ECEn 370
Introduction to Probability
Section 001
Midterm
Winter, 2010

Instructor
Professor Brian Mazzeo

Closed Book
Testing Center Calculator
3-Hour Time Limit

IMPORTANT!

- WRITE YOUR NAME on every page of the exam.
- Answer questions 1-20 (Part I) on the provided bubble sheet.
- Questions 1-20 are worth 1 point each.
- Answer question 21 (Part II) directly on the exam.
- Question 21 has its points specified individually.
- Do not discuss the exam with other students.
Part I

1. A test for the probability-phobia-disease is assumed to be correct 95% of the time. We also assume that the probability of any random person in our class having this disease is 0.01. Given that you just tested positive for this disease, what is the probability that you do not have the disease?

\[ P(C^c | D) = \frac{P(C^c \cap D)}{P(D)} = \frac{P(B^c A^c) P(C^c)}{P(B^c A^c) P(C^c) + P(B A) P(C)} \]

\[ = \frac{(0.05)(0.99)}{(0.05)(0.99) + (0.95)(0.01)} = 0.8390 \]

A) 0.9900
B) 0.9000
C) 0.8390
D) 0.7250
E) 0.0500
F) 0.0100
G) 0.0099
H) 0.0098
I) 0.0001
J) None of the Above

2. I have the sets \( A, B, C \) which are events formed from a sample space, \( \Omega \). Which of the following statements are always true?

I) \( A \cup (A \cap B) = A \cap (A \cup B) \) **True**

II) \( A \cap A^c = \emptyset \) **True**

III) \( (A \cap B)^c = A^c \cup B \) **False**

IV) \( A \cup (B \cap C) = (A \cup B) \cap (A \cup C) \) **True**

V) \( A^c = \emptyset \) **False**

A) I only.
B) I and II only.
C) I and III only.
D) I, II, and IV only.
E) II and III only.
F) I, II, III, and V only.
G) IV only.
H) II and IV only.
I) I, II, III, IV, and V.
J) None of the Above.

3. It is Friday and after ECEn 370 you want to find a date for tonight and there are two people, \( A \) and \( B \), you would like to ask. The probability that \( A \) will accept an invitation is \( \frac{3}{10} \). The probability that \( B \) will accept an invitation is \( \frac{2}{3} \). You randomly select one of the two and extend an invitation. If you are rejected, then you go to the other person and extend an invitation. We make the following assumptions: (1) you can only end up with one date tonight and (2) once you are rejected, you can’t ask the same person again tonight. What is the probability that you will have a date with person \( B \) tonight?

\[ \left( \frac{1}{2} \right) \left( \frac{2}{3} \right) + \left( \frac{1}{2} \right) \left( \frac{7}{10} \right) \left( \frac{2}{3} \right) \]

\[ = \frac{1}{3} + \frac{7}{30} = \frac{17}{30} \]

A) 1/5
B) 2/3
C) 3/10
D) 17/30
E) 23/30
F) 17/23
G) 7/10
H) 8/9
I) 1/3
J) None of the Above.
4. Three students invite six people to an event tonight. The probability that any one of the six people will show up is $\frac{1}{6}$, independent of the others. What is the probability that more than three people show up?

A) $\frac{21}{32}$
B) $\frac{1}{8}$
C) $\frac{1}{4}$
D) $\frac{1}{2}$
E) $\frac{9}{32}$
F) $\frac{3}{16}$
G) $\frac{3}{32}$
H) $\frac{1}{32}$
J) None of the Above.

\[ \sum_{k=4}^{6} \binom{6}{k} \left( \frac{1}{2} \right)^k \left( \frac{1}{2} \right)^{6-k} = \left[ \binom{6}{4} + \binom{6}{5} + \binom{6}{6} \right] \frac{1}{64} = \left( \frac{6 \cdot 5}{2} + 6 + 1 \right) \left( \frac{1}{64} \right) = \frac{22}{64} = \frac{11}{32} \]

