Homework 2 Solutions
ECEn 670, Fall 2009

2.27 Suppose that** $p(k)$ **is a geometric pmf. Define** $q(k) = (p(k) + p(-k))/2$. **Show that this is a pmf and find its mean and variance. Find the probability of the the sets** $\{k : |k| \geq K\}$ **and** $\{k : k \text{ is a multiple of } 3\}$**. Find the probability of the sets** $\{k : k \text{ is odd}\}$**.

The properties of a pmf are that it is nonnegative and sums to 1. Because the geometric pmf is always nonnegative, that is not a problem. We can show that it sums to 1:

$$\sum_k q(k) = \frac{1}{2}\sum_k p(k) + \frac{1}{2}\sum_k p(-k) = 1$$

Thus, $q(k)$ is a valid pmf. The mean is computed by

$$\sum_k kq(k) = \frac{1}{2}\sum_k kp(k) + \frac{1}{2}\sum_k kp(-k) = \frac{1}{2}\sum_k kp(k) + \frac{1}{2}\sum(-k)p(k) = 0$$

The second moment is computed by

$$\sum_k k^2 q(k) = \frac{1}{2}\sum_k k^2 p(k) + \frac{1}{2}\sum_k k^2 p(-k) = \sum_k k^2 p(k)$$

which is the second moment of the original distribution.
The mean of the original geometric pmf is $1/p$ and its variance is $(1-p)/p^2$. This indicates that the second moment of the geometric pmf is $m^{(2)} = \sigma^2 + m^2 = \frac{1-p}{p^2} + \frac{1}{p^2} = \frac{2-p}{p^2}$. This is the same as the variance of $q(k)$. Thus the variance of $q(k)$ and of symmetrizing in this fashion is equal to the second moment of $p(k)$.

We will use the relation

$$\sum_{k=n}^{N+n} a^k = a^n \cdot \frac{1-a^{N+1}}{1-a}$$

to help us to find the probability of the sets.

For $\{k : |k| \geq K\}$,

$$\sum_{k:|k|\geq K} = \frac{1}{2}\sum_{k=K}^{\infty} p(k) + \frac{1}{2}\sum_{k=-K}^{-\infty} p(-k) = \sum_{k=K}^{\infty} p(k)$$

$p(k) = (1-p)^{k-1}p$; k = 1, 2, ...
$\sum_{k=K}^{\infty} p(k) = \sum_{k=K}^{\infty} (1-p)^{k-1} p = p\sum_{k=K-1}^{\infty} (1-p)^k = p(1-p)^{K-1} \cdot \frac{1}{1-(1-p)} = \frac{p(1-p)^{K-1}}{p-1}$

For $\{k : k \text{ is a multiple of } 3\}$,

$$\sum_{k:k \text{ is a multiple of } 3} q(k) = \frac{1}{2}\sum_{k=1}^{\infty} p(3k) + \frac{1}{2}\sum_{k=-1}^{-\infty} p(-3k) = \sum_{k=1}^{\infty} p(3k) = \sum_{k=1}^{\infty} (1-p)^{3k-1} p$$

$$= \frac{p}{1-p} \cdot \sum_{k=1}^{\infty} (1-p)^{3k} = \frac{p}{1-p} \cdot (1-p)^3 \cdot \frac{1}{1-(1-p)^3} = \frac{p(1-p)^2}{1-(1-p)^3}$$

For $\{k : k \text{ is odd}\}$,

$$\sum_{k:k \text{ is odd}} q(k) = 1 - \sum_{k:k \text{ is even}} q(k) = 1 - \sum_{k:k \text{ is multiple of } 2} q(k) = \frac{1}{2}\sum_{k=1}^{\infty} p(2k) + \frac{1}{2}\sum_{k=-1}^{-\infty} p(-2k) = \sum_{k=1}^{\infty} (1-p)^{2k-1} p$$

$$= \frac{p}{1-p} \cdot \sum_{k=1}^{\infty} (1-p)^{2k} = \frac{p}{1-p} \cdot (1-p)^2 \cdot \frac{1}{1-(1-p)^2} = \frac{p(1-p)}{1-(1-p)^2}$$

