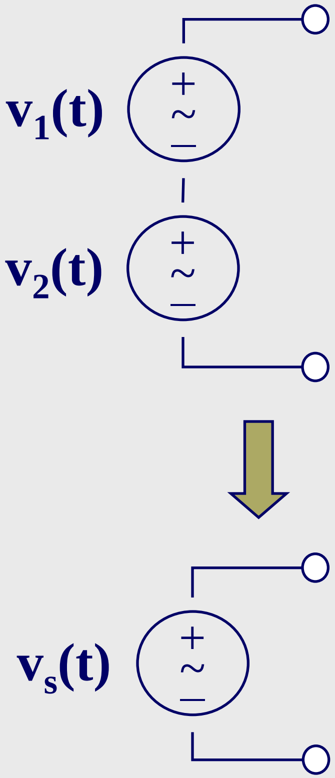
$$v_S(t) = 28.98\cos\left(377t + \frac{\pi}{6}\right)$$

Bring **ωt** back

NB: the answer is **NOT** simply the addition of the amplitudes of $\mathbf{v_1(t)}$ and $\mathbf{v_2(t)}$ (i.e. 15 + 15), and the addition of their phases (i.e. π/4 + π/12)

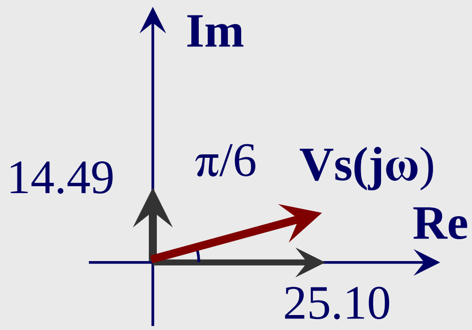
Phasors

- **Example1**: compute the phasor voltage for the equivalent voltage $\mathbf{v_s(t)}$
  - $\mathbf{v_1(t) = 15\cos(377t+\pi/4)}$
  - $\mathbf{v_2(t) = 15\cos(377t+\pi/12)}$



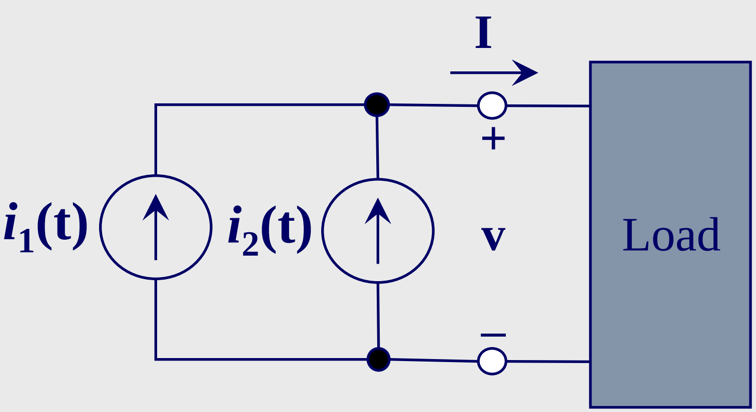
$$V_S(j\omega) = 28.98\angle\frac{\pi}{6}$$

$$v_S(t) = 28.98\cos\left(377t + \frac{\pi}{6}\right)$$



Phasors of Different Frequencies

<u>Superposition of AC signals</u>: when signals do **not have the same frequency** (ω) the $\mathbf{e^{j\omega t}}$ term in the phasors can no longer be implicit



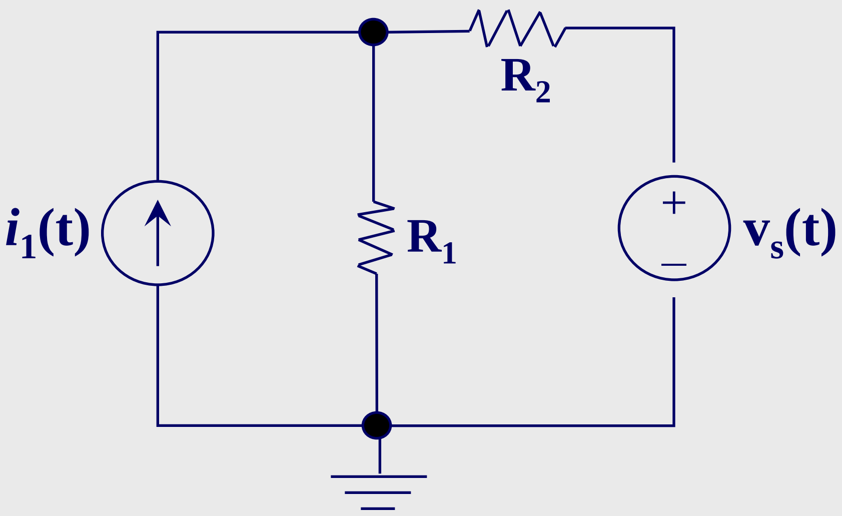
$$i(t) = i_1(t) + i_2(t)$$

$$\begin{aligned} I(j\omega) &= I_1(j\omega_1) + I_2(j\omega_2) \\ &= A_1 e^{j0} e^{j\omega_1 t} + A_2 e^{j0} e^{j\omega_2 t} \\ &\neq A_1 e^{j0} + A_2 e^{j0} \end{aligned}$$