5. You have the following joint PMF of random variables $X$ and $Y$:

<table>
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<th></th>
<th>1/0</th>
<th>1/10</th>
<th>0</th>
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<td>2</td>
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<tr>
<td>3</td>
<td>1/10</td>
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Evaluate the following expression:

\[ E[X] + E[Y] + \text{var}(X) = \frac{25}{10} + \frac{17}{10} + \frac{125}{100} = \frac{250 + 170 + 125}{100} = \frac{545}{100} \]

A) $5.4500$
B) $4.2000$
C) $3.6500$
D) $3.2500$
E) $2.7000$
F) $2.5000$
G) $1.7000$
H) $1.2000$
I) $0.7500$
J) None of the Above.

\[ E[X] = \binom{\frac{3}{10}}{1} (1) + \binom{\frac{3}{10}}{2} (2) + \binom{\frac{3}{10}}{3} (3) + \binom{\frac{3}{10}}{4} (4) = \frac{3}{10} (3 + 2 + 12 + 8) = \frac{25}{10} \]

\[ E[Y] = \binom{\frac{5}{10}}{1} (1) + \binom{\frac{5}{10}}{2} (2) + \binom{\frac{5}{10}}{3} (3) = \frac{5}{10} (5 + 6 + 6) = \frac{17}{10} \]

\[ E[X^2] = \binom{\frac{5}{10}}{1} (2) + \binom{\frac{5}{10}}{2} (4) + \binom{\frac{5}{10}}{3} (9) + \binom{\frac{5}{10}}{4} (16) = \frac{1}{10} (3 + 4 + 6 + 16) = \frac{27}{10} \]

\[ \text{var}(X) = E[X^2] - \left( E[X] \right)^2 = \frac{27}{10} - \left( \frac{25}{10} \right)^2 = \frac{270 - 625}{100} = \frac{-355}{100} \]

\[ E[X^2] = E[X] + \text{var}(X) = \frac{25}{10} + \frac{-355}{100} = \frac{250 - 355}{100} = \frac{-105}{100} \]

6. Suppose that $X$, $Y$, and $Z$ are independent random variables. $E[X] = 3$, $E[Y] = 2$, and $E[Z] = 1$. var($X$) = 3, var($Y$) = 2, and var($Z$) = 1. Evaluate the following expression:

\[ E[X^2 + Y^2 + \text{var}(X-Y)] = \text{var}(X) + \text{var}(Y) = 3 + 7 = 10 \]

\[ E[X^2 + Y^2 + \text{var}(X)] = E[X^2] + E[Y^2] + \text{var}(X) = \frac{27}{10} + \frac{27}{10} + \frac{-355}{100} = \frac{270 - 355}{100} = \frac{-85}{100} \]

A) $5$
B) $7$
C) $10$
D) $11$
E) $16$
F) $17$
G) $23$
H) $25$
I) $28$
J) None of the Above.
7. Laman produces 1,000 widgets, 100 of which are defective. Lemuel produces 2,000 widgets, 150 of which are defective. The widgets from both are combined and a widget is selected at random and found to be defective. What is the probability it came from Laman?

\[
P(C|B) = \frac{P(C\cap B)}{P(B)} = \frac{\frac{100}{250}}{\frac{300}{5000}} = \frac{100}{250} = 0.4
\]

A) 0.0333  B) 0.0500  C) 0.0833  D) 0.2000  E) 0.2500  F) 0.2333  G) 0.3000

\[\rightarrow H) 0.4000  I) 0.4333  J) None of the Above.\]

8. Of a group of 100 students, 20 students are women. Two students are selected at random, without replacement, from the group. What is the probability that the second student selected is a woman, given that the first student selected was a woman?