2.29 A probability space consists of a sample sapce Ω = all pairs of positive integers (that is, $\Omega = \{1,2,3,\dots\}^2$) and a probability measure P described by the pmf p defined by

$$p(k,m) = p^2(1-p)^{k+m-2}.$$

***(a) Find** $P(\{(k,m) : k \geq m\})$.*

$$\begin{aligned}
P(\{(k,m):k\geq m\}) &= \sum_{m=1}^{\infty}\sum_{k=m}^{\infty} p(k,m) = \sum_{m=1}^{\infty}\sum_{k=m}^{\infty} p^2(1-p)^{k+m-2} \\
&= \sum_{m=1}^{\infty}\left(p^2(1-p)^{m-2}\sum_{k=m}^{\infty}(1-p)^k\right) \\
&\overset{(l=k-m)}{=} \sum_{m=1}^{\infty}\left(p^2(1-p)^{m-2}\sum_{l=0}^{\infty}(1-p)^{l+m}\right) \\
&= \sum_{m=1}^{\infty}\left(p^2(1-p)^{2m-2}\sum_{l=0}^{\infty}(1-p)^{l}\right) \\
&= \sum_{m=1}^{\infty}\left(p^2(1-p)^{2m-2}\cdot\frac{1}{1-(1-p)}\right) \\
&= \sum_{m=1}^{\infty}\left(p^2(1-p)^{2m-2}\cdot\frac{1}{p}\right) \\
&= \sum_{m=1}^{\infty}\left(p(1-p)^{2m-2}\right) \\
&= p(1-p)^{-2}\sum_{m=1}^{\infty}(1-p)^{2m} \\
&= p(1-p)^{-2}\left[\left(\sum_{m=0}^{\infty}(1-p)^{2m}\right)-1\right] \\
&= p(1-p)^{-2}\left[\frac{1}{1-(1-p)^2}-1\right] \\
&= p(1-p)^{-2}\left[\frac{1}{2p-p^2}-1\right] \\
&= p(1-p)^{-2}\left[\frac{1-2p+p^2}{2p-p^2}\right] \\
&= \frac{p}{2p-p^2} = \frac{1}{2-p}
\end{aligned}$$

(b) Find the probability** $P(\{(k,m):k+m=r\})$ **as a function of** r **for** $r=2,3,\dots$ **Show the result is a pmf.

$$\begin{aligned}
p(r) &= P(\{(k,m):k+m=r\}) \\
&= P(\{(k,m):k=r-m\}) \\
&= p(r-1,1)+p(r-2,2)+p(r-3,3)+\cdots+p(2,r-2)+p(1,r-1) \\
&= \sum_{m=1}^{r-1}p(r-m,m) = \sum_{m=1}^{r-1}p^2(1-p)^{r-m+m-2} = \sum_{m=1}^{r-1}p^2(1-p)^{r-2} \\
&= p^2(1-p)^{r-2}(r-1)
\end{aligned}$$

To show that this is a pmf is a more difficult prospect. We need to show that $\sum_{r=2}^{\infty}p(r)=1$.

$$\begin{aligned}
\sum_{r=2}^{\infty}p(r) &= \sum_{r=2}^{\infty}p^2(1-p)^{r-2}(r-1) \\
&= p^2\sum_{r=2}^{\infty}(1-p)^{r-2}(r-1) \\
&\overset{(n=r-2)}{=} p^2\sum_{n=0}^{\infty}(1-p)^n(n+1) \\
&= p^2\sum_{n=0}^{\infty}n(1-p)^n + p^2\sum_{n=0}^{\infty}(1-p)^n \\
&= p^2(1-p)\sum_{n=0}^{\infty}n(1-p)^{n-1} + p^2\sum_{n=0}^{\infty}(1-p)^n \\
&= p^2(1-p)\cdot\frac{1}{(1-(1-p))^2} + p^2\cdot\frac{1}{1-(1-p)} \\
&= p^2(1-p)\cdot\frac{1}{p^2} + p^2\cdot\frac{1}{p} \\
&= 1-p+p = 1
\end{aligned}$$

To solve this, the identity $\sum_{n=0}^{\infty}nx^{n-1} = \frac{1}{(1-x)^2}$ was used.

