ECEn 370
Introduction to Probability
Section 001

Midterm
Winter, 2009

Instructor
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Closed Book
Non-graphing Calculator Allowed
No Time Limit

IMPORTANT!

- WRITE YOUR NAME on every page of the exam.
- Answer questions 1-17 (Part I) on the provided bubble sheet.
- Questions 1-17 are worth 1 point each.
- Answer questions 18-19 (Part II) directly on the exam.
- Questions 18-19 have their points specified individually.
- Do not discuss the exam with other students.
Part I

1. A test for the probability-phobia-disease is assumed to be correct 99% of the time. We also assume that the probability of any random person in our class having this disease is 0.0001. Given that you just tested positive for this disease, what is the probability that you actually have the disease?

   A) 0.9900  
   B) 0.0100  
   C) 0.0099  
   D) 0.0098  
   E) 0.0001  
   F) None of the Above  

   Correct Answer: D

   Let event $A = \{\text{having the disease}\}$ and event $B = \{\text{testing positive for disease}\}$

   \[
   P(A|B) = \frac{P(A)P(B|A)}{P(A)P(B|A) + P(A^c)P(B|A^c)} = \frac{0.0001 \cdot 0.99}{0.0001 \cdot 0.99 + 0.9999 \cdot 0.01} = 0.0098
   \]

2. Border Patrol. If an intruder is present in a certain area, our monitoring system detects him and generates an alarm signal with probability 0.99. We assume that an intruder is present with probability 0.01. What is the probability of an intruder being present and no alarm given?

   A) 0.9900  
   B) 0.0100  
   C) 0.0990  
   D) 0.0099  
   E) 0.0001  
   F) None of the Above

   Correct Answer: E

   Let event $A = \{\text{intruder present}\}$ and event $B = \{\text{system generates alarm}\}$.

   $P(A) = 0.01$  
   $P(B|A) = 0.99$  
   $P(B^c|A) = 1 - 0.99 = 0.01$  
   $P(A \cap B^c) = P(A)P(B^c|A) = 0.01 \cdot 0.01 = 0.0001$

3. Consider two independent fair coin tosses, and the following events:

   \[H_1 = \{\text{1st toss is a head}\}\]  
   \[H_2 = \{\text{2nd toss is a head}\}\]  
   \[D = \{\text{the two tosses have different results}\}\]

   Which of the following statements are true?

   1) $H_1$ and $H_2$ are pairwise independent events.
   2) $H_1$ and $D$ are pairwise independent events.
   3) $H_2$ and $D$ are pairwise independent events.
   4) $H_1$ and $H_2$ and $D$ are independent events.

   A) 1 only.  
   B) 1 and 2 only.  
   C) 1 and 3 only.  
   D) 1, 2, and 3 only.  
   E) 2 and 3 only.  
   F) 1, 2, 3, and 4.  
   G) 4 only.  
   H) None of the Above.

   Correct Answer: D. 1, 2, and 3 are true, but $P(H_1 \cap H_2 \cap D) = 0 \neq P(H_1)P(H_2)P(D)$
4. An internet service provider in a very small town has installed 10 modems to serve the needs of a population of 13 dialup customers. It is known that at a given time, each customer will need a modem connection with probability 0.6. What is the probability that there are more customers needing a connection than there are modems?

A) 0.0013
B) 0.0113
C) 0.0126
D) 0.0453
E) 0.0579
F) 0.1436
G) None of the Above.

Correct Answer: E

This is a typical binomial random variable problem. We want to find the probability of event \(A = \{\text{more customers than modems}\}\)

\[
P(A) = \sum_{k=11}^{13} p(k)
\]

where

\[
p(k) = \binom{n}{k} p^k (1-p)^{n-k}
\]

\[
P(A) = \binom{13}{11} (0.6)^{11}(0.4)^2 + \binom{13}{12} (0.6)^{12}(0.4)^1 + \binom{13}{13} (0.6)^{13}(0.4)^0
\]

\[
P(A) = 0.0453 + 0.0113 + 0.0013 = 0.0579
\]

5. A source transmits a string of symbols consisting of 0 and 1s through a noisy communication channel. Symbols are received correctly except for the following errors:

Given that a 0 is transmitted, a 1 is received with probability 0.01.

Given that a 1 is transmitted, a 0 is received with probability 0.02.

Errors in different symbol transmissions are independent. If the string of symbols 10011 is transmitted, what is the probability that it is received correctly?