A) 0.100  B) 0.192  C) 0.200  D) 0.215  E) 0.250  F) 0.300  G) 0.333  H) 0.350  I) 0.400  J) None of the Above.

\[\rightarrow \frac{99\text{ students left}}{19 \text{ are women}} \Rightarrow \frac{19}{99} \approx 0.1919\]

9. Let \(A\) and \(B\) be two independent events in \(S\). It is known that \(P(A \cap B) = 0.16\) and \(P(A \cup B) = 0.64\). Calculate \(P(A)\).

\[P(A \cup B) = P(A) + P(B) - P(A \cap B)\]

\[0.64 = P(A) + P(B) - 0.16\]

\[0.8 = P(A) + P(B)\]

\[P(A)P(B) = 0.16 \text{ from independence}\]

\[\therefore P(A) = P(B) = 0.4\]

A) 0.160  B) 0.200  C) 0.250  D) 0.300  E) 0.350  \[\rightarrow F) 0.400\]

G) 0.450  H) 0.500  I) 0.640  J) None of the Above.
10. A binary source generates digits 1 and 0. The probability of a 1 is 0.6 and the probability of a 0 is 0.4. A five-digit sequence is generated. What is the probability that two 1s and three 0s will occur in the sequence?

\[
\binom{5}{2} (0.6)^2 (0.4)^3 = \frac{5!}{2!3!} (0.6)^2 (0.4)^3
\]

\[
= \frac{5 \cdot 4 \cdot 3}{2 \cdot 1} \cdot (0.6)^2 \cdot (0.4)^3
\]

\[
= 10 \cdot (0.6)^2 \cdot (0.4)^3 \approx 0.2304
\]

A) 0.100
B) 0.150
C) 0.230
D) 0.250
E) 0.300
F) 0.317
G) 0.600
H) 0.683
I) 0.700
J) None of the Above.

11. The number of packets arriving at a server during any 1-second period is known to be Poisson distributed with \( \lambda = 2 \) arrivals per period. Find the probability that more than three packets will arrive during any 1-second period.

\[
P(X > 3) = 1 - P(X \leq 3)
\]

\[
= 1 - \sum_{k=0}^{3} \frac{\lambda^k e^{-\lambda}}{k!}
\]

\[
= 1 - e^{-2} \left( 1 + \frac{2}{1} + \frac{2^2}{2} + \frac{2^3}{6} \right)
\]

\[
= 1 - e^{-2} \left( 1 + 2 + 2 + \frac{8}{6} \right)
\approx 1 - e^{-2} \left( 5 + \frac{4}{3} \right)
\approx 0.1429
\]

A) 0.933
B) 0.124
C) 0.135
D) 0.140
E) 0.143
F) 0.148
G) 0.149
H) 0.150
I) 0.153
J) None of the Above.

12. A production line manufactures 1000-lb. weights that have 10 percent tolerance. Let \( X \) denote the weight of a manufactured weight. Assuming that \( X \) is a normal random variable with mean 1000 lb and variance 2500-lb, find the probability that a weight picked at random will be rejected. \( \Phi(y) \) is the cumulative distribution function of a standard normal random variable.

\[
\text{std dev} = 50
\]

\[
\bar{X} \cdot \mu = 1000, \; \sigma^2 = 2500
\]

\[
P(\text{rejected}) = 1 - P(900 \leq X \leq 1100)
\]

\[
= 1 - P\left( \frac{900 - 1000}{50} \leq \frac{X - \mu}{\sigma} \leq \frac{1100 - 1000}{50} \right)
\]

\[
= 1 - P\left( -2 \leq \frac{X - \mu}{\sigma} \leq 2 \right)
\]

\[
= 2 \left( 1 - \Phi(2) \right)
\]