***(c) Find the probability** $P(\{(k,m):k \text{ is an odd number}\})$.*

$$\begin{aligned}
P(\{(k,m):k\text{ is odd}\}) &= \sum_{k\text{ is odd}}\sum_{m=1}^{\infty}p(k,m) = \sum_{i=0}^{\infty}\sum_{m=1}^{\infty}p(2i+1,m) \\
&= \sum_{i=0}^{\infty}\sum_{m=1}^{\infty}p^2(1-p)^{2i+1+m-2} \\
&= \sum_{i=0}^{\infty}\sum_{m=1}^{\infty}p^2(1-p)^{2i+m-1} \\
&= \sum_{i=0}^{\infty}p^2(1-p)^{2i-1}\sum_{m=1}^{\infty}(1-p)^m \\
&\overset{(n=m-1)}{=} \sum_{i=0}^{\infty}p^2(1-p)^{2i-1}\sum_{m=1}^{\infty}(1-p)^{n+1} \\
&= \sum_{i=0}^{\infty}p^2(1-p)^{2i}\sum_{m=1}^{\infty}(1-p)^{n} \\
&= \sum_{i=0}^{\infty}p^2(1-p)^{2i}\cdot\frac{1}{1-(1-p)} = p\sum_{i=0}^{\infty}(1-p)^{2i} \\
&= p\cdot\frac{1}{1-(1-p)^2} = \frac{p}{2p-p^2} = \frac{1}{2-p}
\end{aligned}$$

(d) Define the event $F = \{(k,m) : k \geq m\}$. Find the conditional pmf $p_F(k,m) = P(\{k,m\}|F)$. Is this a product pmf?

$$p_F(k,m) = P(\{k,m\}|F) = \begin{cases} 0 & \text{if } k < m \\ \frac{P(\{k,m\}\cap F)}{P(F)} & \text{if } k \geq m \end{cases}$$
$$= \begin{cases} 0 & \text{if } k < m \\ \frac{P(\{k,m\})}{P(F)} & \text{if } k \geq m \end{cases}$$
$$= \begin{cases} 0 & \text{if } k < m \\ \frac{p^2(1-p)^{k+m-2}}{1/(2-p)} & \text{if } k \geq m \end{cases}$$
$$= \begin{cases} 0 & \text{if } k < m \\ p^2(2-p)(1-p)^{k+m-2} & \text{if } k \geq m \end{cases}$$

To see if this is a product pmf, then we can split this into $p_F(k,m) = p_F(k)\,p_F(m)$. We can see that a knowledge of k greatly influences the distribution of m and vice versa so this is not a product pmf.

2.32 Rita and Ravi are starting a company. Together they must raise at least $100,000. Each rasises money with a uniform distribution between $0 and $100,000. (Assume that money is continuous and that this is a uniform pdf - it is easier that way). If either of them individually raises more than $75,000 they have to fill out extra Internal Revenue Service forms. What is the probability that they raise enough money but neither has to fill out extra forms?

We can view this as a square formed by two uniform distributions where the two sides are uniform distributions from 0 to 100,000. The area that corresponds to them raising at least 100,000 is found by $X + Y \geq 100,000$. This is the upper right hand corner of the square. The area corresponding to both of them raising less than 75,000 is a square bounded on both sides by 0 to 75,000. The intersection of these areas is then a right triangle triangle with a size formed by 75,000 - 25,000 = 50,000. The area of this triangle is thus 1/2 * 50,000 * 50,000 = 1,250,000,000. The proportion of this area to the whole area, 10^10 is 1/8.

2.33 The probability that a man has a particular disease is 1/20. John is tested for the disease but the test is not totally accurate. The probability that a person with the disease tests negative is 1/50 while the probability that a person who does not have the disease tests positive is 1/10. John's test returns positive.
(a) Find the probability that John has the disease.
Let D_J denote the event that John has the disease, and $\overline{D_J}$ the event that John does not have the disease. Let T_J denote the event that the test is positive and $\overline{T_J}$ the event that the test is negative. From the information given we know that

$$P(D_J) = \frac{1}{20} \quad P\left(\overline{D_J}\right) = \frac{19}{20} \quad P(T_J|D_J) = \frac{49}{50} \quad P\left(T_J|\overline{D_J}\right) = \frac{1}{10}$$

We want to find $P(D_J|T_J)$. By Bayes Rule,

$$P(D_J|T_J) = \frac{P(T_J|D_J)\,P(D_J)}{P(T_J|D_J)\,P(D_J) + P\left(T_J|\overline{D_J}\right)P\left(\overline{D_J}\right)}$$

$$P(D_J|T_J) = \frac{\frac{49}{50}\cdot\frac{1}{20}}{\frac{49}{50}\cdot\frac{1}{20} + \frac{1}{10}\cdot\frac{19}{20}} = \frac{49}{144}$$

(b) You are now told that this disease is hereditary. The probability that a son suffers from the disease if his father does is 4/5, the probability that a son is infected with the disease even though his father is not is 1/95. What is the probability that Max has the disease given that his son Peter has the disease? (Note: you may assume that the disease affects only males so you can ignore the dependence on Peter's mother's health.)