NB: $\mathbf{e^{j\omega t}}$ can no longer be implicit

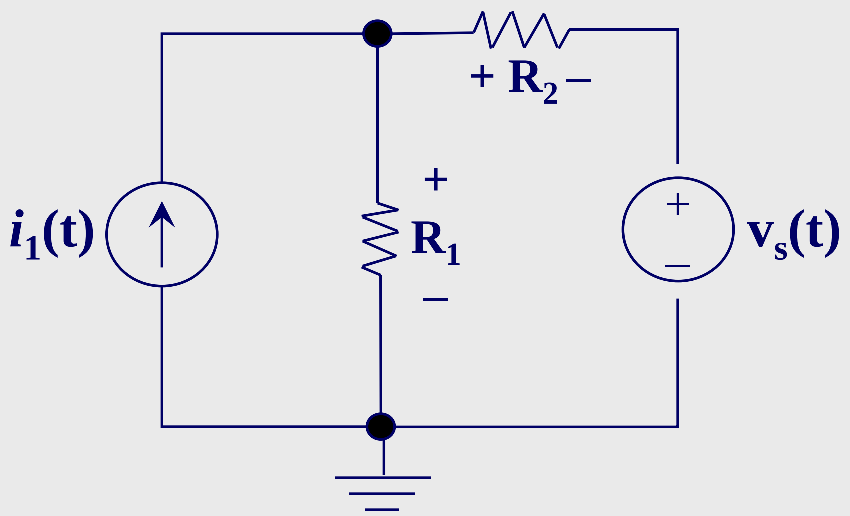
Phasors of Different Frequencies

<u>Superposition of AC signals</u>: when signals do **not have the same frequency** (ω) solve the circuit separately for each different frequency (ω) – then add the individual results



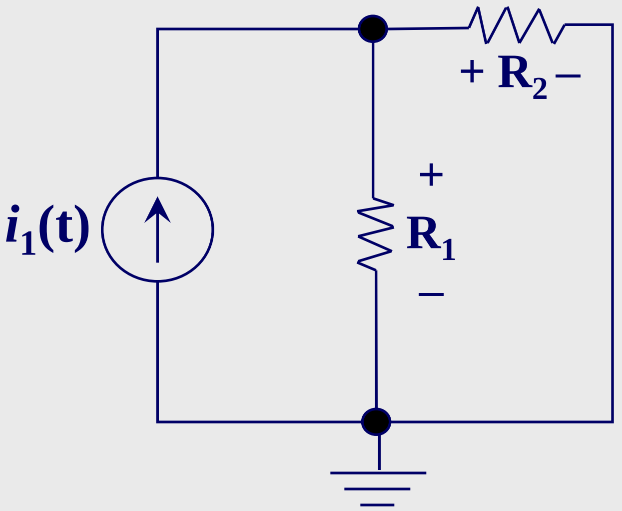
Phasors of Different Frequencies

- **Example2**: compute the resistor voltages
  - $i_s(t) = 0.5\cos[2\pi(100t)]$ A
  - $v_s(t) = 20\cos[2\pi(1000t)]$ V
  - $R_1 = 150\Omega$, $R2 = 50\ \Omega$



Phasors of Different Frequencies

- **<u>Example2</u>**: compute the resistor voltages
  - $\boldsymbol{i_s(t) = 0.5\cos[2\pi(100t)]}$ **A**
  - $\boldsymbol{v_s(t) = 20\cos[2\pi(1000t)]}$ **V**
  - $\boldsymbol{R_1 = 150\Omega}$**, R2 = 50 Ω**

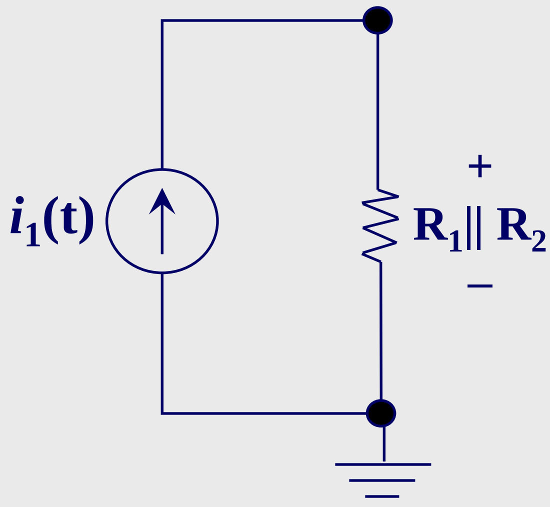


1. Since the sources have different frequencies ($\omega_1 = 2\pi*100$) and ($\omega_2 = 2\pi*1000$) use superposition
   - first consider the ($\omega_1 = 2\pi*100$) part of the circuit
   - When $v_s(t) = 0$ – short circuit

Phasors of Different Frequencies

- **<u>Example2</u>**: compute the resistor voltages
  - $i_s(t) = 0.5\cos[2\pi(100t)]$ A
  - $v_s(t) = 20\cos[2\pi(1000t)]$ V
  - $R_1 = 150\Omega$, $R_2 = 50\ \Omega$

1. Since the sources have different frequencies ($\omega_1 = 2\pi*100$) and ($\omega_2 = 2\pi*1000$) use superposition
   - first consider the ($\omega_1 = 2\pi*100$) part of the circuit



$$I_s(j\omega) = 0.5\angle 0$$

$$V_{I1}(j\omega) = V_{I2}(j\omega) = I_s \cdot R_1 \| R_2$$

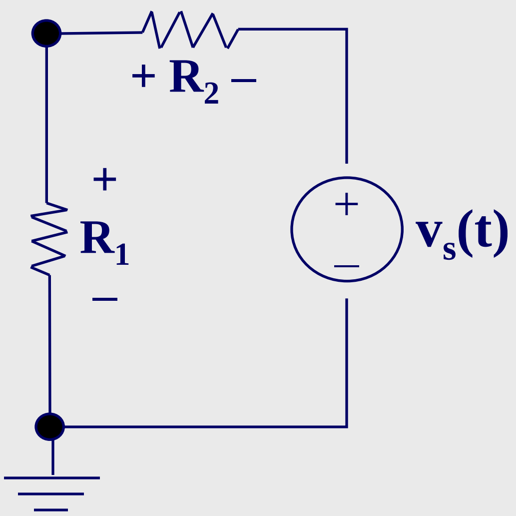
$$= I_s \cdot \frac{R_1 R_2}{R_1 + R_2}$$

$$= 0.5\angle 0 \cdot \frac{(50)(150)}{(50)+(150)}$$

$$= 18.75\angle 0$$

Phasors of Different Frequencies

- **Example2**: compute the resistor voltages
  - $i_s(t) = 0.5\cos[2\pi(100t)]$ A
  - $v_s(t) = 20\cos[2\pi(1000t)]$ V
  - $R_1 = 150\Omega$, $R_2 = 50\ \Omega$



