ECEn 370

Homework Problem Set 12

Due on Friday, March 30, 2012.

From Bertsekas and Tsitsiklis, *Introduction to Probability, 2nd Ed.* Note that these are labeled incorrectly as chapter 7 problems in the book.

1. (5 pts) Chapter 6 Problem 1.
2. (5 pts) Chapter 6 Problem 2.
3. (5 pts) Chapter 6 Problem 3.
4. (5 pts) Chapter 6 Problem 8.
5. (5 pts) Chapter 6 Problem 10.
6. (5 pts) Chapter 6 Problem 13.
7. (15 pts) Chapter 6 Problem 14.
8. MATLAB Problem (worth 30 points)
   This is modeled after Steven Kay's Poisson Random Process example.
   Suppose you have cars arriving at an intersection going east and cars arriving at the same intersection going north. The stop-signs are missing so cars just pass through the intersection at full speed. An accident will occur if two arrivals are within one-half second of each other. We can see that if more cars arrive at the intersection per unit of time, then the probability of an accident increases.

   Let T_i^{EW} be the arrival times of cars going east. They arrive with parameter λ.
   Let T_i^{NS} be the arrival times of cars going north. They also arrive with parameter λ.

   An accident occurs whenver $\left|T_i^{EW} - T_j^{NS}\right| \leq \tau$ for all i, j where τ is the time it takes for a car to clear the intersection.
   Let I_i be an indicator variable for an accident defined as
   $$I_i = \begin{cases} 1 & \text{if there is at least one NS arrival with } \left|T_i^{EW} - T^{NS}\right| \leq \tau \\ 0 & \text{otherwise} \end{cases}$$
   Formulated this way, T^{NS} can be any NS arrival time.

   The number of accidents in a time-interval $[0,\, t]$ is given by $X(t) = \sum_{i=1}^{N(t)} I_i$ where $N(t)$ is the Poisson counting random process for the EW traffic.
   The expected value of $X(t)$ is given by $\mathrm{E}\left[X(t)\right] = \lambda t \mathrm{E}\left[I_1\right]$.

   a) Show the following (these are intermediate steps to find $\mathrm{E}\left[X(t)\right]$ and $\mathrm{E}\left[X(t)\right]/t$):
   - i) $\mathrm{E}\left[I_i\right] = \mathrm{P}\left[\left|T_i^{EW} - T^{NS}\right| \leq \tau\right]$
   - ii) $\mathrm{P}\left[\left|T_i^{EW} - T^{NS}\right| \leq \tau\right] = \int_0^\infty \mathrm{P}\left[\left|T_i^{EW} - T^{NS}\right| \leq \tau \,\middle|\, T_i^{EW} = t\right] f_{T_i}(t)dt$
   - iii) You need this to solve the integral above: $\mathrm{P}\left[t - \tau \leq T^{NS} \leq t + \tau\right] = 1 - \exp(-2\lambda\tau)$
   - iv) $\mathrm{E}\left[I_i\right] = 1 - \exp(-2\lambda\tau)$
   - v) $\mathrm{E}\left[X(t)\right] = \lambda t \left(1 - \exp\left(-2\lambda\tau\right)\right)$
   - vi) $\mathrm{E}\left[X(t)\right]/t = \lambda \left(1 - \exp\left(-2\lambda\tau\right)\right)$

   b) For our $\tau = 0.5$ seconds above, plot the average number of accidents per hour for 0 to 60 arrivals per hour.

   c) Now simulate the Poisson process for this at 10, 20, 30, 40, 50, and 60 arrivals per hour.

- Generate vectors T^{EW} and T^{NS} of 10,000 arrival times by summing exponential random variables in the following manner:
$T_i = T_{i-1} + Z_i$ where $T_0 = 0$ and Z_i are exponential random variables.
- Plot some representative graphs of EW and NS arrivals so you can how they compare (~ over an hour).
- For each element of T^{EW} determine if an accident occured by scanning across the T^{NS} vector to see if there is an arrival within 0.5 seconds.
- Count the number of accidents that occured over the 10,000 arrival times.
- Plot the accident rate for the different arrival rates.
- Compare the simulated accident rate with the accident rate computed in part (b) above