Let D_M denote the event that Max has the disease, and $\overline{D_M}$ the event that Max does not have the disease. Let D_P denote the event that Peter has the disease, and $\overline{D_P}$ the event that Peter does not have the disease. We know

$$P(D_P) = P(D_M) = \frac{1}{20} \quad P(D_P|D_M) = \frac{4}{5} \quad P\left(D_P|\overline{D_M}\right) = \frac{1}{95}$$

$$P(D_M|D_P) = \frac{P(D_P|D_M)\,P(D_M)}{P(D_P)} = \frac{\frac{4}{5}\cdot\frac{1}{20}}{\frac{1}{20}} = \frac{4}{5}$$

(c) Michael is also tested but he worries about the accuracy of the test so he take the test 10 times. One of the ten tests turns out positive, the other nine negative. What is the probability that Michael has the disease?

Let D_L denote the event that Michael has the disease, and $\overline{D_L}$ the event that Michael does not have the disease. Let T_L denote the event that one test is positive and $\overline{T_L}$ the event that one test is negative. Let $T_L^{9/10}$ denote the event that one test is positive out of 10. We know

$$P(T_L|D_L) = \frac{49}{50} \quad P\left(\overline{T_L}|D_L\right) = \frac{1}{50} \quad P\left(T_L|\overline{D_L}\right) = \frac{1}{10} \quad P\left(\overline{T_L}|\overline{D_L}\right) = \frac{9}{10}$$

$$P\left(T_L^{9/10}|D_L\right) = \binom{10}{1} P(T_L|D_L)\cdot\left(P\left(\overline{T_L}|D_L\right)\right)^9 = \binom{10}{1}\frac{49}{50}\cdot\left(\frac{1}{50}\right)^9$$

$$P\left(T_L^{9/10}|\overline{D_L}\right) = \binom{10}{1} P\left(T_L|\overline{D_L}\right)\cdot\left(P\left(\overline{T_L}|\overline{D_L}\right)\right)^9 = \binom{10}{1}\frac{1}{10}\cdot\left(\frac{9}{10}\right)^9$$

$$\begin{aligned} P\left(D_L|T_L^{9/10}\right) &= \frac{P\left(T_L^{9/10}|D_L\right)\cdot P(D_L)}{P\left(T_L^{9/10}|D_L\right)\cdot P(D_L) + P\left(T_L^{9/10}|\overline{D_L}\right)\cdot P\left(\overline{D_L}\right)} \\ &= \frac{\binom{10}{1}\frac{49}{50}\cdot\left(\frac{1}{50}\right)^9\cdot\frac{1}{20}}{\binom{10}{1}\frac{49}{50}\cdot\left(\frac{1}{50}\right)^9\cdot\frac{1}{20} + \binom{10}{1}\frac{1}{10}\cdot\left(\frac{9}{10}\right)^9\cdot\frac{19}{20}} \\ &\approx 6.8\times 10^{-16} \end{aligned}$$

2.35 Given the uniform pdf on $[0, 1]$, $f(x) = 1;\ x \in [0, 1]$, ***find an expression for*** $P((a, b))$ ***for all real*** $b > a$. ***Define the cumulative distribution function or cdf*** F ***as the probability of the event*** $\{x : x \le r\}$ ***as a function of*** $r \in \Re$:

$$F(r) = P((-\infty, r]) = \int_{-\infty}^{r} f(x)\,dx.$$

Find the cdf for the uniform pdf. Find the probability of the event

$$\begin{aligned} G &= \left\{\omega : \omega \in \left[\tfrac{1}{2^k}, \tfrac{1}{2^k} + \tfrac{1}{2^{k+1}}\right) \text{ for some even } k\right\} \\ &= \textstyle\bigcup_{k \text{ even}} \left[\tfrac{1}{2^k}, \tfrac{1}{2^k} + \tfrac{1}{2^{k+1}}\right). \end{aligned}$$

