Final Exam Helpful Information Sheet

Discrete Uniform over \([a, b]\):

\[ p_X(k) = \begin{cases} \frac{1}{b-a+1}, & \text{if } k = a, a+1, \ldots, b, \\ 0, & \text{otherwise}, \end{cases} \]

\[ E[X] = \frac{a+b}{2}, \quad \text{var}(X) = \frac{(b-a)(b-a+2)}{12}, \quad M_X(s) = \frac{e^{sa}(e^{s(b-a+1)} - 1)}{(b-a+1)(e^s - 1)}. \]

Bernoulli with Parameter \(p\):

\[ p_X(k) = \begin{cases} p, & \text{if } k = 1 \\ 1-p, & \text{if } k = 0, \end{cases} \]

\[ E[X] = p, \quad \text{var}(X) = p(1-p), \quad M_X(s) = 1 - p + pe^s. \]

Binomial with Parameters \(p\) and \(n\):

\[ p_X(k) = \binom{n}{k}p^k(1-p)^{n-k}, \quad k = 0, 1, \ldots, n, \]

\[ E[X] = np, \quad \text{var}(X) = np(1-p), \quad M_X(s) = (1 - p + pe^s)^n. \]

Geometric with Parameter \(p\):

\[ p_X(k) = (1-p)^{k-1}p, \quad k = 1, 2, \ldots, \]

\[ E[X] = \frac{1}{p}, \quad \text{var}(X) = \frac{(1-p)}{p^2}, \quad M_X(s) = \frac{pe^s}{1 - (1-p)e^s}. \]

Poisson with Parameter \(\lambda\):

\[ p_X(k) = e^{-\lambda}\frac{\lambda^k}{k!}, \quad k = 0, 1, \ldots, \]

\[ E[X] = \lambda, \quad \text{var}(X) = \lambda, \quad M_X(s) = e^{\lambda(e^s - 1)}. \]

Continuous Uniform Over \([a, b]\):

\[ f_X(x) = \begin{cases} \frac{1}{b-a}, & \text{if } a \leq x \leq b, \\ 0, & \text{otherwise}, \end{cases} \]

\[ E[X] = \frac{a+b}{2}, \quad \text{var}(X) = \frac{(b-a)^2}{12}, \quad M_X(s) = \frac{esb - esa}{s(b-a)}. \]

Exponential with Parameter \(\lambda\):

\[ f_X(x) = \begin{cases} \lambda e^{-\lambda x}, & \text{if } x \geq 0, \\ 0, & \text{otherwise}, \end{cases} \]

\[ E[X] = \frac{1}{\lambda}, \quad \text{var}(X) = \frac{1}{\lambda^2}, \quad M_X(s) = \frac{\lambda}{\lambda - s}. \]

Normal with Parameters \(\mu\) and \(\sigma^2 > 0\):

\[ f_X(x) = \frac{1}{\sqrt{2\pi}\sigma}e^{-(x-\mu)^2/2\sigma^2} \]

\[ E[X] = \mu, \quad \text{var}(X) = \sigma^2, \quad M_X(s) = e^{(\sigma^2s^2/2) + \mu s}. \]
Conditional Probability:
\[ P(A|B) = \frac{P(A \cap B)}{P(B)} \]

Total Probability Theorem:
\[ P(B) = P(A_1)P(B|A_1) + \cdots + P(A_n)P(B|A_n) \]

Bayes’ Rule:
\[ P(A_i|B) = \frac{P(A_i)P(B|A_i)}{P(B)} \]

Expectation:
\[ E[X] = \sum x p_X(x) \quad \quad E[X] = \int_{-\infty}^{\infty} x f_X(x)dx \]

Variance:
\[ \text{var}(X) = E[(X - E[X])^2] \]

Cumulative Distribution Function:
\[ F_X(x) = P(X \leq x) \]

Covariance:

Correlation Coefficient:
\[ \rho(X,Y) = \frac{\text{cov}(X,Y)}{\sqrt{\text{var}(X)\text{var}(Y)}} \]

Law of Iterated Expectations:
\[ E[E[X|Y]] = E[X] \]