1. Since the sources have different frequencies ($\omega_1 = 2\pi*100$) and ($\omega_2 = 2\pi*1000$) use superposition
   - first consider the ($\omega_1 = 2\pi*100$) part of the circuit
   - Next consider the ($\omega_2 = 2\pi*1000$) part of the circuit

$$V_s(j\omega) = 20\angle 0$$

$$V_{V1}(j\omega) = V_s \cdot \frac{R_1}{R_1 + R_2} = 20\angle 0 \cdot \frac{(150)}{(50)+(150)} = 15\angle 0$$

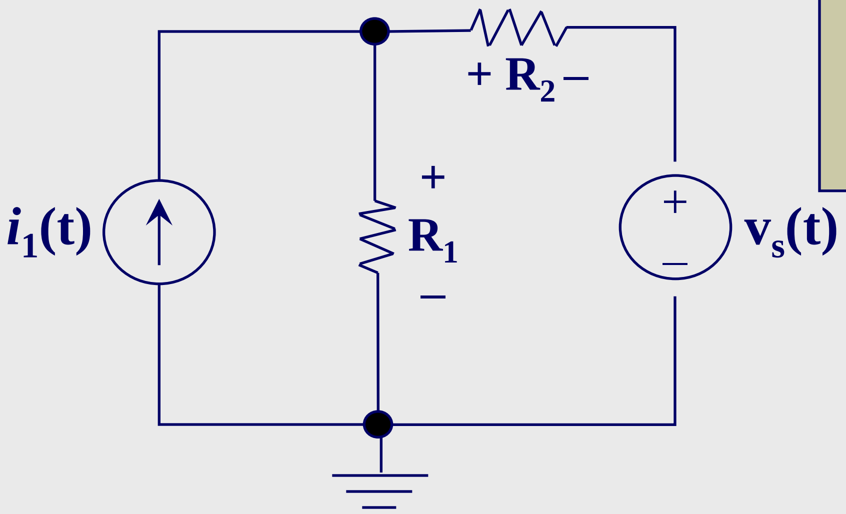
$$V_s(j\omega) = 20\angle 0$$

$$V_{V2}(j\omega) = -V_s \cdot \frac{R_2}{R_1 + R_2} = -20\angle 0 \cdot \frac{(50)}{(50)+(150)} = -5\angle 0 = 5\angle \pi$$

BYU Electrical Engineering Computer Engineering

Phasors of Different Frequencies

- **<u>Example2</u>**: compute the resistor voltages
  - $i_s(t) = 0.5\cos[2\pi(100t)]$ A
  - $v_s(t) = 20\cos[2\pi(1000t)]$ V
  - $R_1 = 150\Omega$, $R_2 = 50\ \Omega$



1. Since the sources have different frequencies ($\omega_1 = 2\pi*100$) and ($\omega_2 = 2\pi*1000$) use superposition
   - first consider the ($\omega_1 = 2\pi*100$) part of the circuit
   - Next consider the ($\omega_2 = 2\pi*1000$) part of the circuit
   - Add the two together

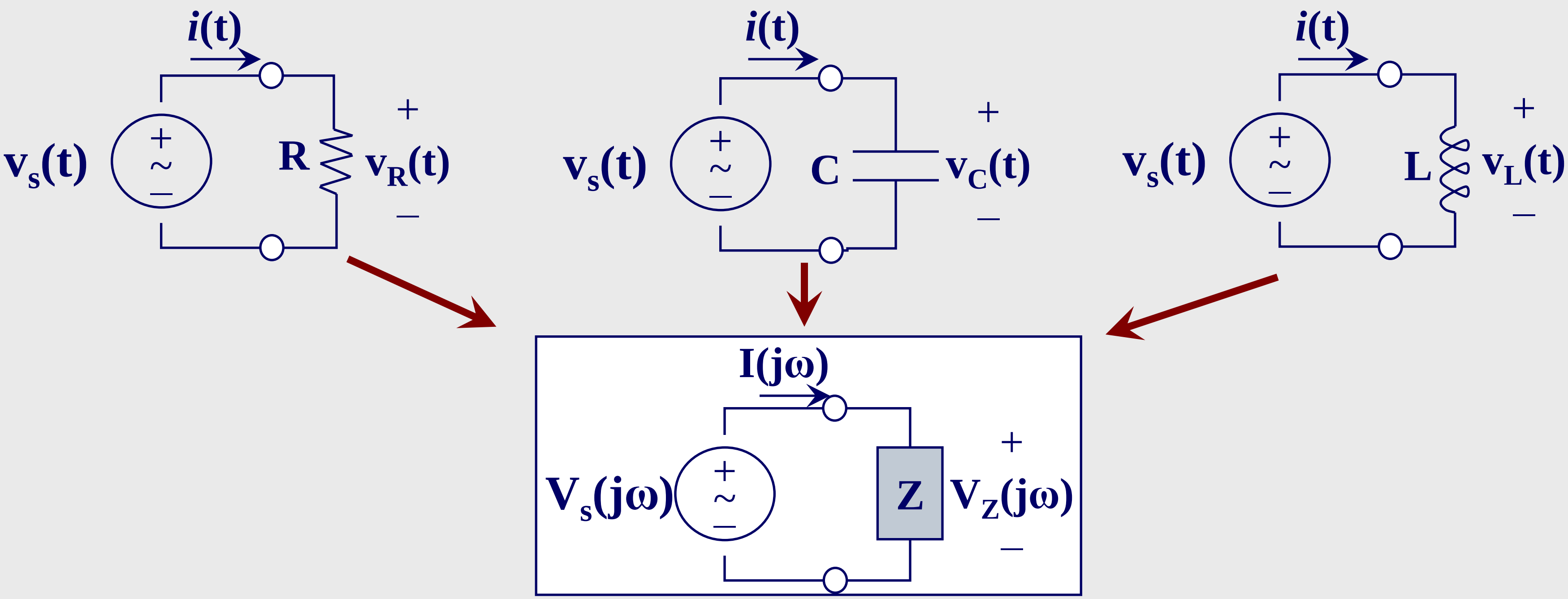
$$V_1(j\omega) = V_{I1}(j\omega) + V_{V1}(j\omega)$$
$$= 18.75\angle 0 + 15\angle 0$$
$$v_1(t) = 18.75\cos[2\pi(100t)] + 15\cos[2\pi(1000t)]$$