$$P((a,b)) = \begin{cases} 0 & b < 0 \\ \int_0^b 1dx = b & a < 0 < b \le 1 \\ \int_a^b 1dx = b - a & 0 \le a < b \le 1 \\ \int_a^1 1dx = 1 - a & 0 \le a \le 1 \le b \\ 0 & 1 \le a < b \end{cases}$$

$$P(\{x : x \le r\}) = P((-\infty, r]) = \int_{-\infty}^{r} f(x)dx = \begin{cases} 0 & r < 0 \\ r & 0 \le r \le 1 \\ 1 & 1 < r \end{cases}$$

$$\begin{aligned} P\left(\bigcup_{k \text{ even}} \left[\tfrac{1}{2^k}, \tfrac{1}{2^k} + \tfrac{1}{2^{k+1}}\right)\right) &= \textstyle\sum_{k=2,4,6,\ldots} \frac{1}{2^{k+1}} \\ &= \tfrac{1}{2} \textstyle\sum_{k=1}^{\infty} \frac{1}{2^{2k}} \\ &= \tfrac{1}{8} \textstyle\sum_{k=1}^{\infty} \frac{1}{2^{2(k-1)}} \\ &= \tfrac{1}{8} \textstyle\sum_{k=0}^{\infty} \frac{1}{4^k} = \frac{1}{8} \cdot \frac{1}{1-1/4} = \frac{1}{6} \end{aligned}$$

2.38 Let $\Omega = \Re^2$ and suppose we have a pdf $f(x,y)$ such that

$$f(x,y) = \begin{cases} C & \text{if } x \geq 0,\ y \geq 0,\ x+y \leq 1 \\ 0 & \text{otherwise} \end{cases}$$

Find the probability $P(\{(x,y) : 2x > y\})$. Find the probability $P(\{(x,y) : x \leq \alpha\})$ for all real α. Is f a product pdf?

$$1 = C \int_0^1 \int_0^{1-x} dx dy = C \int_0^1 dx\, (1-x) = \frac{C}{2}$$

and thus $C = 2$.

$$\begin{aligned} P(\{(x,y) : 2x > y\}) &= C \textstyle\int_0^{1/3} dy \int_{y/2}^{1-y} dx \\ &= C \textstyle\int_0^{2/3} dy\, (1 - y - y/2) \\ &= C \textstyle\int_0^{2/3} dy\, (1 - 3y/2) \\ &= C \left(y - 3y^2/4\right)\big|_0^{2/3} = 2/3 \end{aligned}$$

Assume that $0 < \alpha < 1$.

$$\begin{aligned} P(\{(x,y) : x \leq \alpha\}) &= C \textstyle\int_0^{\alpha} dx \int_0^{1-x} dy \\ &= C \textstyle\int_0^{\alpha} dx\, (1-x) \\ &= C \left(x - \frac{x^2}{2}\right)\Big|_0^{\alpha} \\ &= 2\alpha\,(1 - \alpha/2) \end{aligned}$$

This is not a product pdf because we cannot split the pdf into two functions that when multiplied yield the original pdf.

2.40 Given the one-dimensional exponential pdf, find $P(\{x : x > r\})$ and the cumulative distribution function $P(\{x : x \leq r\})$ for $r \in \Re$.

$$\begin{aligned} P(\{x : x > r\}) &= \textstyle\int_r^{\infty} \lambda e^{-\lambda x} dx \\ &= e^{-\lambda x}\big|_r^{\infty} \\ &= e^{-\lambda r},\ r \geq 0 \end{aligned}$$

If $r < 0$ then $P(\{x : x > r\}) = 1$.

$$P(\{x : x \leq r\}) = 1 - P(\{x : x > r\}) = \begin{cases} 1 - e^{-\lambda r} & r \geq 0 \\ 0 & r < 0 \end{cases}$$

2.45 Let $\Omega = \Re^2$ and suppose we have a pdf such that

$$f(x,y) = \begin{cases} C|x| & -1 \leq x \leq 1;\ -1 \leq y \leq x \\ 0 & \text{otherwise} \end{cases}$$

Find the constant C. Is f a product pdf?

$$\begin{aligned}\int dx \int dy\ f(x,y) &= \int_{-1}^{1} dx \int_{-1}^{x} dy C|x| \\ &= \int_{-1}^{1} dx C|x|(x+1) \\ &= C\left(\int_0^1 dx\ x(-x+1) + \int_0^1 dx\ x(x+1)\right) \\ &= C2\left(\left.\frac{x^2}{2}\right|_0^1\right) = C\end{aligned}$$

This means that $C = 1$. This is not a product pdf because x and y are not independent.