Sum of Random Number of Independent Random Variables:
\[ E[Y] = E[N]E[X] \]
\[ \text{var}(Y) = E[N]\text{var}(X) + (E[X])^2\text{var}(N) \]
\[ M_Y(s) = M_N(\log M_X(s)) \]

Law of Total Variance:
\[ \text{var}(X) = E[\text{var}(X|Y)] + \text{var}(E[X|Y]) \]

Moment Generating Function (Transform):
\[ M_X(s) = E[e^{sX}] \]
Markov Inequality:

\[ P(X \geq a) \leq \frac{E[X]}{a}, \quad \text{for all } a > 0. \]

Chebyshev Inequality:

\[ P(|X - \mu| \geq c) \leq \frac{\sigma^2}{c^2}, \quad \text{for all } c > 0 \]

Convergence in Probability:

\[ \lim_{n \to \infty} P(|Y_n - a| \geq \epsilon) = 0 \]

Convergence in Mean Square:

\[ \lim_{n \to \infty} E[(X_n - c)^2] = 0 \]

Central Limit Theorem: Let \( X_1, X_2, \ldots \) be a sequence of independent identically distributed random variables with common mean \( \mu \) and variance \( \sigma^2 \), and define

\[ Z_n = \frac{X_1 + \cdots + X_n - n\mu}{\sigma\sqrt{n}}. \]

Then, the CDF of \( Z_n \) converges to the standard normal CDF

\[ \Phi(z) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-x^2/2} dx, \]

in the sense that

\[ \lim_{n \to \infty} P(Z_n \leq z) = \Phi(z), \quad \text{for every } z. \]
Pascal PMF of order $k$:

$$p_{Y_k}(t) = \binom{t-1}{k-1} p^k (1-p)^{t-k}, \quad t = k, k+1, \ldots$$

Poisson with parameter $\lambda\tau$:

$$p_{N_{\tau}}(k) = P(k, \tau) = e^{-\lambda\tau} \frac{(\lambda\tau)^k}{k!}, \quad k = 0, 1, \ldots,$$

$$\mathbb{E}[N_{\tau}] = \lambda\tau \quad \text{var}(N_{\tau}) = \lambda\tau \quad M_{N_{\tau}}(s) = e^{\lambda\tau(e^s-1)}.$$

Erlang PDF of order $k$:

$$f_{Y_k}(y) = \frac{\lambda^k y^{k-1} e^{-\lambda y}}{(k-1)!}, \quad y \geq 0$$

Chapman-Kolmogorov Equation for the $n$-Step Transition Probabilities:

$$r_{ij}(n) = \sum_{k=1}^{m} r_{ik}(n-1) p_{kj}, \quad \text{for } n > 1, \text{ and all } i, j,$$

starting with

$$r_{ij}(1) = p_{ij}.$$

Steady State Convergence Theorem:

$$\lim_{n \to \infty} r_{ij}(n) = \pi_j, \quad \text{for all } i.$$

$$\pi_j = \sum_{k=1}^{m} \pi_k p_{kj}, \quad j = 1, \ldots, m,$$

$$1 = \sum_{k=1}^{m} \pi_k.$$

Absorption Probability Equations:

$$a_s = 1,$$

$$a_i = 0, \quad \text{for all absorbing } i \neq s,$$

$$a_i = \sum_{j=1}^{m} p_{ij} a_j, \quad \text{for all transient } i.$$

Equations for Expected Time to Absorption:

$$\mu_i = 0, \quad \text{for all recurrent states } i,$$

$$\mu_i = 1 + \sum_{j=1}^{m} p_{ij} \mu_j, \quad \text{for all transient states } i.$$

Steady-State Convergence Theorem for Continuous-Time Markov Chain:

$$\lim_{t \to \infty} \mathbf{P}(X(t) = j | X(0) = i) = \pi_j, \quad \text{for all } i.$$

$$\pi_j \sum_{k \neq j} q_{jk} = \sum_{k \neq j} \pi_k q_{kj}, \quad j = 1, \ldots, m,$$

$$1 = \sum_{k=1}^{m} \pi_k.$$