$$V_2(j\omega) = V_{I2}(j\omega) + V_{V2}(j\omega)$$
$$= 18.75\angle 0 - 5\angle 0$$
$$v_1(t) = 18.75\cos[2\pi(100t)] - 5\cos[2\pi(1000t)]$$

Impedance

Impedance: complex resistance (has no physical significance)

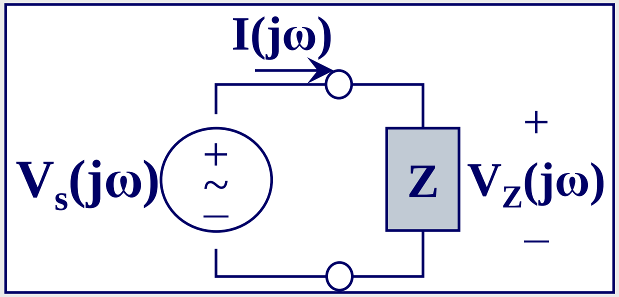
- will allow us to use network analysis methods such as node voltage, mesh current, etc.
- Capacitors and inductors act as **frequency-dependent** resistors



Impedance – Resistors

Impedance of a Resistor:

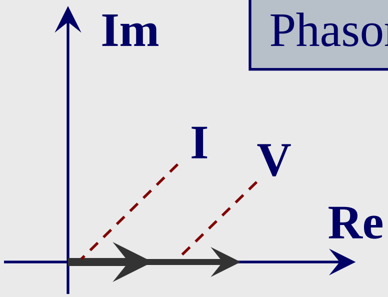
- Consider Ohm's Law in phasor form:



$$v_s(t) = A\cos(\omega t)$$
$$i(t) = \frac{v_s(t)}{R}$$
$$= \frac{A}{R}\cos(\omega t)$$

Phasor

$$V_Z(j\omega) = A\angle 0$$
$$I_Z(j\omega) = \frac{A}{R}\angle 0$$



Phasor domain

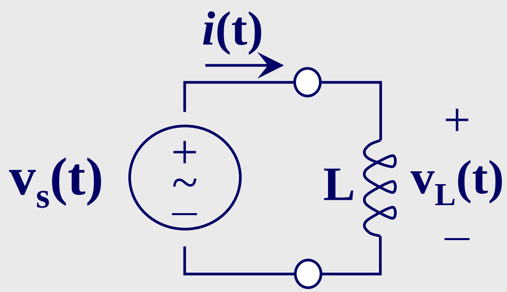
$$Z_R(j\omega) = \frac{V_Z(j\omega)}{I_Z(j\omega)} = R$$

NB: Ohm's Law works the same in DC and AC

Impedance – Inductors

<u>Impedance of an Inductor</u>:

- First consider voltage and current in the **time-domain**



$$v_L(t) = L\frac{di_L(t)}{dt} = v_S(t)$$

$$i_L(t) = \frac{1}{L}\int v_L(\tau)d\tau$$

$$= \frac{1}{L}\int A\cos(\omega\tau)d\tau$$

$$= \frac{A}{\omega L}\sin(\omega t)$$

NB: current is shifted 90 from voltage

$$v_S(t) = v_L(t) = A\cos(\omega t)$$

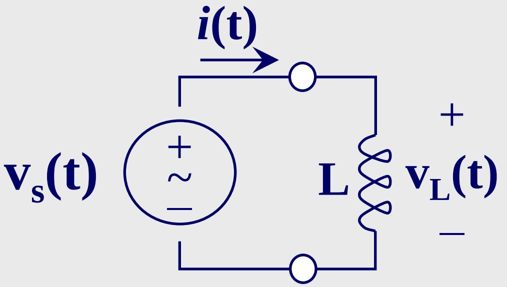
$$i_L(t) = \frac{A}{\omega L}\sin(\omega t)$$

$$= \frac{A}{\omega L}\cos\left(\omega t - \frac{\pi}{2}\right)$$

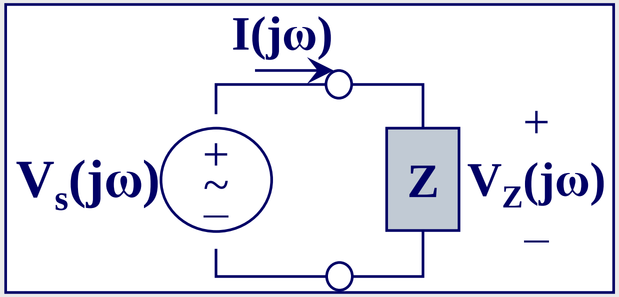
Impedance – Inductors

Impedance of an Inductor:

- Now consider voltage and current in the **phasor-domain**



Phasor



Phasor domain

$$v_S(t)=v_L(t)=A\cos(\omega t)$$

$$i_L(t)=\frac{A}{\omega L}\sin(\omega t)$$

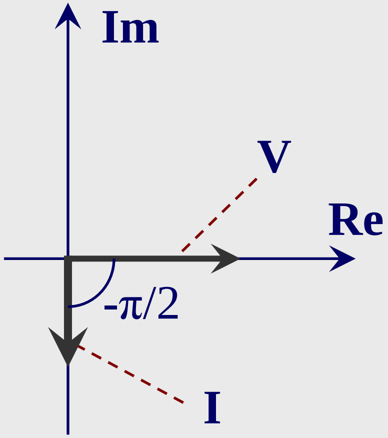
$$=\frac{A}{\omega L}\cos\left(\omega t-\frac{\pi}{2}\right)$$