3.1 Given the probability space $(\Re, \mathcal{B}(\Re), m)$***, where*** m ***is the probability measure induced by the uniform pdf*** f ***on [0,1] (that is,*** $f(r) = 1$ ***for*** $r \in$***[0, 1] and is 0 otherwise), find the pdf's for the following random variables defined on this space:***
(a) $X(r) = r^2$***,***
(b) $Y(r) = |r|^{1/2}$***,***
(c) $Z(r) = \ln|r|$***,***
(d) $V(r) = ar + b$***, where*** a ***and*** b ***are fixed constants.***
(e) Find the pmf for the random variable $W(r) = 3$ ***if*** $r \geq 2$ ***and*** $W(r) = 1$ ***otherwise.***

Let R be a random variable with pdf $f_R(r)$, which for this problem is the uniform pdf on [0,1]. Thus $X = X(R)$ and each part will be drived first for a general pdf f_R and then specialized to the uniform case.

$$F_R(r) = \begin{cases} 0, & r < 0 \\ r, & 0 \leq r \leq 1 \\ 1, & 1 < r \end{cases}$$

(a) Observe that $X = R^2$, so X can take only non-negative values. As a first step in finding the pdf of X we will find its cdf. For $x \geq 0$ we have

$$\begin{aligned}F_X(x) = P(X \leq x) &= P\left(R^2 \leq x\right) \\ &= P\left(|R|^2 \leq x\right) \\ &= P\left(|R| \leq \sqrt{x}\right) \\ &= P\left(-\sqrt{x} \leq R \leq \sqrt{x}\right) \\ &= F_R(\sqrt{x}) - F_R(-\sqrt{x}) \\ &= \left(\begin{cases} 0, & \sqrt{x} < 0 \\ \sqrt{x}, & 0 \leq \sqrt{x} \leq 1 \\ 1, & 1 < \sqrt{x} \end{cases}\right) - \left(\begin{cases} 0, & -\sqrt{x} < 0 \\ -\sqrt{x}, & 0 \leq -\sqrt{x} \leq 1 \\ 1, & 1 < -\sqrt{x} \end{cases}\right) \\ &= \left(\begin{cases} 0, & x < 0 \\ \sqrt{x}, & 0 \leq x \leq 1 \\ 1, & 1 < x \end{cases}\right) - 0 \\ &= \begin{cases} 0, & x < 0 \\ \sqrt{x}, & 0 \leq x \leq 1 \\ 1, & 1 < x \end{cases}\end{aligned}$$

$$f_X(x) = \frac{dF_X(x)}{dx} = \begin{cases} \frac{1}{2}x^{-1/2}, & 0 \leq x \leq 1 \\ 0, & \text{otherwise} \end{cases}$$

(b) $Y = |R|^{1/2}$, so Y can take only non-negative values.

$$\begin{aligned}
F_Y(y) = P(Y \le y) &= P\left(|R|^{1/2} \le y\right) \\
&= P\left(|R| \le y^2\right) \\
&= P\left(-y^2 \le R \le y^2\right) \\
&= F_R\left(y^2\right) - F_R\left(-y^2\right) \\
&= \left(\begin{cases} 0, & y^2 < 0 \\ y^2, & 0 \le y^2 \le 1 \\ 1, & 1 < y^2 \end{cases}\right) - \left(\begin{cases} 0, & -y^2 < 0 \\ -y^2, & 0 \le -y^2 \le 1 \\ 1, & 1 < -y^2 \end{cases}\right) \\
&= \left(\begin{cases} 0, & y^2 < 0 \\ y^2, & 0 \le y^2 \le 1 \\ 1, & 1 < y^2 \end{cases}\right) - 0 \\
&= \begin{cases} y^2, & 0 \le y \le 1 \\ 1, & 1 < y \end{cases}
\end{aligned}$$

$$f_Y(y) = \frac{df_Y(y)}{dy} = \begin{cases} 2y, & 0 \le y \le 1 \\ 0, & 1 < y \end{cases}$$