A) 0.9010
B) 0.9039
C) 0.9319
D) 0.9510
E) 0.9992
F) None of the Above

Correct Answer: F

\[
P\{\text{1 received correctly}\} = 1 - 0.02 = 0.98
\]

\[
P\{\text{0 received correctly}\} = 1 - 0.01 = 0.99
\]

\[
P\{\text{10011 received correctly}\} = 0.98 \cdot 0.99 \cdot 0.99 \cdot 0.98 \cdot 0.98 = 0.9225
\]

6. If at first you don't succeed, try, try, try again. A computer will successfully send a message across a network with probability 0.7. The computer will retry sending the message until it is successfully sent. Given that we know that the computer will successfully transmit the message on or before the fourth attempt, what is the probability that the computer successfully sends the message on the first attempt?

A) 0.2100
B) 0.6943
C) 0.7000
D) 0.7057
E) 0.8215
F) None of the Above

Correct Answer: D
This is an example of conditioning a random variable on an event. This type of random variable is geometric.

\[ p_{X \mid A}(k) = \begin{cases} \frac{(1-p)p^{k-1}}{\sum_{m=1}^{k-1}(1-p)^m}, & \text{if } k = 1, 2, 3, 4 \\ 0, & \text{otherwise} \end{cases} \]

\[ p_{X \mid A}(k) = \frac{0.3^0 \cdot 0.7}{0.3^0 \cdot 0.7 + 0.3^1 \cdot 0.7 + 0.3^2 \cdot 0.7 + 0.3^3 \cdot 0.7} = 0.7057 \]

7. Suppose that \( X, Y, \) and \( Z \) are independent random variables. \( E[X] = 1, E[Y] = 2, \) and \( E[Z] = 3. \) \( \text{var}(X) = 1, \text{var}(Y) = 2, \) and \( \text{var}(Z) = 3. \) Compute the value of the following expression:

\[ E[2X + 3Y − Z − 10] + \text{var}(3Y − 2X + 2Z) \]

A) 5
B) 13
C) -3
D) 10
E) None of the Above

Correct Answer: B

\[ E[2X + 3Y − Z − 10] + \text{var}(3Y − 2X + 2Z) = 2E[X] + 3E[Y] − E[Z] − 10 + \text{var}(Y) + 4\text{var}(X) + 4\text{var}(Z) \]

\[ = 2(1) + 3(2) − 3 − 10 + 2 + 4(1) + 4(3) = 13 \]

8. You ask a person on a date over and over, and each time there is a probability 0.10 that he or she accepts, independent of previous attempts. While there are no guarantees that he or she will accept, how many tries is it expected that it will take until he or she accepts?

A) 10
B) 100
C) 1000
D) 5
E) None of the Above

Correct Answer: A

This is a geometric random variable with parameter \( p = 0.1. \) The expected value, \( E[X] = \frac{1}{p} = 10. \)

9. The time until a lightbulb burns out is modeled by an exponential variable \( X \) with an expected lifetime of 1 year. What is the probability that the bulb will burn out between 2 and 3 years?

A) 0.0067
B) 0.0498
C) 0.0855
D) 0.1353
E) 0.3679
F) None of the Above

Correct Answer: C

\[ P(2 \leq X \leq 3) = \int_{2}^{3} e^{-\lambda x} dx = [−e^{-\lambda x}]_{2}^{3} = −e^{-3} + e^{-2} = 0.0855 \]

10. The PDF of a random variable \( X \) is given by

\[ f_{X}(x) = \begin{cases} \frac{2}{3^2}, & 0 \leq x \leq 3, \\ 0, & \text{otherwise}. \end{cases} \]

Compute \( E[X] + \text{var}(X). \)

A) 1/2
B) 1
C) 3/2
D) 2
E) $\frac{5}{2}$
F) 3
G) $\frac{7}{2}$
H) $\frac{9}{2}$
I) None of the Above

Correct Answer: E

\[ E[X] = \int_0^3 x f_X(x) dx = \int_0^3 \frac{2}{9} x^2 dx = \left[ \frac{2}{27} x^3 \right]_0^3 = \frac{2}{27} (27) = 2. \]

\[ E[X^2] = \int_0^3 x^2 f_X(x) dx = \int_0^3 \frac{2}{9} x^3 dx = \left[ \frac{2}{36} x^4 \right]_0^3 = \frac{1}{18} (81) = \frac{9}{2} \]

\[ \text{var}(X) = E[X^2] - (E[X])^2 = \frac{9}{2} - 4 = \frac{1}{2} \]

11. On any given day a student’s score on an exam is a normal random variable with an average of 70 and a standard deviation of 10. The student takes an exam today. What is the probability that the student scores between 60 and 90? $\Phi(y)$ is the cumulative distribution function of a standard normal random variable.