Phasor

$$V_Z(j\omega)=A\angle 0$$

$$I_Z(j\omega)=\frac{A}{\omega L}\angle -\frac{\pi}{2}$$

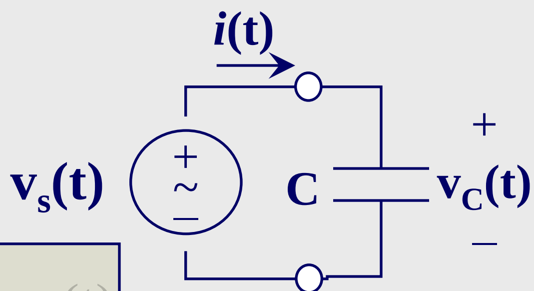
$$Z_L(j\omega)=\frac{V_Z(j\omega)}{I_Z(j\omega)}=j\omega L$$



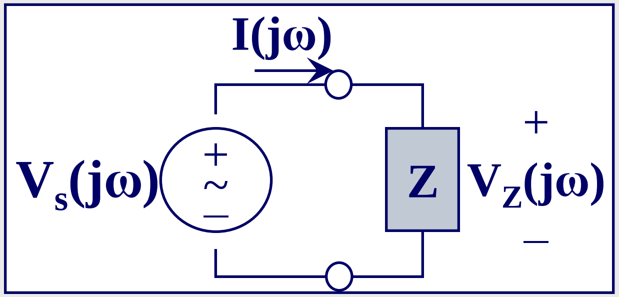
Impedance – Capacitors

Impedance of a capacitor:

- First consider voltage and current in the **time-domain**



Phasor



$$v_C(t) = \frac{1}{C}\int i_C(\tau)d\tau = v_S(t)$$

$$i_C(t) = C\frac{dv_C(t)}{dt}$$

$$= C\frac{d}{dt}[A\cos(\omega t)]$$

$$= -C[A\omega\sin(\omega t)]$$

$$= \omega CA\cos\left(\omega t + \frac{\pi}{2}\right)$$

Phasor

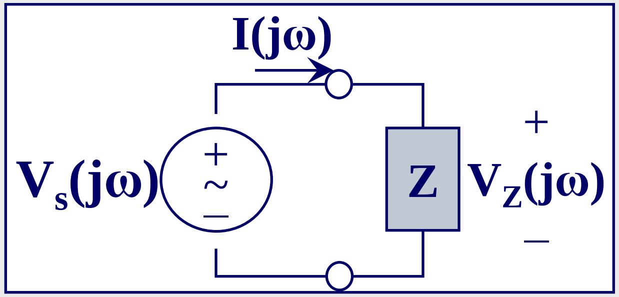
$$V_Z(j\omega) = A\angle 0$$

$$I_Z(j\omega) = \omega CA\angle\frac{\pi}{2}$$

Impedance – Capacitors

Impedance of a capacitor:

- Next consider voltage and current in the **phasor-domain**



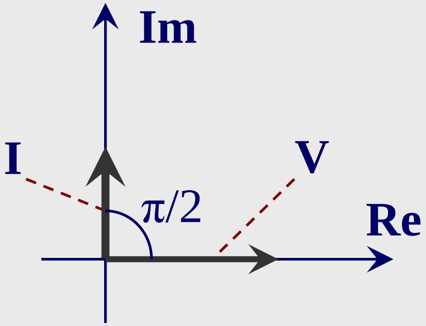
$$V_Z(j\omega) = A\angle 0$$

$$I_Z(j\omega) = \omega C A\angle \frac{\pi}{2}$$

$$Z_L(j\omega) = \frac{V_Z(j\omega)}{I_Z(j\omega)} = \frac{1}{\omega C}\angle -\frac{\pi}{2} = \frac{-j}{\omega C} = \frac{1}{j\omega C}$$

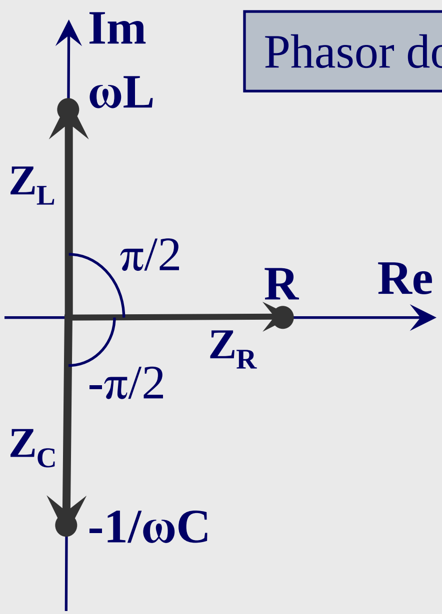
$$-j = e^{-j\frac{\pi}{2}} = \frac{1}{j}$$