(c) $Z = \ln|R|$

$$\begin{aligned}
F_Z(z) = P(Z \le z) &= P(\ln|R| \le z) \\
&= P\left(e^{\ln|R|} \le e^z\right) \\
&= P(|R| \le e^z) \\
&= P(-e^z \le R \le e^z) \\
&= F_R(e^z) - F_R(-e^z) \\
&= \left(\begin{cases} 0, & e^z < 0 \\ e^z, & 0 \le e^z \le 1 \\ 1, & 1 < e^z \end{cases}\right) - \left(\begin{cases} 0, & -e^z < 0 \\ -e^z, & 0 \le -e^z \le 1 \\ 1, & 1 < -e^z \end{cases}\right) \\
&= \begin{cases} 0, & e^z < 0 \\ e^z, & 0 \le e^z \le 1 \\ 1, & 1 < e^z \end{cases} \\
&= \begin{cases} e^z, & -\infty < z \le 0 \\ 1, & 0 < z \end{cases}
\end{aligned}$$

$$f_Z(z) = \frac{df_Z(z)}{dz} = \begin{cases} e^z, & -\infty < z \le 0 \\ 0, & 0 < z \end{cases}$$

(d)$V(r) = ar + b$, where a and b are fixed constants.

$$\begin{aligned}
F_V(r) = P(V \le v) &= P(aR + b \le v) \\
&= P(aR \le v - b) \\
&= \begin{cases} P\left(R \le \frac{v-b}{a}\right) & \text{for } a > 0 \\ P\left(R \ge \frac{v-b}{a}\right) & \text{for } a < 0 \end{cases} \\
&= \begin{cases} F_R\left(\frac{v-b}{a}\right) & \text{for } a > 0 \\ 1 - F_R\left(\frac{v-b}{a}\right) & \text{for } a < 0 \end{cases}
\end{aligned}$$

$$f_V(v) = \frac{dF_V(v)}{dv} = \begin{cases} \frac{1}{a} f_R\left(\frac{v-b}{a}\right) & \text{for } a > 0 \\ -\frac{1}{a} f_R\left(\frac{v-b}{a}\right) & \text{for } a < 0 \end{cases}$$

$$f_V(v) = \frac{1}{|a|} f_R\left(\frac{v-b}{a}\right), \ b \le v \le a + b$$

(e) $p_W(w) = \begin{cases} 1, & \text{for } w = 1 \\ 0, & \text{otherwise} \end{cases}$

3.4 A random variable X has a uniform pdf on $[0, 1]$. What is the probability density function for the volume of a cube with sides of length X?

Let $Y = X^3$.

$$F_X(x) = \begin{cases} 0, & x < 0 \\ x, & 0 \le x \le 1 \\ 1, & 1 < x \end{cases}$$

$$F_Y(y) = P(Y \le y) = P\left(X^3 \le y\right) = P\left(X \le y^{1/3}\right) = F_X\left(y^{1/3}\right)$$

$$F_Y(y) = \begin{cases} 0, & y^{1/3} < 0 \\ y^{1/3}, & 0 \le y^{1/3} \le 1 \\ 1, & 1 < y^{1/3} \end{cases} = \begin{cases} 0, & y < 0 \\ y^{1/3}, & 0 \le y \le 1 \\ 1, & 1 < y \end{cases}$$

$$f_Y(y) = \frac{dF_Y(y)}{dy} = \begin{cases} \frac{1}{3}y^{-2/3}, & 0 \le y \le 1 \\ 0, & \text{otherwise} \end{cases}$$

3.6 Use the properties of probability measures to prove the following facts about cdf's: if F is the cdf of a random variable, then
(a) $F(-\infty) = 0$ and $F(\infty) = 1$.
(b) $F(r)$ is a monotonically nondecreasing function, that is, if $x \ge y$, then $F(x) \ge F(y)$.
(c) F is continuous from the right, that is, if ϵ_n, n = 1, 2, ... is a sequence of positive numbers decreasing to zero, then

$$\lim_{n\to\infty} F(r + \epsilon_n) = F(r).$$

Note that continuity from the right is a result of the fact that we defined a cdf as the probability of an event of the form $(-\infty, r]$. If instead we had defined it as the probability of an event of the form $(-\infty, r)$ (as is often done in Eastern Europe), then cdf's would be continous from the left instead of from the right. When is a cdf continuous from the left? When is it discontinuous?