A) \( \Phi(0.04) \)
B) \( \Phi(0.08) \)
C) 1 - \( \Phi(0.04) \)
D) 1 - \( \Phi(0.08) \)
E) 2(1-\( \Phi(0.04) \))
F) 2(1 - \( \Phi(0.08) \))
G) 1 - \( \Phi(2) \)
H) 2(1 - \( \Phi(2) \))
I) 2(1 - \( \Phi(4) \))
J) None of the Above.
13. A random variable is called a Rayleigh random variable if its pdf is given by

\[ f_X(x) = \begin{cases} \frac{x}{\sigma^2} e^{-x^2/(2\sigma^2)} & x > 0 \\ 0 & x < 0 \end{cases} \]

The radial distance [in meters (m)] of the landing point of a parachuting sky diver from the center of a target area is known to be a Rayleigh random variable with parameter \( \sigma^2 = 100 \). Find the probability that the sky diver will land within a radius of 10 m from the center of the target area.

\[ F_X(x) = \int_0^x \frac{t}{\sigma^2} e^{-t^2/(2\sigma^2)} dt \]
\[ = e^{-100} - e^{-200} = 1 - e^{-0.5} \approx 0.39347 \]

- A) 0.393
- B) 0.396
- C) 0.398
- D) 0.400
- E) 0.402
- F) 0.405
- G) 0.408
- H) 0.410
- I) 0.413
- J) None of the Above.

14. If at first you don't succeed, try, try, try again. A computer will successfully send a message across a network with probability 0.7. The computer will retry sending the message until it is successfully sent. Given that we know that the computer will successfully transmit the message on or before the fourth attempt, what is the probability that the computer successfully sends the message on the first attempt?

\[ A = \text{transmitted on or before } 4 \text{th attempt} \]
\[ P(A) = \sum_{k=1}^{4} p_X(k) = \sum_{k=1}^{4} (1-p)^{k-1} p = (\frac{7}{10}) (\frac{3}{10}) (\frac{7}{10}) (\frac{7}{10}) = 0.630 \]
\[ (\frac{7}{10}) (\frac{7}{10}) = 0.189 \]

\[ P(\exists = 1|A) = \frac{7}{9} \quad \frac{7}{9919} \approx 0.7057 \]

- A) 0.2100
- B) 0.6221
- C) 0.6233
- D) 0.6943
- E) 0.7000
- F) 0.7057
- G) 0.7222
- H) 0.7234
- I) 0.8215
- J) None of the Above.

15. You take the bus, walk, or ride your bike to school. In the morning you toss a fair, six-sided die.
If you get a 1 or 2, or 3 you take the bus.
If you get a 4, or 5 you walk.
If you get a 6 you ride your bike.
On average, it takes you 10 minutes if you take the bus, 15 minutes if you walk, and 6 minutes if you ride your bike. What is the expected amount of time it takes you to get to campus?

\[ \frac{1}{2} (10) + \frac{1}{3} (15) + \frac{1}{6} (6) = \frac{5}{2} + \frac{5}{2} + 1 = 11 \]

- A) 6 minutes
- B) 7 minutes
- C) 8 minutes
- D) 9 minutes
- E) 10 minutes
- F) 11 minutes
- G) 12 minutes
- H) 13 minutes
- I) 14 minutes
- J) None of the Above.
16. A random variable $X$ has a probability density function given by

$$f_X(x) = \begin{cases} \frac{2(x-5)}{9}, & \text{if } 5 \leq x \leq 8, \\ 0, & \text{otherwise.} \end{cases}$$

Find the cumulative distribution function, $F_X(x)$, and compute the value of the following expression:

$$F_X(1) + F_X(7) + F_X(9) = 0 + \frac{4}{9} + 1 = \frac{13}{9}$$

17. A joint probability density function is given by the following:

$$f_{X,Y}(x, y) = \begin{cases} c, & \text{if } 0 \leq x \leq 1 \text{ and } 0 \leq y \leq 1, \\ \frac{1}{4}, & \text{if } 2 \leq x \leq 3 \text{ and } 2 \leq y \leq 7, \\ 0, & \text{otherwise.} \end{cases}$$

What is the value of $c$ that will make this a legitimate PDF?