Phasor domain

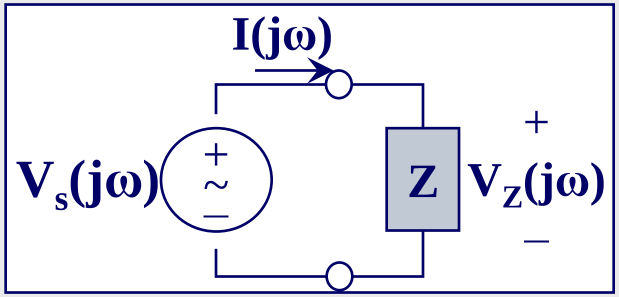


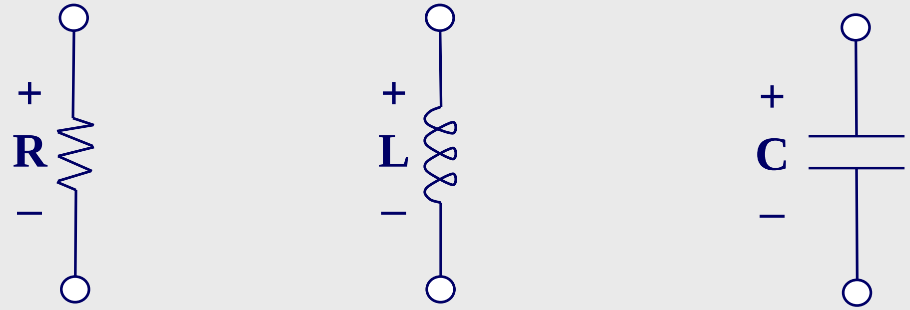
Impedance

Impedance of resistors, inductors, and capacitors



Phasor domain





$$Z_R(j\omega) = R$$

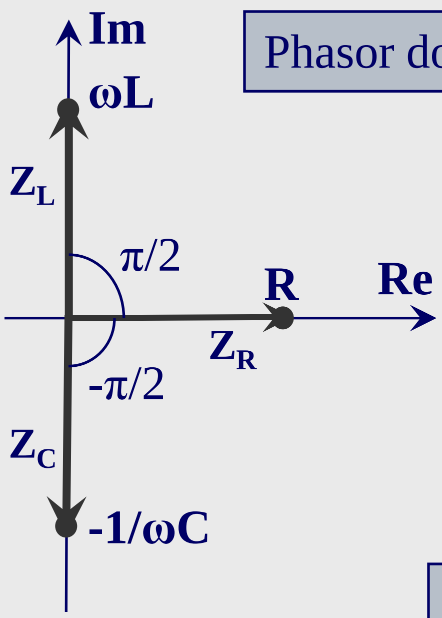
$$Z_L(j\omega) = j\omega L$$

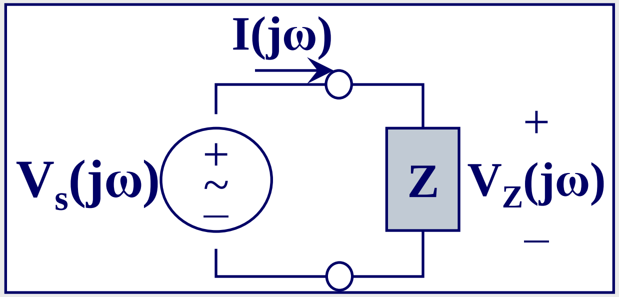
$$Z_C(j\omega) = \frac{1}{j\omega C}$$

Impedance

Impedance of resistors, inductors, and capacitors

Phasor domain





Impedence in general:

$$Z(j\omega) = R(j\omega) + jX(j\omega)$$

Not a phasor but a complex number

AC resistance

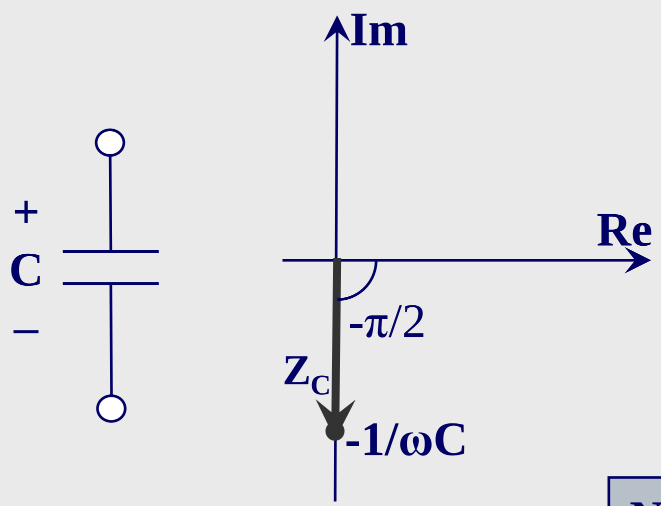
reactance

Impedance

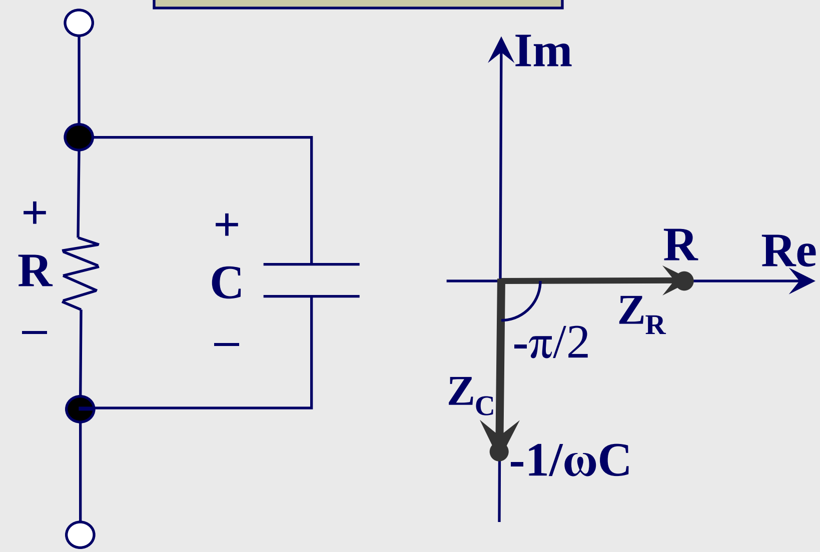
<u>Practical capacitors</u>: in practice capacitors contain a real component (represented by a resistive impedance $\mathbf{Z_R}$)

- At **high frequencies** or **high capacitances**
  - **ideal** capacitor acts like a **short circuit**
- At **low frequencies** or **low capacitances**
  - **ideal** capacitor acts like an **open circuit**