A) $\Phi(30)$
B) $\Phi(3)$
C) $\Phi(90) - \Phi(60)$
D) $\Phi(2) - (1 - \Phi(1))$
E) $\Phi(2) - \Phi(1)$
F) $(1 - \Phi(2)) - \Phi(1)$
G) $\Phi(90) - (1 - \Phi(60))$
H) None of the Above

Correct Answer: D

\[ P(60 \leq X \leq 90) = P \left( \frac{60 - \mu}{\sigma} \leq Y \leq \frac{90 - \mu}{\sigma} \right) = P(-1 \leq Y \leq 2) = \Phi(2) - \Phi(-1) = \Phi(2) - (1 - \Phi(1)) \]

12. You take the bus, walk, or ride your bike to school. In the morning you toss a fair, six-sided die.
If you get a 1 or 2, you take the bus.
If you get a 3, 4, or 5, you walk.
If you get a 6, you ride your bike.

On average, it takes you 12 minutes if you take the bus, 14 minutes if you walk, and 6 minutes if you ride your bike. What is the expected amount of time it takes you to get to campus?

A) 9 minutes
B) 10 minutes
C) 11 minutes
D) 12 minutes
E) 13 minutes
F) 14 minutes
G) None of the Above

Correct Answer: D

\[ E[X] = \sum_{i=1}^n P(A_i) E[X|A_i] = \frac{1}{3} \cdot 12 + \frac{1}{2} \cdot 14 + \frac{1}{6} \cdot 6 = 4 + 7 + 1 = 12 \]

13. A joint probability density function is given by the following:

\[ f_{X,Y}(x,y) = \begin{cases} c, & \text{if } 0 \leq x \leq 1 \text{ and } 0 \leq y \leq 1, \\ \frac{1}{5}, & \text{if } 1 \leq x \leq 3 \text{ and } 1 \leq y \leq 3, \\ 0, & \text{otherwise}. \end{cases} \]
What is the value of $c$ that will make this a legitimate PDF?
A) $1/5$
B) $2/5$
C) $3/5$
D) $4/5$
E) 1
F) 0
G) None of the Above
Correct Answer: A

$$\int_{-\infty}^{\infty} f_{X,Y}(x,y)dxdy = 1 = c(1)(1) + \frac{1}{5}(2)(2) = c + \frac{4}{5}$$

$$\therefore c = \frac{1}{5}$$

14. You have a signal $s = +1$. It is corrupted by the addition of noise, $N$, which is uniformly distributed over $[-2,2]$. Thus the received signal is $s + N$. What is the probability that the received signal is less than zero?
A) $1 - \Phi\left(\sqrt{\frac{3}{2}}\right)$
B) $\Phi\left(\sqrt{\frac{3}{2}}\right)$
C) $1/8$
D) $1/4$
E) $1/2$
F) $3/4$
G) None of the Above
Correct Answer: D

$$P(s + N < 0) = P(N < -1) = \frac{1}{4}$$

15. A random variable $X$ has a probability mass function given by

$$p_X(x) = \begin{cases} 
\frac{1}{4}, & \text{if } x = 1, \\
\frac{1}{2}, & \text{if } x = \frac{3}{2}, \\
\frac{1}{4}, & \text{if } x = 3, \\
0, & \text{otherwise.}
\end{cases}$$

Find the cumulative distribution function, $F_X(x)$, of $X$. Compute the value of the following expression:

$$E[X] + F_X(2)$$

A) $3/4$
B) $7/4$
C) 2
D) $5/2$
E) 3
F) $13/4$
G) $15/4$
H) None of the Above
Correct Answer: D

$$E[X] = \frac{1}{4} \cdot 1 + \frac{1}{2} \cdot \frac{3}{2} + \frac{1}{4} \cdot 3 = \frac{1}{4} + \frac{3}{4} + \frac{3}{4} = \frac{7}{4}$$

$$F_X(2) = P(X \leq 2) = \frac{1}{4} + \frac{1}{2} = \frac{3}{4}$$
16. A random variable $X$ has a probability density function given by

$$f_X(x) = \begin{cases} \frac{2(x-2)}{9}, & \text{if } 2 \leq x \leq 5, \\ 0, & \text{otherwise}. \end{cases}$$

Find the cumulative distribution function, $F_X(x)$, and compute the value of the following expression:

$$F_X(0) + F_X(4) + F_X(6)$$

A) 0  
B) 1  
C) $\frac{13}{9}$  
D) $\frac{5}{3}$  
E) $\frac{17}{9}$  
F) 2  
G) 3  
H) None of the Above

Correct Answer: C

$$F_X(x) = \int_2^x \frac{2(t-2)}{9} \, dt = \left[ \frac{(t-2)^2}{9} \right]_2^x = \frac{(x-2)^2}{9} \text{ for } 2 \leq x \leq 5$$

$$F_X(0) + F_X(4) + F_X(6) = 0 + \frac{4}{9} + 1 = \frac{13}{9}$$

17. Suppose the probability of at least one flash flood in Provo is 0.01 this year, the probability of at least one major blizzard in Provo is 0.1 this year, and the probability of at least one tornado in Provo is 0.05 this year. Assume that these disasters are independent. What is the probability that this year we will have at least one flood, blizzard, or tornado in Provo this year?