$$\frac{1}{4} (1) (5) + c = 1$$

$$\frac{1}{4} + c = 1$$

$$c = \frac{3}{4}$$

18. You ask a person on a date over and over, and each time there is a probability 0.01 that he or she accepts, independent of previous attempts. While there are no guarantees that he or she will accept, how many tries is it expected that it will take until he or she accepts?

$$p = 0.01$$

$$\frac{1}{p} = 100$$
19. A source transmits a string of symbols consisting of 0 and 1s through a noisy communication channel. Symbols are received correctly except for the following errors:
Given that a 0 is transmitted, a 1 is received with probability 0.02.
Given that a 1 is transmitted, a 0 is received with probability 0.01.
Errors in different symbol transmissions are independent. If the string of symbols 00110 is transmitted, what is the probability that it is received correctly?

\[
\begin{align*}
A) & \ 0.918 \\
B) & \ 0.920 \\
C) & \ 0.922 \\
D) & \ 0.924 \\
E) & \ 0.932 \\
F) & \ 0.934 \\
G) & \ 0.941 \\
H) & \ 0.943 \\
I) & \ 0.951 \\
J) & \ \text{None of the Above}
\end{align*}
\]

\[
\begin{align*}
&\text{For a 1, } 0.98 \\
&\text{For a 0, } 0.98 \\
&\left(0.98 \times 0.98 \times 0.99 \times 0.98 \times 0.98\right) = 0.92246
\end{align*}
\]

20. Suppose that this year the probability of at least one date is 0.9, the probability of at least one missing homework is 0.5, and the probability of at least one bombed test is 0.8. What is the probability that this year you will have at least one date or at least one missing homework or at least one bombed test?

A) 0.04 \\
B) 0.09 \\
C) 0.36 \\
D) 0.50 \\
E) 0.72 \\
F) 0.80 \\
G) 0.90 \\
H) 0.95 \\
I) 0.96 \\
\rightarrow J) \ \text{None of the Above}

\[
\begin{align*}
&\text{\textit{Miss-key}} \\
P\left(\text{No date } \land \text{ No missing HW } \land \text{ No Bombed Test}\right) \\
&\quad = 0.1 \times 0.5 \times 0.2 = 0.0100 \\
&1 - P(A) = 0.99
\end{align*}
\]
21. (5 points) Secret Diplomacy. You are hired by the State Department to analyze the probability of the following transactions. The CIA continually sends messages to the Ambassador in Tehran. To do this, the CIA makes two copies of the message. They send one to Dick Tracy with probability of success 0.7 and the other to James Bond with probability of success 0.8. Dick Tracy gets the message to the Ambassador with probability of success of 0.8 and James Bond with probability of success of 0.9. The probability of success of each leg of the journey is independent of the other legs.

a) What is the probability that a single message makes it to the Ambassador?

b) James Bond is being followed. If he has a message, there is a 0.3 probability that his message will be copied without his knowledge and delivered to Mr. X (i.e. he may still deliver his message to the Ambassador). If you know that Mr. X received a particular message from the CIA, what is the probability that the Ambassador received the same message?

c) The CIA decides to send an important message repeatedly to make sure that the Ambassador gets it. What is the probability that it takes three or less attempts to get the message to the Ambassador?

\[ P(\bar{X} \leq 3) = \sum_{k=1}^{3} q^k p = p + (1-p)p + (1-p)^2 p \approx 0.998 \]

\[ \text{This is a sum of ten Bernoulli random variables with parameter } p = (0.8)(0.3) = 0.24 \]

\[ E[\bar{X}] = E[\sum_{i=1}^{10} X_i] = 10 E[X] = 10 (0.24) = 2.4 \]

\[ \sigma^2 = \sqrt{\text{Var}(\bar{X})} = 1.35 \]

This is a sum of ten Bernoulli random variables with parameter \( p = (0.8)(0.3) = 0.24 \)