Ideal Capacitor



Practical Capacitor



NB: the ratio of $\mathbf{Z_C}$ to $\mathbf{Z_R}$ is highly frequency dependent

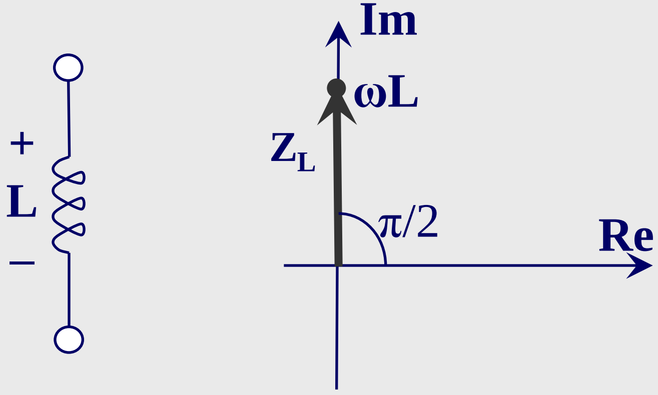
BYU Electrical Engineering Computer Engineering

Impedance

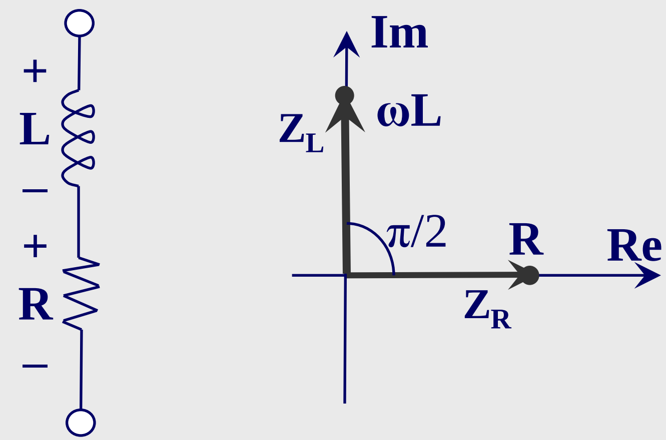
<u>Practical inductors</u>: in practice inductors contain a real component (represented by a resistive impedance $\mathbf{Z_R}$)

- At **low frequencies** or **low inductances** $\mathbf{Z_R}$ has a strong influence
  - **Ideal** inductor acts like a **short circuit**
- At **high frequencies** or **high inductances** $\mathbf{Z_L}$ dominates $\mathbf{Z_R}$
  - **Ideal** inductor acts like an **open circuit**
  - At high frequencies a capacitor is also needed to correctly model a practical inductor

Ideal Inductor



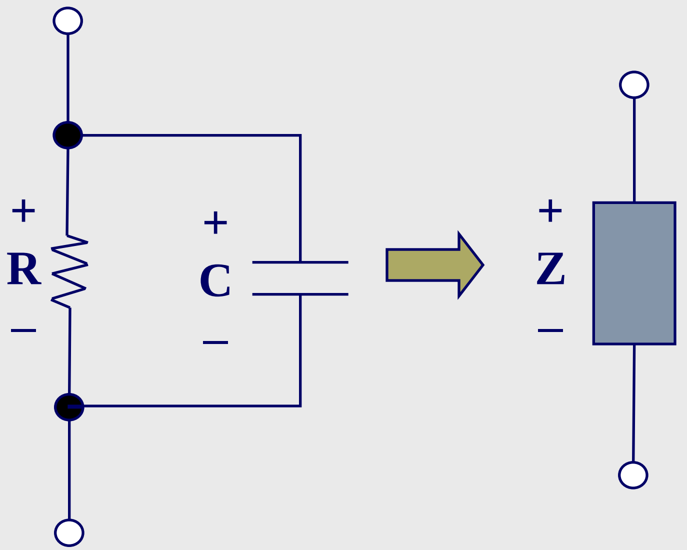
Practical Inductor



NB: the ratio of $\mathbf{Z_L}$ to $\mathbf{Z_R}$ is highly frequency dependent

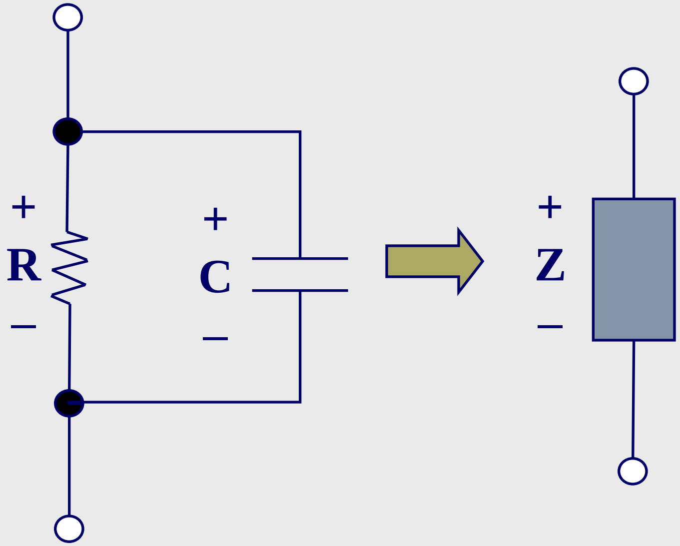
Impedance

- **Example3**: impedance of a practical capacitor
  - Find the impedance
  - **ω** = 377 rads/s, **C** = 1nF, **R** = 1MΩ