A) 0.0001  
B) 0.0100  
C) 0.0600  
D) 0.1500  
E) 0.1600  
F) 0.1736  
G) 0.2315  
H) None of the Above

Correct Answer: H

$$P(\{\text{no disasters}\}) = (1 - 0.01)(1 - 0.1)(1 - 0.05) = 0.99 \cdot 0.9 \cdot 0.95 = 0.8464$$

$$P(\{\text{at least one disaster}\}) = 1 - P(\{\text{no disasters}\}) = 1 - 0.8464 = 0.1536$$
Part II

18. (4 points) Network connectivity problem. You are at home and have rigged up quite a strange collection of wireless switches to try to connect your laptop computer to the internet through your router. On any given attempt to connect to the internet, you have the following connection probabilities as indicated in the figure. Assume that the connection probabilities are independent from each other and that the attempts are independent from each other.

(a) What is the probability that on a given attempt you will successfully connect from your laptop to the internet (to two significant digits)?

Answer:
\[ P(SW_2 \leftrightarrow SW_5) = 1 - [1 - (0.9)(0.8)][1 - (0.9)(0.7)] = 0.8964 \]
\[ P(LT \leftrightarrow SW_5) = 1 - [1 - (0.8)P(SW_2 \leftrightarrow SW_5)][1 - (0.9)(0.5)] = 0.8444 \]
\[ P(\text{connection}) = (0.3)P(LT \leftrightarrow SW_5) = 0.2533 \approx 0.25 \]

(b) What is the probability that it takes three or less attempts to connect to the internet?

Answer:
This can be modeled by a geometric random variable \( X \) with parameter \( p = \frac{1}{4} \).

\[ P(X \leq 3) = \sum_{k=1}^{3} p_X(k) = \sum_{k=1}^{3} (1-p)^{k-1}p = \frac{1}{4} + \frac{3}{4} \cdot \frac{1}{4} + \frac{9}{16} \cdot \frac{1}{4} = \frac{16 + 12 + 9}{64} = \frac{37}{64} \approx 0.58 \]

(c) How many attempts on average does it take to connect to the internet?

Answer:
This is the expectation of the geometric random variable \( X \) described in part b.

\[ E[X] = \frac{1}{p} = 4 \]

We would expect it to take about 4 attempts on average to connect to the internet.

(d) What is the probability that out of ten attempts to connect to the internet, you are successful five times?

Answer:
This can be modeled by a binomial random variable \( Y \) with parameters \( p = \frac{1}{4} \) and \( n = 10 \).

\[ P(Y = 5) = p_Y(5) = \binom{10}{5} \left( \frac{1}{4} \right)^5 \left( \frac{3}{4} \right)^5 = 0.0584 \]
19. (3 points) You have the following joint probability mass function (PMF) of random variables $X$ and $Y$:

![Joint probability mass function diagram]

(a) Find the marginal PMFs of $X$ and $Y$, $p_X(x)$ and $p_Y(y)$.

$$p_X(x) = \sum_y p_{X,Y}(x,y) = \begin{cases} \frac{3}{10}, & x = 1 \\ \frac{3}{10}, & x = 2 \\ \frac{2}{5}, & x = 3 \\ 0, & \text{otherwise} \end{cases}$$

$$p_Y(y) = \sum_x p_{X,Y}(x,y) = \begin{cases} \frac{1}{5}, & y = 1 \\ \frac{3}{10}, & y = 2 \\ \frac{5}{10}, & y = 3 \\ 0, & \text{otherwise} \end{cases}$$

(b) Find the conditional PMFs $p_{X|Y}(x|2)$ and $p_{Y|X}(y|3)$.

$$p_{X|Y}(x|2) = \begin{cases} \frac{1}{3}, & x = 1 \\ \frac{1}{3}, & x = 2 \\ \frac{1}{3}, & x = 3 \\ 0, & \text{otherwise} \end{cases}$$

$$p_{Y|X}(y|3) = \begin{cases} 0, & y = 1 \\ \frac{1}{4}, & y = 2 \\ \frac{1}{4}, & y = 3 \\ \frac{1}{2}, & y = 4 \\ 0, & \text{otherwise} \end{cases}$$

(c) Find the joint cumulative distribution function (CDF) of $X$ and $Y$, $F_{X,Y}(x,y)$, and compute $F_{X,Y}(2.5, 2.5)$. The joint distribution is found by $P(X \leq x, Y \leq y)$. Thus, we have:

$$F_{X,Y}(2.5, 2.5) = p_{X,Y}(1, 1) + p_{X,Y}(1, 2) + p_{X,Y}(2, 1) + p_{X,Y}(2, 2) = 4 \cdot \frac{1}{10} = \frac{2}{5}$$