(a)

$$F(-\infty) = \Pr(X \le -\infty) = P(\emptyset) = 0$$
$$F(\infty) = \Pr(X \le \infty) = P((-\infty, \infty)) = P(\Omega) = 1$$

(b) $x \ge y \Rightarrow (-\infty, y) \subset (-\infty, x) \Rightarrow P((-\infty, y)) \le P((-\infty, x)) \Rightarrow F(x) \le F(y)$

(c)

$$\begin{aligned} \lim_{n\to\infty} F(r + \epsilon_n) &= \lim_{n\to\infty} P((-\infty, \epsilon_n]) \\ &= P(\lim_{n\to\infty}(-\infty, \epsilon_n]) \text{ by continuity of probability} \\ &= P((-\infty, r]) \\ &= F(r) \end{aligned}$$

A cdf will be continuous from the left at a point r if the probability of the point r is zero, otherwise there will be a jump in the cdf equal to the probability of the point.

3.7 Say we are given an arbitrary cdf F for a random variable and we would like to simulate an experiment by generating one of these random variables as input to the experiment. As is typical of computer simulations, all we have available is a uniformly

distributed random variable U; that is, U has the pdf of 3.1. This problem explores a means of generating the desired random variable from U (this method is occasionally used in computer simulations). Given the cdf F , define the inverse cdf $F^{-1}(r)$ as the smallest value of $x \in \Re$ for which $F(x) \geq r$. We specify "smallest" to ensure a unique definition since F may have the same value for an interval of x. Find the cdf of the random variable Y defined by $Y = F^{-1}(U)$.
Suppose next that X is a random variable with cdf $F_X(\alpha)$. What is the distribution of the random variable $Y = F_X(X)$? This mapping is used on individual picture elements (pixels) in an image enhancement technique known as "histrogram equalization" to enhance contrast.

Note that U is a uniform random variable in [0, 1], so its pdf is $f_U(u) = 1$ for $0 \leq u \leq 1$ and its cdf is $F_U(u) = u$, for $0 \leq u \leq 1$, as well. In the first part of this problem, we are given U and a non-decreasing function $F(x)$ and we are asked to generate a random variable that has $F(x)$ as its cdf.

If we define $Y = F^{-1}(U)$ - where $F^{-1}(r)$ is the smallest value of $x \in \Re$, for which $F(x) \geq r$ - then we will prove that Y has the desired cdf, $F_Y(y) = F(y)$ and thus, it will be suitable for our simulation experiment.

Indeed

$$F_Y(y) = P(Y \leq y) = P\left(F^{-1}(U) \leq y\right)$$

Since F is a non-decreasing function, $F^{-1}(U) \leq y$ implies that $F\left(F^{-1}(U)\right) \leq F(y)$, or $U \leq F(y)$. So, it is

$$F_Y(y) = P(Y \leq y) = P(U \leq F(y)) = F_U(F(y)) = F(y)$$

In conclusion, by defining $Y = F^{-1}(U)$, we've managed to produce a random variable with a prescribed cdf, by means of a mapping on a uniform random variable U.

Suppose that X is a random variable with a known cdf $F_X(\alpha)$. If we define the random variable Y, as $Y = F_X(X)$, then for its cdf we have

$$F_Y(y) = P(Y \leq y) = P(F_X(X) \leq y)$$

where $0 \leq y \leq 1$.

If we define as $F^{-1}(y)$ the smallest value of $x \in \Re$ for which $F_X(x) \geq y$, then we can write

$$P(F_X(x) \leq y) = P\left(X \leq F_X^{-1}(y)\right) + P\left(F_X^{-1}(y) \leq X \leq \hat{F}_X^{-1}(y)\right)$$

where $\hat{F}_X^{-1}(y)$ is defined as the largest value of $x \in \Re$ for which $F_X(x) = y$. However,

$$P\left(F_X^{-1}(y) \leq X \leq \hat{F}_X^{-1}(y)\right) = F_X\left(\hat{F}_X^{-1}(y)\right) - F_X\left(F_X^{-1}(y)\right) = y - y = 0$$

This means that

$$F_Y(y) = P(F_X(X) \leq y) = P\left(X \leq F_X^{-1}(y)\right) = F_X\left(F_X^{-1}(y)\right) = y.$$