Impedance

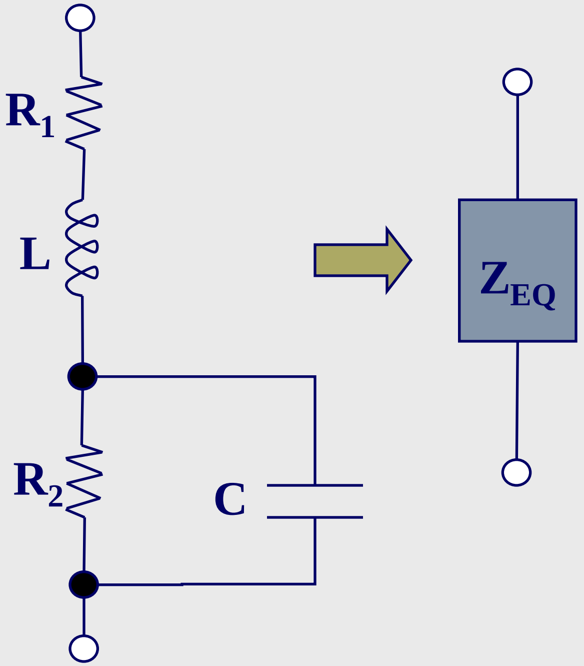
- **Example3**: impedance of a practical capacitor
  - Find the impedance
  - **ω** = 377 rads/s, **C** = 1nF, **R** = 1MΩ



$$
\begin{aligned}
Z &= Z_R \parallel Z_C \\
&= \frac{R \cdot (1/j\omega C)}{R + (1/j\omega C)} \\
&= \frac{R}{1 + j\omega CR} \\
&= \frac{10^6}{1 + j(377)(10^{-9})(10^6)} \\
&= \frac{10^6}{1 + j0.377} \\
&= 9.36 \times 10^5 \angle(-0.36)\ \Omega
\end{aligned}
$$

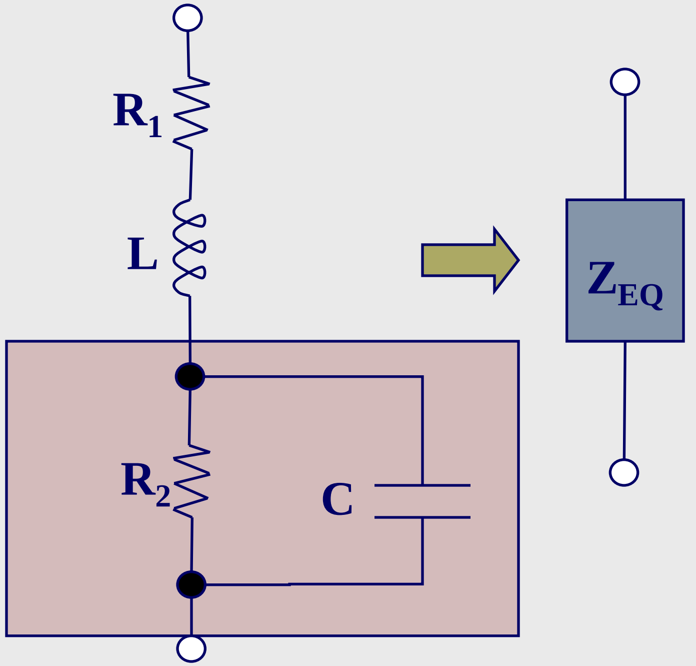
Impedance

- **Example4**: find the equivalent impedance (Z_{EQ})
  - $\omega = 10^4$ rads/s, **C** = 10uF, $R_1 = 100\Omega$, $R_2 = 50\Omega$, **L** = 10mH



Impedance

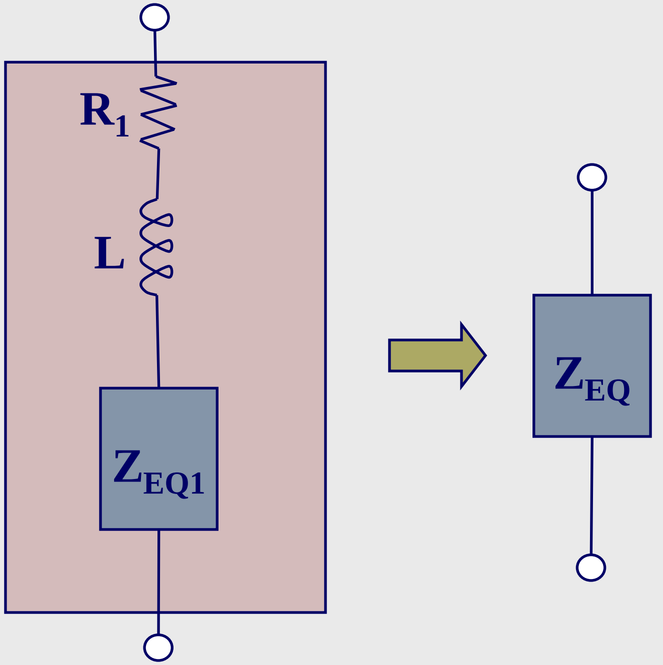
- **Example4**: find the equivalent impedance (**Z_{EQ}**)
  - **ω** = 10^4 rads/s, **C** = 10uF, **R_1** = 100Ω, **R_2** = 50Ω, **L** = 10mH



$$
\begin{aligned}
Z_{EQ1} &= Z_{R2} \parallel Z_C \\
&= \frac{R_2(1/j\omega C)}{R_2 + (1/j\omega C)} \\
&= \frac{R_2}{1 + j\omega C R_2} \\
&= \frac{50}{1 + j(10^4)(10 \times 10^{-6})(50)} \\
&= \frac{50}{1 + j5} \\
&= 1.92 - j9.62 \\
&= 9.81\angle(-1.37)\ \Omega
\end{aligned}
$$

Impedance

- **Example4**: find the equivalent impedance (Z_{EQ})
  - **ω** = 10^4 rads/s, **C** = 10uF, **R_1** = 100Ω, **R_2** = 50Ω, **L** = 10mH



$$
\begin{aligned}
Z_{EQ} &= Z_{R1} + Z_L + Z_{EQ1} \\
&= R_1 + j\omega L + 9.81\angle(-1.37) \\
&= 100 + j(10^4)(10^{-2}) + 1.92 - j9.62 \\
&= 101.92 + j90.38 \\
&= 136.2\angle 0.723\ \Omega
\end{aligned}
$$

NB: at this frequency (**ω**) the circuit has an **inductive impedance** (reactance or phase is positive)

$$Z_{EQ1} = 9.81\angle(-1.37)\ \Omega$$